Math 240 Practice Problems Set 2, March 2015

1. (a) Give an example of a finite dimensional $\mathbb{C}$-vector space V and a linear operator $T : V \to V$ such that $\mathrm{Ker}(T) \neq (0)$ and the range/image of T is equal to V. If such an example does not exist, explain the reason.

(b) Give an example of a $\mathbb{C}$-vector space V, not necessarily finite dimensional, and a linear operator $T : V \to V$ such that $\mathrm{Ker}(T) \neq (0)$ and the range/image of T is equal to V. If such an example does not exist, explain the reason.

2. (a) Find the 2×2 matrix A such that $\vec{x} \mapsto A \cdot \vec{x}$ for $\vec{x} \in \mathbb{R}^2$ is the counter-clockwise rotation about the origin by 45°.

(b) Does there exist an invertible 2×2 matrix C with real entries such that $C^{-1} \cdot A \cdot C$ is a diagonal matrix? Find such a matrix C if there is one, or explain why such a matrix C does not exist.

(c) Does there exist an invertible 2×2 matrix C with complex entries such that $D^{-1} \cdot A \cdot D$ is a diagonal matrix? Find such a matrix D if there is one, or explain why such a matrix D does not exist.

3. Let $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 5 & 0 \\ -4 & 0 & 9 \end{pmatrix}$. Compute e^{tA} explicitly.

5. Let $A = \begin{pmatrix} -2 & 0 & 4 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Determine whether e^{tA} is bounded as $t \to \infty$, i.e. whether there exist positive integers $M, N > 0$ such that all entries of $\exp(tA)$ are $\leq M$ for all $t \geq N$.

6. Find the general solution of the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x} + \cos x - 1$$

7. Find the general solution of the differential equation

$$\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 5\right)^2 y = e^{(-1+2\sqrt{-1})x}$$

9. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x}$$

on the half-line $x > 0$.

10. Determine all solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = e^{-3x}$$

such that $\lim_{x\to\infty} y(x) = 0$, or explain why no such solution exists.

11. Is there a solution $y(x)$ of the differential equation

$$\left(\frac{d^2}{dx^2} + 4\right)^2 y = \sin(2x)$$

such that $y(x)$ is *bounded* on $\mathbb{R}$ (in the sense that there exists a constant $C > 0$ such that $|y(x)| \le C$ for all $x \in \mathbb{R}$)? Find all bounded solutions if they exist, and explain why every solution is unbounded.

12. True or False: If A is a 5×5 matrix such that the real part of each of its complex eigenvalues are ≤ 0, then all entries of $\exp(tA)$ are bounded as $t \to \infty$.

13. True or False: If A is a 5×5 matrix such that $(x-1)^2$ divides the characteristic polynomial $\det(x \cdot \mathrm{I}_5 - A)$ of A and $\dim(\mathrm{Ker}(\mathrm{I}_5 - A)) = 1$, then $e^{-t} \cdot \exp(tA)$ is unbounded as $t \to \infty$.

14. True or False: Suppose that $P(u)$ is a polynomial in u and the real part of all complex roots of $P(u)$ are less than or equal to 0. Then every solution $y(t)$ of the differential equation $P(\frac{d}{dt})y = 0$ is bounded as $t \to \infty$.