Least Squares and Markov Processes

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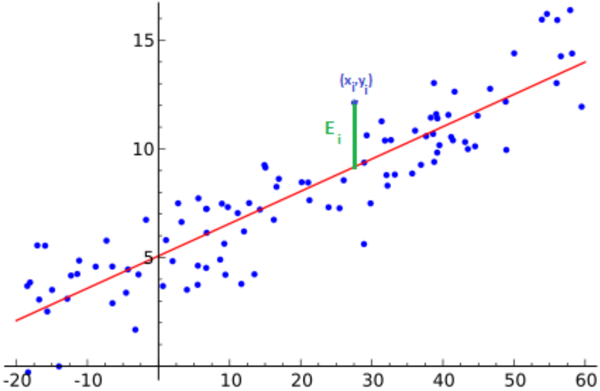
University of Pennsylvania

Math 115

Least Squares (linear regression)

Find the "best" line fitting a collection of data points (in the plane).



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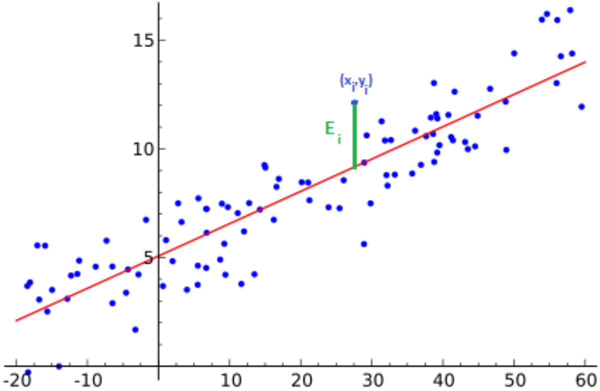

The **Error** E in using the line will be $E = {E_1}^2 + {E_2}^2 + \ldots {E_n}^2$ where E_i is the vertical distance from the ith point (x_i, y_i) to the line.

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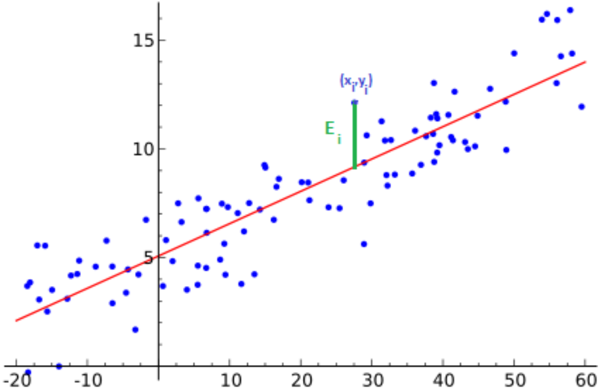

The **Error** E in using the line will be $E = E_1{}^2 + E_2{}^2 + \ldots E_n{}^2$ where E_i is the vertical distance from the ith point (x_i, y_i) to the line. Usually there are lots of data points and you use **Maple** to solve.

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Use this to do our example.

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Problem: A food truck introduces cheese-steaks to its menu. The owner tries different prices to see what the market is like. The following chart represents five weeks of sales:

Week	Price	Total sales
1	\$3.50	113
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Set up a chart:

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x	y	xy	x^2
3.50	113	395.5	12.25
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So our estimate for sales at \$3.60 is $-70.4(3.60) + 355.2 = 101.76$.

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Problem:: Some residents of a certain planet choose to wear hats and some don't. On planet X 80% of the daughters of women who wear hats also wear hats and 30% of the daughters of women who don't wear hats choose to wear them. Assume that this trend remains unchanged from one generation to the next. What is the corresponding **transition matrix**?

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Any such matrix is called **Stochastic**

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If A is the transition matrix then we see (check it out for our problem):

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Note that A^n is another Stochastic matrix!

Problem Taxis pick up and deliver passengers in a city that is divided into three zones. Records kept by drivers show that of the passengers picked up in zone I, 50% are taken to zone I, 40% to zone II, and 10% to zone III. Of the passengers picked up in zone II, 40% go to zone I, 30% go to zone II, and 30% to zone III. Of the passengers picked up in zone III, 20% go to zone I, 60% to zone II, and 20% to zone III.

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The interesting question is "What happens in the long run?". The process often settles down.

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The initial state is $\begin{bmatrix} .25 \\ .75 \end{bmatrix}_0$, while $A^n \begin{bmatrix} .25 \\ .75 \end{bmatrix}_0 = \begin{bmatrix} \\ \end{bmatrix}_n$.
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This happens a lot!!!

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- All columns of the stable matrix equal the stable distribution (and hence are all equal).
- The stable distribution X satisfies the system of linear equations given by $AX = X$ and the sum of the entries of X is equal to 1.

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The last statement lets us find the stable distribution and hence the stable matrix.

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$$A = \begin{bmatrix} .5 & .1 & 0 & .5 \\ 0 & .2 & 0 & .2 \\ .5 & .3 & 1 & .2 \\ 0 & .4 & 0 & .1 \end{bmatrix}$$

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is a Stochastic matrix where the third state is absorbing.
Consider the Stochastic matrix:

$$A = \begin{bmatrix} 1 & .1 & 0 & 0 \\ 0 & .2 & 0 & 1 \\ 0 & .3 & 1 & 0 \\ 0 & .4 & 0 & 0 \end{bmatrix}$$

What are the absorbing states?

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is a Stochastic matrix where the third state is absorbing.
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What are the absorbing states?
State 1 is, state 2 is not, state 3 is, state 4 is not.

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It turns out that if A is an absorbing stochastic matrix then A^n also settles down to a limit called the **Stable matrix**.

Definition: An **Absorbing stochastic matrix** is a Stochastic matrix that satisfies:

- 1) There is at least one absorbing state.
- 2) From any state you can get to an absorbing state directly or through one or more intermediate states.

Check the two examples above. Yes to both.
Consider

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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It turns out that if A is an absorbing stochastic matrix then A^n also settles down to a limit called the **Stable matrix**. To analyze this, first reorder your states so that the absorbing ones come first.

Absorbing Stochastic Matrices

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Our second example (after switching second and third states) becomes:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & .3 & 0 \\ 0 & 0 & .2 & 1 \\ 0 & 0 & .4 & 0 \end{bmatrix}$$

Absorbing Stochastic Matrices

Problem: In a new voting scheme each voter can go to the polls many times. At the poll they can listen to arguments for and against the two candidates. They can choose during any visit to cast a vote (which they can not now change). Over time it is discovered that during a visit to the polls an Undecided voter decides to vote for candidate A 20% of the time, for candidate B 30% of the time, and stays Undecided 50% of the time.

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Notice that for absorbing matrices the long term trend depends on the initial state!

Example Try the initial states $\begin{bmatrix} .4 \\ .4 \\ .2 \end{bmatrix}$ and $\begin{bmatrix} .2 \\ .3 \\ .5 \end{bmatrix}$.

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No mater what the initial states the long term trend sends all objects to the absorbing states.

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What about our previous example:

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END OF COURSE – GOOD LUCK ON THE FINAL!!!