MATH 240 ASSIGNMENT 6, SPRING 2018

Due in class on Friday, March 2

Part 1. Read DELA 7.6, 7.7 and 8.1–8.3.

Part 2. Do and write up the following problems from DELA:

- (True/False problems) "additional problems" 17 and 18 in § 7.7, p. 490
- "additional problems" 15 in § 7.7, p. 490

Part 3. Problems from old final exams.

- fall 2016 final exam, problem 4
- fall 2016 final exam, problem 9. (Note: The domain of the function $y(x)$ in this problem is $\{x \in \mathbb{R} \mid x > 0\}$. The form of this equation is named "Cauchy–Euler" in the book. Under the change of variables $x = e^t$, so that $x\frac{d}{dx} = \frac{d}{dt}$, the equation in question reduces to a linear ODE with constant coefficients.)
- fall 2016 final exam, problem 11
- fall 2015 final exam, problem 6
- fall 2015 final exam, problem 9
- spring 2016 final exam, problem 11
- spring 2016 final exam, problem 12

Part 4. Extra credit problem
Consider the following four-dimensional vector space $\mathbb{H}$ over $\mathbb{R}$ with a given basis denoted by $1, i, j, k$, so that every element z of $\mathbb{H}$ can be written as

$$z = a + bi + cj + dk$$

for uniquely determined real numbers a, b, c, d. In addition to vector additions and scalar multiplications, there is a "vector multiplication" operation

$$\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{H}, \qquad (z, w) \mapsto \mu(z, w)$$

on $\mathbb{H}$. We will abbreviate $\mu(z, w)$ to $z \cdot w$. (Do not confuse the "dot" as an "inner product".) This product on $\mathbb{H}$ satisfies the distributive law (on both sides), but *not* commutative. To define/specifies this product, we only need to define the product of all pairs of the four basis elements. The element "1" is the unity for multiplication, so it suffices to define/specify the products of any two elements of {i,j,k}; they are given by

$$i^2 = j^2 = k^2 = -1, \qquad i \cdot j = k = -j \cdot i, \quad j \cdot k = i = -k \cdot j, \quad k \cdot i = j = -i \cdot k.$$

(a) Write down explicitly the product of two elements

$$z_1 = a_1 + b_1 i + c_1 j + d_1 k \quad \text{and} \quad z_2 = a_2 + b_2 i + c_2 j + d_2 k$$

in $\mathbb{H}$.

(b) Show that the product defined above is indeed associative.

(c) Let $z = a + bi + cj + dk \neq 0$ be a non-zero element of $\mathbb{H}$. Let

$$\bar{z} = a - bi - cj - dk$$

Show that $z \cdot \bar{z} = \bar{z} \cdot z \in \mathbb{R}_{>0} \cdot 1$ and that z has a multiplicative inverse z^{-1}.

Define the length $|z|$ of any element $z \in \mathbb{H}$ by

$$|z|^2 \cdot 1 = z \cdot \bar{z} = \bar{z} \cdot z.$$

(d) Let $z = a + bi + cj + dk$ be a non-zero element of $\mathbb{H}$. Define a linear operator T_z on $\mathbb{H}$ by

$$T_z(w) := z \cdot w \cdot z^{-1} \qquad \forall w \in \mathbb{H}.$$

Show that T_z is an orthogonal operator, in the sense that

$$|T_z(w)| = |w| \qquad \forall w \in \mathbb{H}$$

and that the three dimensional vector subspace $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ is closed under T_z. Can you identify the restriction of T_z to $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k \cong \mathbb{R}^3$ in a geometric way?

(e) Interpret the scalar product and cross product on $\mathbb{R}^3$ in terms of the "good multiplication" on $\mathbb{H}$.

(This provides one explanation as to why the cross product on $\mathbb{R}^3$ is a "funny thing".)