Vector spaces attached to an $m \times n$ matrix

Let $A \in \mathrm{M}_{m\times n}(\mathbb{C})$ be an $m \times n$ matrix with coefficients in $\mathbb{C}$. Denote by $\mathbb{C}^n_{\mathrm{r}}$ and $\mathbb{C}^n_{\mathrm{c}}$ the space of all row and column vectors with n entries respectively. Similarly we have n-dimensional vector spaces $\mathbb{C}^m_{\mathrm{r}}$ and $\mathbb{C}^m_{\mathrm{c}}$ over $\mathbb{C}$.

1. A matrix $A \in \mathrm{M}_{m\times n}(\mathbb{C})$ above defines a vector subspace $\mathfrak{R}(A) \subset \mathbb{C}^n_{\mathrm{r}}$ and a vector subspace $\mathfrak{N}(A) \subset \mathbb{C}^n_{\mathrm{c}}$, defined by

$$\mathfrak{R}(A) := \text{the linear span of rows of } A \subset \mathbb{C}^n_{\mathrm{r}},$$
$$\mathfrak{N}(A) \subset \mathbb{C}^n_{\mathrm{c}} = \{\vec{x} \in \mathbb{C}^n_{\mathrm{c}} : A \cdot \vec{x} = \vec{0}\}.$$

The dimension of the row span $\mathfrak{R}(A)$ of A is equal to the rank of A, while the dimension of the null space $\mathfrak{N}(A)$ of A is equal to $n - \mathrm{rk}_{\mathbb{C}}(A)$.

We can compute a basis of $\mathfrak{R}(A)$ and a basis of $\mathfrak{N}(A)$ using elementary row operations.

(1) Use elementary row operations to compute the row reduced echelon form B of A.

(2) The non-zero rows of B is a $\mathbb{C}$-basis of $\mathfrak{R}(A)$.

(3) Let $t_1, \ldots, t_{n-r}$, be the free variables among the variables $x_1, \ldots, x_n$, $r := \mathrm{rk}_{\mathbb{C}}(A)$. The non-free (or constrained) variable are indicated by the locations of the pivots of the row reduced echelon form B of A. Solve the constrained variables in terms of the free variables, so that the general solution of the equation $A \cdot \vec{x} = \vec{0}$ is expressed in the form

$$t_1\, \vec{v}_1 + \cdots + t_{n-r}\, \vec{v}_{n-r}.$$

The vectors $\vec{v}_1, \ldots, \vec{v}_{n-r}$ is a $\mathbb{C}$-basis of the null space $\mathfrak{N}(A)$ of A.

To summarize: The row span $\mathfrak{R}(A)$ of A is an r-dimensional vector subspace of $\mathbb{C}^n_{\mathrm{r}}$. Every element of $\mathfrak{R}(A)$ represents a linear equation generated by the m rows of A. The null space $\mathfrak{N}(A)$ of A are the set of all common solutions in $\mathbb{C}^n_{\mathrm{c}}$ to all equations given by elements of $\mathfrak{R}(A)$. The space $\mathfrak{R}(A)$ of equations is r-dimensional, and the space $\mathfrak{N}(A)$ of solutions is $n-r$-dimensioinal.

2. We also have the *column span* $\mathfrak{C}(A)$ of A, with

$$\mathfrak{C}(A) := \text{the linear span of columns of } A \subset \mathbb{C}^m_{\mathrm{c}},$$

and the *left null space*

$$\mathfrak{N}'(A) := \{\vec{y} \in \mathbb{C}^m_{\mathrm{r}} : \vec{y} \cdot A = \vec{0}\}.$$

of A. The fact that the row rank r of A is equal to the column rank of A tells us that $\dim_{\mathbb{C}} \mathfrak{C}(A) = r = \mathrm{rk}_{\mathbb{C}}(A)$. Correspondingly the dimension of the left null space $\mathfrak{N}'(A)$ is $m - r$.

To compute a $\mathbb{C}$-basis of $\mathfrak{C}(A)$ and a $\mathbb{C}$-basis of $\mathfrak{N}'(A)$, apply *elementary column operations* to A to get the *column reduced echelon form* B' attached to B. (Of course performing elementary column operations to A is equivalent to doing elementary row operations to the transpose A^t of A.) Then the non-zero columns of B' is a $\mathbb{C}$-basis of $\mathfrak{C}(A)$. The general solution of $\vec{y} \cdot A = \vec{0}$ can be written in the form

$$s_1 \cdot \vec{u}_1 + \cdots + s_{m-r} \cdot \vec{u}_{m-r}\,,$$

where $s_1, \ldots, s_{m-r}$ are the free variables among the variables $y_1, \ldots, y_m$, and $\vec{u}_1, \ldots, \vec{u}_{m-r}$ is a $\mathbb{C}$-basis of $\mathfrak{N}'(A)$.

A warning. Performing elementary row opeations on A will (likely) change the column span. As an example, suppose that $r = \mathrm{rk}(A) < \min(m,n)$ and B is the row reduced echelon form of A. The column span of B is the space of all elements of $\mathbb{C}^n_{\mathrm{c}}$ whose last $n - r$ entries are 0. The last subspace of $\mathbb{C}^n_{\mathrm{c}}$ is not equal to the column span $\mathfrak{C}(A)$ of A, unless the last $m - r$ rows of A are all zero.

3. The vector subspace $\mathfrak{C}(A) \subset \mathbb{C}^m$ can be characterized in terms of existence of solutions for a system of linear equations as follows:

$$\mathfrak{C}(A) = \{\vec{b} \in \mathbb{C}^m_{\mathrm{c}} \colon \text{there exists an element } \vec{x} \in \mathbb{C}^n_{\mathrm{c}} \text{ with } A \cdot \vec{x} = \vec{b}\}.$$

The reason is quite simple: $A \cdot \vec{x} = x_1 \cdot (\text{the first column of } A) + \cdots + x_n \cdot (\text{the } n\text{-th colmn of } A)$. Recall that if the equations $A \cdot \vec{x} = \vec{b}$ has a solution, then the general solutions can be computed using elementary row operations and written in the form

$$\vec{x_0} + t_1 \cdot \vec{v}_1 + \cdots t_{n-r} \cdot \vec{v}_{n-r},$$

where $t_1, \ldots, t_{n-r}$ are the free variables, $\vec{x_0}$ is a particular solution, i.e. $A \cdot \vec{x}_0 = \vec{b}_0$, and $\vec{v}_1, \ldots, \vec{v}_{n-r}$ is a basis of $\mathfrak{N}(A)$.

Similarly we have

$$\mathfrak{R}(A) = \{\vec{c} \in \mathbb{C}^n_{\mathrm{r}} \colon \text{there exists an element } \vec{y} \in \mathbb{C}^m_{\mathrm{r}} \text{ with } \vec{y} \cdot A \cdot = \vec{c}\}.$$