MATH 240 ASSIGNMENT 2, SPRING 2018

Due in class on Friday, January 26

Some tips:

- How to compute a basis of the linear span of a subset S of a vector space V. (Say the base field is $\mathbb{C}$.)

  Pick a basis of V, i.e. a linear coordinate system on V. Then V is identified with row vectors $\mathbb{C}^n$, and S is identified with a subset of $\mathbb{C}^n$, and we get a matrix A whose rows correspond to elements of S. Row reduce A to a row-echelon form. The non-zero rows correspond to elements of a basis of the linear span of S.

- How to compute a basis of the columns of a matrix B?

  Column reduce B to a column-echelon form. Equivalently, use row operations to reduce the transpose B^t of B to row-echelon form.

  Warning: Applying row operations to B will change the linear span of columns.

Part 1. (a) Read DELA 4th ed. 2.6, 3.1–3.5, 4.2, 4.3, 4.4, 4.5, 4.6 and make sure you can do all core problems in these sections. It is important to read chapter 3 if you are not familiar with the concept of determinants.

Summary of key definitions.

- A subset W of a vector space V is a vector space is a *vector subspace* if and only if W is stable under vector addition and scalar multiplication, or equivalently W is stable under all linear combinations. If so then W inherits a vector space structure from V

- The *linear span* of a subset S of a vector space V is the subset formed by all linear combinations of elements of S; it is also the smallest vector subspace of V which contains S.

- A subset S of a vector space V is *linearly independent* if and only if every non-trivial linear combination of elements of S is non-zero. (Here "non-trivial" means that some coefficients in the linear combination is non-zero.)

- A subset S of a vector space V if it is linearly independent and its linear span is V. This means that every element of V can be expressed as a linear combination of elements of S in a unique way, so that S determines a *linear coordinate system* of V.

- Any two bases of a vector space V have the same cardinality (number of elements); this number is the *dimension* of V.

(b) Do (but do *not* hand in) the following problems in DELA:

§2.6 Problem 20

§3.3 Problems 35, 66

§3.4 Problems 13, 27

§4.2 T/F Review e, h, i, j

§4.3 T/F Review b, e, f, g

§4.4 T/F Review d, f, g, k; problems 41, 42

§4.5 T/F review d, g, h; problems 29, 30

§4.6 T/F Review d, e, g, h, j; problems 23, 25, 29

Part 2. Do and write up the following problems from DELA:

§3.5 Additional problem 10

§4.4 Problem 40.

§4.6 Problem 42

Part 3. Problems from old final exams. (Short answers are available for many old final exam problems, but you need to provide all details in your write-up.)

- Fall 2016 final exam, problem 1
- Spring 2016 final exam, problems 3, 4, 5
- Fall 2015 final exam, problems 2, 3, 4. (Note that the T/F problem 1 has nine parts.)

Part 3. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$.

(a) Show that the subset W of $\mathrm{M}_2(\mathbb{C})$ consisting of all 2×2 matrices which commute with A is a vector subspace of $\mathrm{M}_2(\mathbb{C})$.

(b) Find a basis of the above vector space W.

Part 4. Extra credit problems

A. This problem is about row echelon matrices and row reduced echelon matrices.

   (a) Give an example of two different row echelon matrices that are row equivalent.

   (b) It is a fact that for any matrix A (with entries in $\mathbb{R}$ or $\mathbb{C}$, there exists one and only one row reduced echelon matrix which is row-equivalent to A. (This is Theorem 2.4.15 of DELA.) Give a proof of the uniqueness part of this statement.

   (Note: The algorithm explained in class shows the existence. The uniqueness part requires more care because you need to show that applying a permutation of rows does not change the resulting row-reduced echelon form.)

B. The $n \times n$ Hilbert matrix H_n is defined in problem 44 of §2.6; its (i, j) entry is $\frac{1}{i+j-1}$. It is a fact that $\det(A_n)$ is a positive rational number of the form $\frac{1}{C_n}$ for some positive integer C_n.

   (a) Verify the above statement for $n = 4, 5, 6$.

   (b) Try to prove this statement.

   [If you succeed in proving this statement, or even the weaker statement that $\det(H_n) \neq 0$ for every n, please come to my office and show me your proof.]