SCATTERING RIGIDITY WITH TRAPPED GEODESICS

CHRISTOPHER CROKE+

ABSTRACT. We prove that the flat product metric on $D^n \times S^1$ is scattering rigid where D^n is the unit ball in $\mathbb{R}^n$ and $n \geq 2$.

The scattering data (loosely speaking) of a Riemannian manifold with boundary is map $S : U^+\partial M \to U^-\partial M$ from unit vectors V at the boundary that point inward to unit vectors at the boundary that point outwards. The map (where defined) takes V to $\gamma_V'(T_0)$ where γ_V is the unit speed geodesic determined by V and T_0 is the first positive value of t (when it exists) such that $\gamma_V(t)$ again lies in the boundary.

We show that any other Riemannian manifold $(M, \partial M, g)$ with boundary ∂M isometric to $\partial(D^n \times S^1)$ and with the same scattering data must be isometric to $D^n \times S^1$.

This is the first scattering rigidity result for a manifold that has a trapped geodesic. The main issue is to show that the unit vectors tangent to trapped geodesics in $(M, \partial M, g)$ have measure 0 in the unit tangent bundle.

1. Introduction

In this paper we prove scattering rigidity (see below) for a number of compact Riemannian manifolds with boundary that have trapped geodesics.

Consider a compact Riemannian manifold $(M, \partial M, g)$ with boundary ∂M and metric g. We will let $U^+\partial M$ represent the space of inwardly pointing unit vectors at the boundary. That is $V \in U^+\partial M$ means that V is a unit vector based at a boundary point and $< V, \eta^+ > \geq 0$ where η^+ is the unit vector of M normal to ∂M and pointing inward. Similarly we let $U^-\partial M$ represent the outward vectors. Note that $U^+\partial M \cap U^-\partial M = U(\partial M)$ the unit tangent bundle of ∂M.

For $V \in U^+\partial M$ let $\gamma_V(t)$ be the geodesic with $\gamma'(0) = V$. We let $TT(V) \in [0, \infty]$ (the travel time) be the first time $t > 0$ when $\gamma_V(t)$ hits the boundary again. If $\gamma_V(t)$ never hits the boundary again then

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$TT(V) = \infty$ while if either $\gamma_V(t)$ does not exist for any $t > 0$ or there are arbitrarily small values of $t > 0$ such that $\gamma(t) \in \partial M$ then we let $TT(V) = 0$. Note that $TT(V) = 0$ implies that $V \in U(\partial M)$.

The scattering map $S : U^+\partial M \to U^-\partial M$ takes a vector $V \in U^+\partial M$ to the vector $\gamma'(TT(V)) \in U^-\partial M$. It will not be defined when $TT(V) = \infty$ and will be V itself when $TT(V) = 0$. If another manifold $(M_1, \partial M_1, g_1)$ has isometric boundary to $(M, \partial M, g)$ in the sense that $(\partial M, g)$ (g restricted to ∂M) is isometric to $(\partial M_1, g_1)$ then we can identify $U^+\partial M_1$ with $U^+\partial M$ and $U^-\partial M_1$ with $U^-\partial M$. We say that $(M, \partial M, g)$ and $(M_1, \partial M_1, g_1)$ have the same scattering data if they have isometric boundaries and under the identifications given by the isometry they have the same scattering map. If in addition the travel times $TT(V)$ coincide then they are said to have the same lens data.

A compact manifold $(M, \partial M, g)$ is said to be scattering (resp. lens) rigid if for any other manifold $(M_1, \partial M_1, g_1)$ with the same scattering (resp. lens) data there is an isometry from M_1 to M that agrees with the given isometry of the boundaries.

Theorem 1.1. *For any $n \geq 2$ the flat product metric on $D^n \times S^1$ is scattering rigid where D^n is a ball in $\mathbb{R}^n$.*

The fact that not all manifolds are scattering rigid was pointed out in [Cr91]. For $\frac{1}{4} > \epsilon > 0$ let $h(t)$ be a small smooth bump function which is 0 outside $(-\epsilon, \epsilon)$ and positive in $(-\epsilon, \epsilon)$. For $s \in (-1 + 2\epsilon, 1 - 2\epsilon)$ consider surfaces of revolution g_s with smooth generating functions $F_s(t) = 1 + h(s + t)$ for $t \in [-1, 1]$. These surfaces of revolution look like flat cylinders with bumps on them that are shifted depending on s but otherwise look the same (see figure 1). The Clairaut relations show that, independent of s, geodesics entering one side with a given initial condition exit out the other side after the same distance at the same point with the same angle. Hence all metrics have the same scattering data (and in fact lens data) but are not isometric. A much larger class of examples was given in section 6 of [Cr-Kl94]. All of the examples have in common that there are trapped geodesics.

The scattering and lens rigidity problems are closely related to other inverse problems. In particular the boundary rigidity problem is equivalent to the lens rigidity question in the Simple and SGM cases. See [Cr91] and [Cr04] for definitions and relations to some other problems. There is a vast literature on these problems (see for example [Be83, Bu-Iv06, Cr91, Cr90, Gr83, Mi81, Mu77, Ot90, Pe-Sh88, Pe-Uh05]). However all of the results to date concern manifolds without trapped geodesics. The results in this paper constitute the first examples of scattering rigid manifolds that have trapped geodesics.

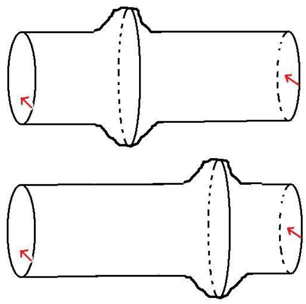

FIGURE 1.1. not isometric but same scattering and lens data

The key difficulty in our case is to show that the set of unit vectors tangent to trapped geodesic rays in the metric g_1 has measure 0 in the unit tangent bundle. This allows us (with an application of Santaló's formula) to conclude that g and g_1 have the same volumes. Since the metric g has a real factor (i.e. $D^n \subset \mathbb{R} \times \mathbb{R}^{n-1}$) we can use a result from [Cr-Kl98] to complete the argument. In fact, the argument in Theorem 1.1 extends (see section 3) to the case where D^n above is replaced by a ball in $\mathbb{R} \times N^{n-1}$ where N is a complete simply connected Riemannian manifold with nonpositive curvature. (In fact with more work one could extend this to the case of no conjugate points but we chose not to give the slightly different arguments here.)

One case that was not dealt with in Theorem 1.1 is the two dimensional case, namely the flat cylinder $[-1, 1] \times S^1$ and the Möbius strip. There are ways in which this case is easier and ways in which it is harder. The major differences are that the scattering data does not determine the lens data and we cannot conclude that the C^∞ jets of the metrics agree at the boundary. The problem of lens rigidity in the two dimensional case will be taken up in a future paper with Pilar Herreros. In particular, it turns out that the Möbius strip is not scattering rigid if $(M_1, \partial M_1, g_1)$ is allowed to be C^1.

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2. The $D^n \times S^1$ for $n \geq 2$ case

In this section we prove of Theorem 1.1. We consider generalizations in Section 3. Throughout the section $n \geq 2$ and g will be the standard flat product metric on $M = D^n \times S^1$. For concreteness we will take D^n to be the unit ball and S^1 to have length 2π. $(M_1, S^{n-1} \times S^1, g_1)$ will be another Riemannian metric on a manifold M_1 whose boundary is isometric to that of M. We use this isometry to identify the two boundaries. We assume that g_1 has the same scattering data as g.

The first thing to note is that the C^∞ jets of g and g_1 agree at the boundary. This follows from [L-S-U03],[Uh-Wa03], or [Zh11] since, for the flat metric g, the second fundamental form of the boundary has a positive eigenvalue at every point. (Note that this argument wont work in the two dimensional case $n = 1$.) This in particular means that we can glue $(\mathbb{R}^n - D^n) \times S^1$ along the boundary of M_1 to yield a smooth metric M_1^{ext} which is isometric to $\mathbb{R}^n \times S^1$ outside of M_1.

The scattering data assumption tells us that the only geodesic loops at a point $p \in \partial M_1$ are the multiples of the S^1 running through p (which stay in the boundary). One consequence of the above is that every geodesic loop at a point $p \in \partial M_1$ is locally minimizing in the space of loops at p. This is true since it is true in M_1^{ext} since there are no conjuagate points along γ because all the sectional curvatures are 0.

Lemma 2.1. *$\pi_1(M_1) = \mathbb{Z}$ and the generator is represented by the S^1 factor of the boundary.*

Proof: To see this fix a base point p on the boundary. We have pointed out that the only geodesic loops at p are multiples of S^1. Further the convexity of the boundary guarantees that there is at least one geodesic loop in each homotopy class (the shortest curve in that class). Thus we need only see that any nonzero multiple γ of the S^1 through p is not null homotopic. By the above γ is locally minimizing. If it were null homotopic then a standard minimax argument along with the convexity of the boundary would guarantee a minimax geodesic loop at p. But since all geodesic loops at p are locally minimizing we get the desired contradiction. The lemma follows.

□

Thus the Riemannian universal cover $\widetilde{M_1}$ of M_1 sits naturally in $\widetilde{M_1^{ext}}$ the universal cover of M_1^{ext} and further $\widetilde{M_1^{ext}} - \widetilde{M_1}$ is isometric to $(R^n - D^n) \times \mathbb{R}$. Also $\widetilde{M} = D^n \times \mathbb{R}$ has the same scattering data as $\widetilde{M_1}$. We will slightly abuse notation and call the metrics on the universal covers g and g_1 as well.

Lemma 2.2. *g_1 has the same lens data as g.*

Proof: This is an application of the first variation formula. For $V \in U^+\partial M$ let $G(V) = L(\gamma_V) - L(\gamma_{1V})$. We need to show that $G(V) = 0$ for all V. A smooth curve of initial conditions $V(s)$ in the interior of $U^+\partial M$ gives rise to smooth variations $\gamma_{V(t)}$ through geodesics in M

and $\gamma_{1V(t)}$ through geodesics in M_1 whose initial and final conditions agree. (Note that this uses the convexity of the boundary since for more general manifolds with boundary there may be a discontinuous jump in the endpoints of geodesics.) The first variation formula (along with the fact that the metrics agree at the boundary) tells us that $G(V(s)) = L(\gamma_{V(s)}) - L(\gamma_{1\ V(s)})$ is independent of s. Since $U^+\partial M$ is connected, G is constant. Further when V approaches a non vertical vector (i.e. one not tangent to the S^1 factor) in $\partial(U^+\partial M) = U\partial M$ then $L(\gamma_V)$ and $L(\gamma_{1V})$ approach 0 and hence $G(V)$ approaches 0. Thus $G \equiv 0$ and the lemma follows.

□

Lemma 2.3. *An M_1 geodesic γ between boundary points is the shortest path in its homotopy class (rel boundary points).*

Proof: This is the same as saying that such geodesics in the universal cover are the minimizing paths between the endpoints. This is true for $\widetilde{M}$ where there is a unique geodesic between any two boundary points. Thus there is also a unique geodesic between boundary points in $\widetilde{M}_1$ which must thus be the minimizing geodesic.

□

In fact, this implies that all geodesic segments in $\widetilde{M_1^{ext}}$ are minimizing except possibly in the case that they are segments of geodesics trapped in $\widetilde{M}_1$. If p and q are points in $\widetilde{M_1^{ext}} - \widetilde{M}_1 = \mathbb{R}^{n+1} - D^n \times \mathbb{R}$ then this implies that $d_1(p,q) = d(p,q)$.

A geodesic γ_{1V} will either be trapped or coincide with an oriented Euclidean line L_V outside $\widetilde{M}_1$. By the direction of L_V we mean the oriented line through the origin parallel to L_V.

There are two cases that are exceptional. These are trapped geodesics and vertical geodesics (i.e. $\{x\} \times \mathbb{R}$ for $x \in R^n - D^n$). We will exclude both these cases by the phrase "L_v is not vertical". For $p \in \widetilde{M_1^{ext}}$ we let $\mathbf{A}(p) = \{V \in U_p\widetilde{M_1^{ext}} | L_V \text{ is not vertical}\}$. Note that for $p \in \widetilde{M_1^{ext}} - \widetilde{M}_1$ we have $\mathbf{A}(p)$ is just the unit sphere with the north and south pole removed. $\mathbf{A}$ will represent the union of the $\mathbf{A}(p)$.

Lemma 2.4. *If $V_i \in \mathbf{A}(p)$ converges to a vector $V \in U_p - \mathbf{A}(p)$ then the directions of the lines L_{V_i} become vertical.*

Proof: Assume this is not the case. Then there is a subsequence of the V_i (which we will still call V_i) such that the directions of the lines L_{V_i} converge to a non vertical direction L. We claim that a subsequence of these L_{V_i} converge to a line L_W. To see this we only need to note that the lines L_{V_i} intersect a common compact set. Now the length of $\gamma_{V_i} \cap \widetilde{M_i}$ is the same as the length of $L_{V_i} \cap D^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ which is uniformly bounded above (say by B) since the directions of the L_{V_i} converge to L. Thus all the L_{V_i} intersect the boundary of $\widetilde{M_i}$ inside the compact ball about p of radius B. This means that the geodesics γ_{V_i} (for the subsequence) converge to L_V outside $\widetilde{M_i}$ but they converge to γ_V which is supposed to be trapped. This yields the desired contradiction.

□

We next see that even though $\widetilde{M_1^{ext}}$ might a-priori have conjugate points (along geodesics trapped in $\widetilde{M_1}$), Busemann functions along rays where L_V is not vertical behave like those in manifolds without conjugate points. In particular they are $C^{1,1}$ smooth, $|\nabla b_{1V}| = 1$, and the Lipshitz constant for ∇b_{1V} is uniformly bounded.

For $V \in U\widetilde{M_1^{ext}}$, let $b_{1V} : \widetilde{M_1^{ext}} \to \mathbb{R}$ be the Busemann function defined by V. I.e.

$$b_{1V}(p) = lim_{t\to\infty} d_1(p, \gamma_v(t)) - t.$$

Since $d_1(p, q) = d(p, q)$ when p and q are points in $\widetilde{M_1^{ext}} - \widetilde{M_1}$, b_{1V} coincides with the Euclidean b_V outside $\widetilde{M_1}$ as long as γ_{1V} is not trapped. That is b_{1V} will coincide with the hight function (up to a constant) in the direction L_V.

For all reals s we will let $H_V(s) = \{p \in \widetilde{M_1^{ext}} | b_{1V}(p) = s\}$ be the s level set of b_{1V}. Of course, outside of $\widetilde{M_1}$, $H_V(s)$ is just a hyperplane perpendicular to L_V.

Lemma 2.5. *For all $V \in \mathbf{A}$, b_{1V} is $C^{1,1}$ and the Lipshitz constant of ∇b_{1V} is bounded by a constant independent of V.*

Proof: The proof is the usual proof that such a statement holds on manifolds without conjugate points. This is done by showing that the approximating functions, $f_t(p) = d_1(p, \gamma_{1V}(t)) - t$, are C^∞ have $|\nabla f_t| = 1$ and have uniformly bounded Hessian. If $\gamma_{1V}(t) \in \widetilde{M_1^{ext}} - \widetilde{M_1}$ then all geodesics from $\gamma_{1V}(t)$ minimize so the distance function from $\gamma_{1V}(t)$ is C^∞ for large t. The fact that $|\nabla f_t| = 1$ is clear. The uniform control on the Hessian is also the same as we will see. Fix a number r less than the convexity radius of M_1^{ext} - which exist since it is flat outside M_1. Since there is a compact set K of base points such that for $q \notin K$ the ball $B(q, r)$ is flat we conclude that the eigenvalues of the second fundamental forms of the boundaries of $B(q, r)$ are uniformly bounded above and below independent of q. This same bound applies to balls in the universal cover $\widetilde{M_1^{ext}}$. Now to bound the Hessian of f_t at

$q \in \widetilde{M_1^{ext}}$ let $\tau(s)$ be the (unique) geodesic from $\gamma_{1V}(t)$ to $q = \tau(s_0)$ (we can assume $s_0 >> r$ since we will be taking the limit as $t \to \infty$). Then by the triangle inequality the level set of f_t at q (i.e. $\partial B(\gamma_{1V}(t), s_0)$) lies outside both $B(\tau(s_0 - r), r)$ and $B(\tau(s_0 + r), r)$ which are tangent to the level set at q. Hence the second fundamental forms of the level sets are uniformly bounded and hence so is the Hessian. Thus the lemma follows.

□

The usual properties of Busemann functions (see [Es77] for basic properties of Busemann functions) tell us that if $W(p) = \nabla b_{1V}(p)$ then $\gamma'_{1W}(t) = \nabla b_{1V}(\gamma_{1W}(t))$ for all t. Hence if γ_{1W} is not trapped then L_W will be parallel to L_V. A straightforward open and closed argument shows that for all p, $\gamma_{1W(p)}$ is not trapped.

Lemma 2.6. *Let V and W in $\mathbf{A}$ be such that L_V and L_W are not parallel to each other and W is not horizontal. Then for any given s the maximum and minimum values of b_{1V} on the compact $H_W(s) \cap \widetilde{M}_1$ are achieved on the boundary of $H_W(s) \cap \widetilde{M}_1$.*

Proof: We first note that $H_W(s) \cap \widetilde{M}_1$ is compact. If D is the diameter of M_1 then for every $p \in H_W(s) \cap \widetilde{M}_1$ there is a point $q \in \partial(\widetilde{M}_1)$ with $d(p, q) \leq D$. Since b_{1W} has Lipshitz constant 1, we know that $s - D \leq b_{1w}(q) \leq s + D$ and hence p lies in the (compact) set of points that are at distance $\leq D$ from the compact (since W is not horizontal) set of boundary points $\{q \in \partial\widetilde{M}_1 | s-D \leq b_{1W}(q) \leq s+D\} = \{q \in \partial(B^{n-1} \times \mathbb{R}) | s - D \leq b_W(q) \leq s + D\}$.

If the maximum (or minimum) value of b_{1V} occurs in the interior then ∇b_{1V} must be perpendicular to $H_W(s)$ there and hence coincides with ∇b_{1W} at that point. However this contradicts the condition that L_V and L_W are not parallel.

□

We now see that if V and W in $\mathbf{A}$ are such that L_V and L_W are parallel then $b_{1V} - b_{1W}$ is constant. Since they agree with the height functions outside $\widetilde{M}_1$, $b_{1V} - b_{1W} = C$ outside $\widetilde{M}_1$. But then they must also differ by C along any geodesic whose corresponding line is parallel to L_V and L_W. But since such a geodesic passes though every point $p \in \widetilde{M}_1$ (i.e. take the geodesic in the direction of $\nabla b_{1V}(p)$) this says $b_{1V} - b_{1W} = C$ everywhere. In particular for every p there is a unique geodesic passing through p and parallel to a given line. This gives a natural identification of $\mathbf{A}(p)$ with the space of non vertical directions.

Proposition 2.7. *Through every $p \in \widetilde{M}_1$ there is exactly one trapped geodesic.*

Proof: This is equivalent to showing that for every $p \in \widetilde{M_1^{ext}}$, $\mathbf{A}(p)$ consists of the unit sphere $U_p\widetilde{M_1^{ext}}$ minus a pair of antipodal points. Assume that p is a point with more trapped geodesics. Note that if there is a trapped half geodesic at p then the other half must also be trapped by the assumption that the scattering data coincides with the flat case. Thus the tangent directions to trapped geodesics come in antipodal pairs. Of course there is at least one trapped geodesic through p since one could take the limit of a subsequence of geodesics from p to a boundary points q_i where q_i runs off to infinity. We only need to consider p in the interior of $\widetilde{M}_1$.

A limiting half geodesic of a sequence of trapped half geodesics starting at p will be a half geodesic starting at p that stays in $\widetilde{M}_1$. (In fact it stays in the interior since the only half geodesics in $\widetilde{M}_1$ that are tangent to the boundary are the vertical ones hence stay in the boundary.) Thus the set of tangent directions to trapped geodesics (i.e. $U_p - \mathbf{A}(p)$) is closed in U_p and thus $\mathbf{A}(p)$ is open and nonempty (by the correspondence with non vertical directions). The set of boundary points of $U_p - \mathbf{A}(p)$ is thus non empty and if it consisted of a single antipodal pair then $U_p - \mathbf{A}(p)$ would also be a single antipodal pair. Thus there is a pair of distinct unit vectors V and W in $U_p - \mathbf{A}(p)$ such that $\langle V, W\rangle = C$ with $-1 < C < 1$ (one could take $C \geq 0$) and such that there exists sequences $V_i \in \mathbf{A}(p)$ and $W_i \in \mathbf{A}(p)$ such that V_i converges to V and W_i converges to W. We extend V_i and W_i to vector fields by letting $V_i(q) = \nabla b_{1V_i}(q)$ and $W_i(q) = \nabla b_{1W_i}(q)$. By the uniform bound on Lipshitz constants on busemann functions, i.e. Lemma 2.5, there is an $\epsilon > 0$ (depending only on C and the Lipshitz constant of the busemann functions but not on i) such that for all $q \in B_p(\epsilon)$ (the ϵ ball about p) and all sufficiently large i we have

$$-\frac{1+C}{2} < \langle V_i(q), W_i(q)\rangle < \frac{1+C}{2}.$$

This holds since for large i we have $\langle V_i(p), W_i(p)\rangle$ is approximately C and then, with respect to a parallel frame along geodesics of length ϵ, the change of V_i and W_i is uniformly bounded by the Lipshitz constant and ϵ. We can further take ϵ less than the distance from p to $\partial\widetilde{M}_1$.

Now consider the busemann function b_{1V_i} on the 0 level set $H_{W_i}(0)$ of b_{1W_i} (i.e. the level set through p). By the inner product condition above we can find unit speed differentiable curves τ_1 and τ_2 in $H_{W_i}(0)$

starting at p of length ϵ (and hence in $H_{W_i}(0) \cap B_p(\epsilon)$) such that

$$\langle \tau_1'(s), V_i(\tau_1(s))\rangle > \bar{C} = \sqrt{1 - \left(\frac{1+C}{2}\right)^2}$$

and

$$\langle \tau_2'(s), V_i(\tau_2(s))\rangle < -\bar{C} = -\sqrt{1 - \left(\frac{1+C}{2}\right)^2}\ .$$

Thus for every sufficiently large i there are points $z_i^1, z_i^2 \in H_{W_i}(0) \cap B_p(\epsilon)$ such that $b_{1V_i}(z_i^1) > \bar{C}\epsilon$ and $b_{1V_i}(z_i^2) < -\bar{C}\epsilon$. Thus by lemma 2.6 for every sufficiently large i there are points y_i^1 and y_i^2 on the boundary of $H_{W_i}(0) \cap \widetilde{M_1}$ with $b_{1V_i}(y_i^1) > \bar{C}\epsilon$ and $b_{1V_i}(y_i^2) < -\bar{C}\epsilon$.

Since V_i and W_i converge to trapped geodesics, Lemma 2.4 says that as $i \to \infty$ the lines L_{V_i} and L_{W_i} converge to vertical. But this means that the the Busemann functions b_{1V_i} and b_{1W_i} (which are height functions outside $\widetilde{M_1}$) approximate the vertical height function. In particular, for i large enough the values of b_{1V_i} on the boundary of $H_{W_i}(0) \cap \widetilde{M_1}$ vary by less than $\bar{C}\epsilon$. This contradicts the simultaneous existence of y_i^1 and y_i^2 for large i.

□

The first consequence of this proposition is that the trapped geodesics are also minimizing (as limits of minimizing geodesics) and hence g has no conjugate points. Start with a large enough flat $n+1$ torus $T^n \times S^1$ so that D^n sits isometrically in T^n. Now replace $D^n \times S^1$ with $(M_1, \partial M_1, g)$ to get an $n+1$ torus with no conjugate points. Then the theorem of Burago-Ivanov [Bu-Iv94] proving the E. Hopf conjecture says that the metric is flat. This gives a proof of Theorem 1.1.

However the above proof does not generalize very far. In the next section we give an alternative proof that does generalize.

3. GENERALIZATIONS

In the previous section we considered only flat metrics so as to make the the proof more transparent. However the arguments extend almost without change to give

Proposition 3.1. *Let D^n be a ball in a complete simply connected manifold N^n with nonpositive curvature, and $(M_1, \partial M_1, g_1)$ a Riemannian manifold with boundary that has the same scattering data as $(D \times S^1, \partial D \times S^1, g)$ where g is the product metric. Then through every point of M_1 there is exactly one trapped geodesic.*

The only change that affects the proof is that geodesics in $\widetilde{M_1^{ext}}$ are not lines outside $\widetilde{M_1}$ but geodesics in N. Geodesic rays in N are thought of as parallel if they have the same limit point at infinity (which means that they stay a bounded distance from each other). This allows us to relate $\mathbf{A}(p)$ to $\mathbf{A}(q)$ and all the arguments go through.

□

Remark 3.2. To extend this to manifolds without conjugate points one first needs to deal with the fact that balls may not be convex. This would seem to give us problems with the differentiability of the metric on $\widetilde{M_1^{ext}}$ at the boundary of $\widetilde{M}_1$. However (except possibly in the case where the boundary of D contains a region that is totally geodesic) since there are no conjugate points Theorem 1 of [St-Uh09] will still tell us that the metric will be smooth. In fact very little of the argument really needs the boundary to be convex (or the metric to be smooth for that mater) but extending the arguments would look somewhat different from the above. Also one has to worry about relating $\mathbf{A}(p)$ and $\mathbf{A}(q)$. This can be done by fixing a base point $x_0 \in \widetilde{M_1^{ext}} - \widetilde{M}_1$ and looking only at Busemann functions defined by vectors in U_{x_0}. In any event the arguments would look somewhat different and we wont pursue them.

We now want to generalize Theorem 1.1 to the nonflat case. The first point to note is that for g and g_1 as in Proposition 3.1 we have

$$Vol(g_1) = Vol(g).$$

Let $\mathbf{T_1} \subset UM_1$ be the set of unit vectors tangent to trapped geodesic rays. Similarly define $\mathbf{T} \subset UM$ (here $M = D \times S^1$). Using the fact that the metrics are lens equivalent we consider the standard measure preserving map $F : UM_1 - \mathbf{T} \to UM - \mathbf{T}$ which assigns to each vector $V \in UM_1 - \mathbf{T}$ the unique vector $W \in UM_1 - \mathbf{T}$ such that $V = \gamma_1'(t)$ and $W = \gamma'(t)$ where γ and γ_1 are geodesics with corresponding initial conditions on the boundary. I.e. $\gamma(0) \in \partial M$ and $\gamma_1(0) \in \partial M_1$ are corresponding points while $\gamma'(0)$ and $\gamma_1'(0)$ are corresponding inwardly pointing unit vectors. That F (which conjugates the geodesic flows) is measure preserving is a standard fact which follows for example from Santaló's formula (see for example [Cr91]). The fact that $\mathbf{T}$ and $\mathbf{T_1}$ have measure 0 tells us that the unit tangent bundles (and hence the manifolds) have the same volume.

The generalization of Theorem 1.1 is

Theorem 3.3. *Let D^n be a ball in $N^{n-1} \times \mathbb{R}$ where N is a complete simply connected manifold with nonpositive curvature. Then $D \times S^1$ is scattering rigid.*

Proof: One proves this via Proposition 2.2 of [Cr-Kl98]. That result compares two metrics g and g_1 on manifolds without conjugate points with an additional condition on g that it contain a real factor (which is satisfied by our assumption on D). Under a weak lens equivalency condition (which is satisfied since the metrics are lens equivalent) one concludes that $Vol(g_1) \geq Vol(g)$ with equality holding if and only if g_1 is isometric to g. This (along with the above fact that $Vol(g_1) = Vol(g)$) proves the Theorem.

□

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Department ofbb: Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395 USA

E-mail address: ccroke@math.upenn.edu