Math 240 practice problems, April 2018

1. Let A be a 5×5 matrix with $\operatorname{rank}(A) = 3$. Which of the following statements are true?

   a. 0 is an eigenvalue of A.
   b. The first four rows of A are linearly dependent.
   c. A is diagaonalizable.
   d. $\operatorname{rank}(A^2) \geq 1$.
   e. A is defective.

2. Let V be the vector space over $\mathbb{R}$ consisting of all polynomials $p(x, y) \in \mathbb{R}[x, y]$ in two variable x and y with real coefficients whose total degree is at most 2. (If $p(x, y) \in V$ is not the zero polynomial, then the $\deg(p(x, y))$ is 0, 1 or 2.)

   (a) Find a basis of V and determine the dimension of V.

   (b) Let $T : V \to V$ be the linear transformation from V to itself, which sends every element $p(x, y) \in V$ to
   $$T(p(x, y)) := y\frac{dp}{dx} + x\frac{dp}{dy}.$$
   Find the matrix representation of this linear transformation T with respect to the basis you picked in (a).

3. Let $A = \begin{pmatrix} 2 & k & 0 \\ 0 & 3 & 0 \\ 0 & 0 & k \end{pmatrix}$, where k is a parameter. For which value of the parameter k is the matrix A diagonalizable?

4. Find the general solution of the ordinary differential equation
$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = (6t - 15t^2)e^{-3t}.$$

5. Find the general solution of the differential equation
$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 4\, e^{-2t} \ln t, \qquad t > 0$$

6. Let $B := \begin{pmatrix} -1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. Find the general solution of the system of first order linear differential equations
$$\frac{d}{dt}\mathbf{x}(t) = B \cdot \mathbf{x}(t), \quad \text{where } \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

7. Find a (particular) solution $y_p(x)$ of the differential equation

$$\left(\frac{d}{dx}+1\right)^3\left(\frac{d}{dx}-1\right)y(x) = -240\,x^2e^{-x} + 120\,e^{-x}.$$

8. Let $A = \begin{pmatrix} 3 & -1 & 1 & -1 \\ 2 & 0 & 2 & -2 \\ -1 & 1 & 1 & 1 \\ -2 & 2 & -2 & 4 \end{pmatrix}$.

   (a) Compute the matrix exponential e^{tA}.

   (b) Find the vector valued function $\vec{x}(t)$ which satisfies

$$\frac{d}{dt}\vec{x}(t) - A\cdot\vec{x}(t), \quad \vec{x}(0) = (1,0,0,0)^T.$$

9. Consider the following differential equation with a real parameter k

$$\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + 8ky = \cos(2t) \tag{1}$$

Determine all values of the parameter $k \in \mathbb{R}$ such that every solution $y(t)$ of the differential equation (1) is bounded as $t \to \infty$ (i.e. there exists a constant C depending on the solution $y(t)$ such that $|y(t)| \le C$ for all $t \ge 0$).M

10. Find all equilibrium points of the differential equation

$$\frac{dx}{dt} = x(t)y(t), \quad \frac{dy}{dt} = \cos(x(t))\,\cos(y(t))$$

and determine/classify each of the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc.

11. Find all equilibrium points of the differential equation

$$\frac{dx}{dt} = y(t), \quad \frac{dy}{dt} = -4\sin(x(t)) - \frac{y(t)}{25}$$

and determine/classify each of the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc.

12. (a) Find the general solution to the system of differential equations:

$$x' = x + 4y \qquad y' = -2x + 3y$$

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions

$$x(0) = 3, \quad y(0) = 0$$

and draw a graph of the trajectory of the point $(x(t), y(t))$ in the xy-plane. Put an arrow on the curve to indicate the direction of increasing t.

13. Find the general solution of the system $\boldsymbol{x}' = A\boldsymbol{x}$, where

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

14. Find and classify all the critical points of the system

$$\frac{dx}{dt} = y^3 - 4y$$
$$\frac{dy}{dt} = x^2 - x$$

Then draw (and label with arrows) several representative trajectories in the phase plane.

15. The differential equation

$$\frac{d^2y}{dx^2} - x^{-1}\frac{dy}{dx} + 4x^2y = 0, \qquad x > 0$$

has a solution $y_1(x) = \sin(x^2)$. Find a solution $y_2(x)$ which is linearly independent of $y_1(x)$.

16. Let P_3 be the vector space of polynomials with real coefficients and degree less than or equal to 3.

(a) What is the dimension of P_3? Give a basis of P_3?

(b) Define the linear map $L\colon P_3 \to \mathbb{R}^2$ by $L(p) = [p(-1), p(1)]$ — in other words, L reports the values of p for $x = -1$ and $x = 1$. Write the matrix of L with respect to the basis you chose in part (a).

(c) What is the rank of L? Give a basis for the nullspace (kernel) of L.

17. Let $\vec{v}$ be the vector $(1, 1, 1)^T$ in $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation of "rotating by $\frac{\pi}{20}$ relative to the line $\mathbb{R} \cdot \vec{v}$. Find the matrix $A \in \mathrm{M}_3(\mathbb{R})$ such that T is equal to "left multiplication by A" on $\mathbb{R}^3$.

(Note: There are actually two such rotations, depending on which direction you turn.)