MATH 240 ASSIGNMENT 3, SPRING 2018

Due in class on Friday, February 2

Part 1. Read DELA 4.7, 4.8, 4.9, 4.10, 6.1, 6.5.

CHANGE OF BASIS WITHOUT MEMORIZATION: Among the above topics, "change of basis" may be conceptually more demanding. The notation "$P_{C \leftarrow B}$" is not a standard or universally adopted one. Personally I would prefer to do it directly from first principals, which is quite straight-forward and more reliable. (I can never remember the definition of "$P_{C \leftarrow B}$"; there are at least two choices and it's unclear what the notation $P_{C \leftarrow B}$ actually suggest.)

Suppose you an "old basis" $o_1, \ldots, o_d$ and a "new basis" $n_1, \ldots, n_d$ of a vector space V. The two coordinate system are related by an invertible $d \times d$ matrix P, so that if a vector is represented in the old coordinate system by a column vector **(old coordinates)**, while in the new coordinate system the same vector is represented by a column vector **(new coordinates)**, then

$$\textbf{(new coordinates)} = P \cdot \textbf{(old coordinates)}$$

What can this change-of-coordinate matrix P be? We can find out by testing a few examples. The vector o_1 is represented in the old coordinate system by the column vector $\mathbf{e}_1 := (1, 0, \ldots, 0)^t$. The new coordinates of this vector is $P \cdot \mathbf{e}_1$, which is the first column $(p_{11}, p_{21}, \ldots, p_{n1})^t$ of P. This means that the first column of of P is determined by

$$o_1 = p_{11} \cdot n_1 + p_{21} \cdot n_2 + \cdots + p_{d1} \cdot n_d$$

(You can also say that $(p_{11}, p_{21}, \ldots, p_{n1})$ are the new coordinates of the first vector o_1 in the old basis.) The other columns of P are determined similarly: the columns of P are the "new coordinates" of the old basis vectors $o_1, \ldots, o_d$. (In the notation of the book, P would be written as $P_{\mathbf{new} \leftarrow \mathbf{old}}$.)

Part 2. Problems from old final exams.

- Spring 2016 final exam, problem 1
- Spring 2016 final exam, problem 7
- Spring 2016 final exam, problem 8
- Fall 2015 final exam, problem 3
- Fall 2015 final exam, problem 5
- Fall 2014 make-up final, problem 2
- Spring 2014 final exam, questions b, d, e, f, h in the T/F problem I. (You need to provide a sufficient reason when the statement is true, and a counter-example when the statement is false.)
- Spring 2014 final exam, problem VII. Also, produce a basis for the source vector space of the linear transformation L and a basis for the target vector space of L, and write down the matrix representation of L with respect to the bases you picked.
- Fall 2012 make up final, the T/F problem 4

- Fall 2012 make up final, the T/F problem 5

Part 3. Extra credit problems.

A. Let m_1, m_2, n be positive integers. Let $A = (a_{ij}) \in \mathrm{M}_{m_1,n}(\mathbb{C}), B = (b_{kj}) \in \mathrm{M}_{m_2,n}(\mathbb{C})$ be two matrices with the same number of columns. Let V be the subset of $\mathbb{C}^n$ consisting of all n-tuples $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that

   (i) $(x_1, \ldots, x_n)$ lies in the linear span of the rows of B, and

   (ii) $\sum_{j=1}^{n} a_{ij} \cdot x_j = 0$ for $i = 1, \ldots, m_1$

   Produce a method/algorithm to compute a $\mathbb{C}$-basis of the vector subspace V of $\mathbb{C}^n$.

B. Let $X = (x_{ij})_{1 \le i,j \le n}$ be an $n \times n$ matrix whose entries $x_{i,j}$ are variables. Let $C, D \in \mathrm{M}_n(\mathbb{C})$ be square matrices with entries in $\mathbb{C}$. Consider the function $\det(CX + D)$ as a function in n^2 variables x_{ij}. Let

$$\partial \det(CX + D) := \left(\frac{\partial}{\partial x_{ij}} \det(CX + D) \right)_{1 \le i,j \le n}$$

   be the $n \times n$ matrix whose entries are the partial derivatives of $\det(CX + D)$. Show that

$$\partial \det(CX + D) = (\mathrm{cof}(CX + D) \cdot C)^t \,,$$

   where $\mathrm{cof}(CX + D)$ is the cofactor matrix of $CX + D$. In other words $\frac{\partial}{\partial x_{ij}} \det(CX + D)$ is equal to the (j,i)-entry of the matrix product $\mathrm{cof}(CX + D) \cdot C$, for every pair (i, j) with $1 \le i, j \le n$. Formally the above formula for $\partial \det(CX + D)$ can then be restated as

$$\partial \log(\det(CX + D)) = \left((CX + D)^{-1} \cdot C \right)^t \,.$$

   (Recall that the cofactor matrix of an $n \times n$ matrix B is defined in such a way that $\mathrm{cof}(B) \cdot B = B \cdot \mathrm{cof}(B) = \det(B) \cdot \mathrm{I}_n$ holds. In the present situation we have

$$\mathrm{cof}(CX + D) \cdot (CX + D) = (CX + C) \cdot \mathrm{cof}(CX + D) = \det(CX + D) \cdot \mathrm{I}_n \,.$$

   So if $\det(CX + D)$ does not vanish identically, then

$$\mathrm{cof}(CX + D) \cdot C = \det(CX + D) \cdot (CX + D)^{-1} \cdot C,$$

   where $(CX + D)^{-1}$ is regarded as a square matrix whose entries are rational functions in the n^2 variables x_{ij}'s.)