MATH 240 ASSIGNMENT 9, SPRING 2018

Due in class on Friday, March 30

This problem, with many parts, is intended as an illustration of the general method of computing the exponential of a square matrix A which we explained in class. In general this method does *not* compute a good basis for the Jordan form of the given square matrix A, but has the benefit that it always works.

Summary of the method. The matrix D is a block-diagonal matrix, i.e. it is composed of square matrices of smaller size placed diagonally. The sizes of the smaller matrices are the multiplicities of the distinct eigenvalues in the characteristic polynomial of A. Say $\lambda_1, \dots, \lambda_r$ are the distinct eigenvalues of A, and let $e_1, \dots, e_r$ be their multiplicities in the characteristic polynomial $\det(x \cdot \mathrm{I}_n - A)$. What you do is the following.

1. For each eigenvalue λ_i, compute a basis of the space of all generalized eigenvectors for λ_i. The latter is by definition $\mathrm{Ker}(A-\lambda_i)^{e_i}$, and its dimension is e_i. Say $\vec{v}_{i,1}, \dots, \vec{v}_{i,e_i}$ a basis of $\mathrm{Ker}(A-\lambda_i)^{e_i}$ you computed by Gaussian elimination.

2. Let i be an integer between 1 and r as in step 1 above. For each of the vectors $\vec{v}_{i,1}, \dots, \vec{v}_{i,e_i}$, compute $A \cdot \vec{v}_{i,j}$ and write it as a linear combination of $\vec{v}_{i,1}, \dots, \vec{v}_{i,e_i}$. Use these coefficients you found for $A \cdot \vec{v}_{i,j}$ as the j-th column of an $e_i \times e_i$ matrix. Do this for $j = 1, \dots, e_i$, and you get an $e_i \times e_i$ matrix D_i.

3. Put the square matrices $D_1, \dots, D_r$ along the diagonal, you get an $n \times n$ matrix in block-diagonal form. Similarly the r groups of vectors

$$\vec{v}_{1,1}, \dots, \vec{v}_{1,e_1}; \cdots ; \vec{v}_{r,1}, \dots, \vec{v}_{i,e_r}$$

   form the columns of an invertible $n \times n$ matrix C. We have

$$A \cdot C = C \cdot D, \quad \text{equivalently } \ A = C \cdot D \cdot C^{-1}$$

4. Each of smaller square matrix D_i has the property that $\exp(tD_i)$ can be computed in closed form. The key property is this: write

$$D_i = \lambda_i \cdot \mathrm{I}_{e_i} + N_i,$$

   then

$$N_i^{e_i} = 0.$$

   Hence

$$\exp(tN_i) = e^{\lambda_i t}\left(\mathrm{I}_{e_i} + tN_i + \frac{t^2}{2}N_i^2 + \cdots + \frac{t^{e_i-1}}{(e_i-1)!}N_i^{e_i-1}\right).$$

   So we get a formula for $\exp(tD)$, and

$$\exp(tA) = C \cdot \exp(tD) \cdot C^{-1}.$$

In this problem/illustration, $n = 6$, and A is the following 6×6 matrix

$$\begin{pmatrix} 1 & -2 & 0 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

(a) Show that the characteristic polynomial is

$$(x+1)^2 \cdot (x-3)^4$$

The general method will produce a suitable basis $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$ of $\mathbb{C}^6_{\text{col}}$; they form the columns of an invertible 6×6 matrix C, such that the matrix exponential $\exp(tD)$ of the matrix $D := C^{-1} \cdot A \cdot C$ is easy to write down directly.

(b) Find a basis $\vec{v}_1, \vec{v}_2$ of the space V_1 of all generalized eigenvectors for the eigenvalue -1. Recall that by definition $V_1 := \operatorname{Ker}\left((A + 1 \cdot \mathrm{I}_6)^2\right)$.

Remark. (b1) The general theory tells us that $\dim(V_1) = 2$, the multiplicity of -1 in the characteristic polynomial.
(b2) There are many choices of possible bases of V_1. In the present situation $\operatorname{Ker}(A + 1 \cdot \mathrm{I}_6)$ is one-dimensional, and often people pick $\vec{v}_1$ to be a non-zero element of $\operatorname{Ker}(A + 1 \cdot \mathrm{I}_6)$.

(c) Compute the 2×2 matrix, say written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, so that

$$A \cdot \vec{v}_1 = a\vec{v}_1 + c\vec{v}_2, \quad A \cdot \vec{v}_2 = b\vec{v}_1 + d\vec{v}_2$$

Call this matrix D_1.

(d) Compute a basis $\vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$ of the space V_2 of all generalized eigenvectors for the eigenvalue 3. Recall that by definition $V_3 := \operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)^4\right)$.

Remark. The general theory tells us that $\dim(V_2) = 4$. However you may save some time if you pay attention in your calculation. Observe that we have

$$\operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)\right) \subseteq \operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)^2\right) \subseteq \operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)^3\right) \subseteq \operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)^4\right)$$

So if $(A - 3 \cdot \mathrm{I}_6)^i$ has rank 2 for some i between 1 and 3, then $V_2 = \operatorname{Ker}\left((A - 3 \cdot \mathrm{I}_6)^i\right)$ and you don't have to compute higher powers of $(A - 3 \cdot \mathrm{I}_6)$.

(e) For each of $\vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$, compute its image under A and write the result as a linear combination of $\vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6$; that gives you a column of the desired 4×4 matrix D_2. For instance, there are uniquely defined numbers c_3, c_4, c_5, c_6 such that

$$A \cdot \vec{v}_3 = c_3\vec{v}_3 + c_4\vec{v}_4 + c_5\vec{v}_5 + c_6\vec{v}_6.$$

These four numbers c_3, c_4, c_5, c_6 form the first column of D_2.

(f) Let $N_2 := D_2 - 3\mathrm{I}_4$. The theory tells us that $N_2^4 = 0$. Find the smallest positive integer e such that $N_2^e = 0$.

(g) Find $\exp(tD_1)$ and $\exp(tD_2)$ explicitly, and write down $\exp(tD)$, where D is the 6×6 matrix in diagonal block form, with D_1, D_2 as the diagonal blocks.

(h) Write down an invertible 6×6 matrix C such that

$$\exp(tA) = C \cdot \exp(tD) \cdot C^{-1}$$