Math 240 Practice Problems, February 2015

1. Find all solutions of the system of equations

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 3 \\ 6 \end{bmatrix}$$

such that $x_3 \cdot x_4 = 0$.

2. Six of the following 7 elements of $\mathbb{R}^3$,

$$(1, 1, -4),\ (3, 5, -26),\ (2, 1, 1),\ (5, 6, -27),\ (4, 1, 5),\ (5, -2, 29),\ (6, 2, 4)$$

lie on a 2-dimensional subspace (i.e. a plane passing through the origin). Which one does not lie on the plane that contains the other six points?

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Ans. Except the point $(2, 1, 1)$, the other six points all lie on the plane $3x - 7y - z = 0$.

4. Let B be the square matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix}$$

Then the trace of B^{-1} is equal to

A. 21

B. 13

C. -15

D. 0

E. -14

F. None of the above.

5. Consider the following system of linear equations in 6 variables

$$\begin{bmatrix} 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(a) What is the rank of the above 3×6 matrix?

(b) Write down the general solution of the above system of equations, in the form of generic linear combination of a number of linearly independent solutions.

6. Let A be the 3×5 matrix $A = \begin{pmatrix} 3 & 2 & 1 & -1 & -2 \\ 1 & 1 & 2 & 1 & 3 \\ 3 & 1 & -4 & -5 & -13 \end{pmatrix}$.

(a) Determine the dimension of the linear span of the columns of A.

(b) For which values of the parameter c will the system of equations

$$A \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ c \end{pmatrix}$$

admit at least a solution?

7. Provide an example in each of the following questions.

(a) Two square matrices A, B of the same size such that $A \cdot B \neq B \cdot A$.

(b) A non-zero square matrix C such that $C^2 = 0$

(c) Exhibit three different bases of $\mathbb{R}^3$.

(d) A *non-linear* map from the real vector space of all real-valued continuous functions on $\mathbb{R}$ to itself.

(e) Two 3×3 matrices A, B such that $\det(A + B) \neq \det(A) + \det(B)$.

8. Which ones of the following limits exit? Explain your reasons for each of the limits.

(a) $\lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n$

(b) $\lim_{n\to\infty} \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}^n$

(c) $\lim_{n\to\infty} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n$

(d) $\lim_{n\to\infty} \begin{pmatrix} 3 & 5 \\ 5 & -3 \end{pmatrix}^n$

(e) $\lim_{n\to\infty} \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}^n$

9. (a) Find a formula for $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}^n$ valid for every positive integer n.

(b) Compute $\exp\left(t \cdot \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\right)$.

10. Let A be the 4×4 matrix

$$A = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Find a basis of $\mathbb{R}^4$ consisting of eigenvectors of A.

11. Let B be the 4×4 matrix

$$B = \frac{1}{2} \cdot \begin{pmatrix} 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix}$$

Does there exist an invertible 4×4 matrix C such that $C^{-1} \cdot B \cdot C$ is diagonal? Find such a matrix C if there is one, or explain why such a matrix C does not exist.

12. (10 pts) Compute the determinant of the matrix

$$\begin{bmatrix} 0 & 2 & 2 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 5 & 0 & 5 \\ 7 & 15 & 8 & 8 \end{bmatrix}$$

Answer: __________

13. (10 pts) Let B be the square matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Then B^{-1} is equal to: ____________________

14. (10 pts) Let B be an (ordered) basis of $\mathbb{R}^2$, consisting of two vectors v_1, v_2. The ordered basis C of $\mathbb{R}^2$ consists of the same vectors in *reverse* order: v_2, v_1.

(a) Determine the change of basis matrix $P_{C \leftarrow B}$.

Answer: __________

(b) Suppose that the matrix representation of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to basis B is given as $[T]_B^B =$

$$[T]_B^B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Find the matrix representation $[T]_C^C$ of T relative to the basis C, where C is the ordered basis as in (a) above.

Answer: ____________________

15. (10 pts) Let Q_2 be the vector space over $\mathbb{R}$ consisting of all polynomials $f(x, y)$ in two variables x, y whose (total) degree is at most 2.

(a) Find a basis of Q_2. (Hint: what are the monials in Q_2?)

(b) Determine the dimension of Q_2.

(c) Let T be the linear operator from Q_2 to itself which sends every element $f(x,y) \in Q_2$ to $\frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y}$. Find the matrix representation of T with respect to the basis you found in (a).

(d) Find a basis of the kernel of T.

(e) Find a basis of the range of T.

(Hint for (d), (e): use the matrix representation you found in (c).)

16. (10 pts) Let A be the following 4×5 matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

(a) Find a $\mathbb{C}$-basis of $N(A)$, the null space of A.
(b) Find a $\mathbb{C}$-basis of the row span of A.
(c) Find a $\mathbb{C}$-basis of the column span of A.

17. Determine whether the following matrices are diagonalizable over the field $\mathbb{C}$ of complex numbers (i.e. $\mathbb{C}^n$ has a basis consisting of eigenvectors)?

(a) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 \\ 0 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

18. (10 pts) Let A be the 5×5 matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ 0 & 3 & 7 & -1 & 9 \\ 0 & 0 & 2 & -7 & 6 \\ 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Let B be the matrix obtained from A by performing the following row operations, in order: $P_{23}, M_5(\frac{1}{2}), A_{24}(7), P_{14}, A_{13}(-6), M_1(5), A_{43}(5)$. What is the determinant of B?