MATH 240 ASSIGNMENT 11, SPRING 2018

Due in class on Friday, April 13

Part 1. Read DELA 9.9–9.11.

Part 2. Problems from old final exams.

- Fall 2015 final exam, problem 10
- Fall 2015 final exam, problem 11
- Spring 2016 final exam, problem 15
- Fall 2014 Make-up final exam, problem 12
- Spring 2014 final exam, problem 9 (This is *not* a problem on the materials of §§9.9–9.10.)
- Spring 2011 final exam, parts A, B, D, E of problem 10

Part 3.

3A. Determine all equilibrium points of the non-linear autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y(x-4) \\ \frac{dy}{dt} &= x - 4y^2 \end{aligned} \tag{1}$$

and describe the type of each equilibrium point as
A) stable node B) unstable node C) saddle point D) center E) stable spiral F) unstable spiral

3B. Find all equilibrium points of the following non-linear autonomous system

$$\begin{aligned} \frac{dx}{dt} &= x + y - x(x^2+y^2) \\ \frac{dy}{dt} &= -x + y - y(x^2+y^2) \end{aligned} \tag{2}$$

and determine their type.

3C. Someone asked a very good question in class. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}$$

and the discriminant of the characteristic polynomial of A is strictly negative, so that the eigenvalues of A form a complex conjugate pair with non-zero imaginary part. Then depending on the sign of $\mathrm{Tr}(A)$, we know that as $t \to \infty$, the trajectories of non-zero solutions of the differential equation as $t \to \infty$ are

$$\frac{d}{dt}\begin{pmatrix} x & y \end{pmatrix} = A \cdot \begin{pmatrix} x & y \end{pmatrix}$$

- spirals going toward the origin if $\mathrm{Tr}(A) < 0$),

- ellipses going around the origin if $\mathrm{Tr}(A) = 0$,
- spirals going away to infinity if $\mathrm{Tr}(A) > 0$.

The question is, how do we tell whether trajectories go counter-clockwise or clockwise?

(i) Show that the quadratic form

$$cx^2 + (d-a)xy - by^2$$

is either positive definite (takes strictly positive values for any pair $(x,y) \neq (0,0)$) or negative definite (takes strictly negative values for any pair $(x,y) \neq (0,0)$).

(ii) (extra credit) Show that the trajectories goes counter-clockwise if the quadratic form in (i) is positive definite, and clock-wise if the quadratic form is negative definite.

Note: This statement is compatible with the examples we did in class. But validity for a few data points is not a sufficient argument nor a proof.

3D. (extra credit) Investigate the non-linear autonomous system in B2.

(a) Draw a phase portrait. It is a good idea to use a computer algebra system (such as Maple, Mathematica or Matlab) if one is available. The phase portrait would make the statement (c) below look plausible.

(b) Show that in the polar coordinate system (r,θ) for $\mathbb{R}^2$, the system of equations (2) is equivalent to

$$\begin{aligned} r\frac{dr}{dt} &= r^2(1-r^2) \\ \frac{d\theta}{dt} &= -1 \end{aligned} \tag{3}$$

(c) Show/explain that there is no *closed* trajectory (integral curve) inside the unit disk centered about the origin.

(b) Show/explain that every trajectory outside the unit circle spirals toward the unit circle.