MATH 240 ASSIGNMENT 12, SPRING 2018

Due in class on Monday, April 23

Part 1.

1A. Give an example of a $\mathbb{C}$-vector space V, not necessarily finite dimensional, and a linear operator $T : V \to V$ such that $\mathrm{Ker}(T) \neq (0)$ and the range/image of T is equal to V. If such an example does not exist, explain the reason.

1B. Let P_3 be the vector space over the field $\mathbb{R}$ of all real numbers consisting of all polynomials $f(x)$ in x of degree at most 3. Let $T : P_3 \to P_3$ be the linear transformation which sends every element $f(x) \in P_3$ to $x^2 \frac{d^2 f}{dx^2} + \frac{df}{dx}$. Find a basis of the kernel of T AND a basis of the range of T.

1C. The set $\mathbb{C}$ of all complex numbers can be regarded as a vector space over $\mathbb{R}$, where vector addition is the addition of complex numbers and scalar multiplication is multiplication of a complex number by a real number.

(i) Prove that 1 and $\sqrt{-1}$ is an $\mathbb{R}$-basis of $\mathbb{C}$.

(ii) Let $w = a + b\sqrt{-1}$ be a complex number, where $a, b \in \mathbb{R}$. Show that the T_w which sends z to $w \cdot z$ for every $z \in \mathbb{C}$ is an $\mathbb{R}$-linear transformation from $\mathbb{C}$ to $\mathbb{C}$.

(iii) Determine the matrix representation of the operator T_w on $\mathbb{C}$ with respect to the $\mathbb{R}$-basis 1, $\sqrt{-1}$.

1D. Let $P(u) = u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0$, where $a_0, \ldots, a_{n-1}$ be real numbers. Consider the map

$$P(\tfrac{d}{dx}) : \mathbb{R}[x] \to \mathbb{R}[x], \quad f(x) \mapsto P(\tfrac{d}{dx})(f(x)) \ \ \forall f(x) \in \mathbb{R}[x].$$

(i) Show that the above map is a linear map T from $\mathbb{R}[x]$ to $\mathbb{R}[x]$.

(ii) Show that the kernel of T is non-trivial (i.e. not equal to $\{0\}$) if and only if $P(u)$ is divisible by u.

(iii) Either show that every element of $\mathbb{R}[x]$ is in the image of T, or give an example of a polynomial $P(u)$ such that the image of the corresponding linear operator T is not equal to $\mathbb{R}[x]$.

Part 2.(extra credit)

2A. Let $A \in \mathrm{M}_n(\mathbb{C})$ be a square matrix with complex coefficients. Let $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ be n linearly independent solutions of the equation

$$\frac{d}{dt}\vec{x}(t) - A \cdot \vec{x}(t) = 0.$$

(i) Let $W(t) := \det(\vec{x}_1(t), \ldots, \vec{x}_n(t))$ be the determinant of the square matrix whose n columns are $\vec{x}_1(t), \ldots, \vec{x}_n(t)$.. Find a first order differential equation satisfied by $W(t)$.

(ii) Find an explicit formula for $W(t)$ up to an constant, by solving the differential equation you found in (i).

2B. Give a *proof* that the $\mathbb{R}$-vector space $\mathbb{R}[x]$ of all polynomials in one variable x with coefficients in $\mathbb{R}$ is *not* a finite dimensional vector space.