SMALL VOLUME ON BIG N-SPHERES

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Abstract. We consider Riemannian metrics on the n-sphere for $n \geq 3$ such that the distance between any pair of antipodal points is bounded below by 1. We show that the volume can be arbitrarily small. This is in contrast to the 2-dimensional case where Berger has shown that $Area \geq 1/2$.

In 1977 Berger [B77] considered the set of Riemannian metrics g on the n-sphere such that $d(x, A(x)) \geq 1$ and defined the constants:

$$h(n) \equiv inf\{Vol(g)\}.$$

We will also consider $\bar{h}(n) \equiv inf\{Vol(g)|A^*g = g\}$. Where you can think of A as the standard antipodal map or simply set A to be any fixed point free diffeomorphism of S^n with $A^2 = Id$.

One of Berger's reasons for his interest in these invariants was their use in getting lower bounds on the volume of balls $B(p, R)$ in a complete Riemannian manifold with R less than half the injectivity radius $inj(M)$. In this case all geodesics in the ball minimize distance. Hence for all $r \leq R$ the distance in $B(p, R)$ between antipodal points of $S(p, r) = \partial B(p, r)$ is $2r$; and thus the intrinsic distance in $S(p, r)$ is even larger. So we see $Vol(S(p, r)) \geq h(n-1)(2r)^{n-1}$ and

$$Vol(B(p, R)) = \int_0^R Vol(S(p, r))dr \geq$$

$$\geq \int_0^R h(n-1)(2r)^{n-1}dr = 2^{n-1}\frac{h(n-1)}{n}R^n.$$

Berger noted that $h(1) = 2$ and proved $h(2) \geq \frac{1}{2}$ to get a (non-sharp) lower bound $Vol(B(p, R)) \geq 2R^2$ on the volume of 2-balls and $Vol(B(p, R)) \geq \frac{2}{3}R^3$ for 3-balls when $R \leq inj(M)/2$.

In this short note we show by relatively easy examples:

Theorem 0.1. $h(n) = 0$ *for* $n \geq 3$.

This question and some history surrounding it was also discussed in section 6 of [CK03]. In particular, Ivanov (see [I98]) has given examples

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of a sequence of metrics on S^3 that Gromov-Housdorff converge to the standard metric but whose volumes go to zero.

The main open question about volume of balls can be stated as a conjecture which is open in all dimensions (even $n = 2!$).

Conjecture 0.2. *If ω_n represents the volume of the unit n-sphere:*
a) For any $r \leq \frac{inj(M)}{2}$, $Vol(B(r)) \geq \frac{\omega_n}{2}(\frac{2}{\pi})^n r^n$ with equality holding only if the ball is isometric to a hemisphere.
b) For any $r \leq inj(M)$, $Vol(B(r)) \geq \frac{\omega_n}{\pi^n} r^n$, where equality holds if and only if M is isometric to the round sphere of injectivity radius r (i.e. extrinsic radius $\frac{r}{\pi}$).

As mentioned above, Berger [B77] gave non-sharp lower bounds for $n = 2$ and $n = 3$. The author gave nonsharp lower bounds in all dimensions in [Cr80] and showed that $b)$ was true for the "average" ball (i.e. $\frac{1}{Vol(M)} \int_M Vol(B(x,r))dx \geq \frac{\omega_n}{\pi^n} r^n$) for any compact manifold M in any dimension [Cr84]. This followed Berger's isoembolic inequality $Vol(M) \geq \frac{\omega_n}{\pi^n} inj(M)^n$, where equality holds if and only if M is isometric to the round sphere.

As to the question of the exact value of $h(2)$ we have the following interesting

Conjecture 0.3. $h(2) = \frac{4}{\pi}$ *and the round sphere is the only space that achieves this.*

The situation for $\bar{h}$ is quite different. By identifying antipodal points we get a metric on $\mathbb{R}P^n$ such that all non contractible closed curves have length at least 1 (i.e. $sys(M) \geq 1$). Hence Pu's Theorem [Pu52] shows $\bar{h}(2) = \frac{4}{\pi}$, while Gromov's theorem [Gr83] gives us nonsharp positive lower bounds for $\bar{h}(n)$ for all n. The question as to the actual values of $\bar{h}(n)$ for $n \geq 3$ is a hard and very interesting one.

One other place where this notion came up is in [Cr02]. There it is shown that for a Riemannian metric on a 3-sphere with $d(x, A(x)) \geq D$ and L = length of the shortest closed geodesic, we have $Vol^{1/3} \geq const \min\{L, 2D\}$. In this note we show that there is no constant such that $Vol^{1/3} \geq constD$. It is still an open (and very interesting) question if there is a constant so that $Vol^{1/3} \geq constL$.

We now construct the examples on the 3-sphere.

We will let MB_ϵ be the ϵ neighborhood in the flat plane $\mathbb{R}^2$ of a tripod (three unit length line segments from the origin making angles $\frac{2\pi}{3}$ with each other). As ϵ goes to 0 the area goes to 0 while the length of the boundary goes to 6. This has been known for a while as an example of a metric on a 2-disc whose area goes to 0 but such that on the boundary antipodal points satisfy $d_{MB_\epsilon}(p, A_\epsilon(p)) \geq d_{\mathbb{R}^2}(p, A_\epsilon(p) \geq 1$. Here $A_\epsilon(p)$ is the point half way around the boundary. (Note that the non convexity of MB_ϵ is not the point here since one could use equilateral triangles in a more and more negatively curved simply connected space form.)

Our examples are just $M^3_\epsilon = \partial(MB_\epsilon \times MB_\epsilon)$, which as the boundary of a 4-ball is a 3-sphere. Since $M_\epsilon = (\partial(MB_\epsilon) \times MB_\epsilon) \cup (MB_\epsilon \times \partial(MB_\epsilon))$, $Vol(M_\epsilon) = 2Area(MB_\epsilon)L(\partial(MB_\epsilon))$. Hence $\lim_{\epsilon\to 0} Vol(M_\epsilon) = 0$.

We need to define the antipodal map $\bar{A}_\epsilon$ on M_ϵ. First extend A_ϵ to a continuous homeomorphism $A_\epsilon : MB_\epsilon \to MB_\epsilon$ with a single fixed point (the origin in $\mathbb{R}^2$ say). One can do this simply by mapping the line segment between the origin and $p \in \partial(MB_\epsilon)$ to the line segment between the origin and $A_\epsilon(p)$. Then let

$$\bar{A}_\epsilon = (A_\epsilon \times A_\epsilon)|_{M_\epsilon}.$$

For a point $(p, q) \in \partial(MB_\epsilon) \times MB_\epsilon \subset M_\epsilon$ we have

$$d_{M_\epsilon}((p, q), \bar{A}_\epsilon(p, q)) = d_{M_\epsilon}((p, q), (A_\epsilon(p), A_\epsilon(q))) \geq$$
$$\geq d_{\mathbb{R}^4}((p, q), (A_\epsilon(p), A_\epsilon(q))) \geq d_{\mathbb{R}^2}(p, A_\epsilon(p)) \geq 1.$$

Similarly if $(p, q) \in MB_\epsilon \times \partial(MB_\epsilon) \subset M_\epsilon$ then $d_{M_\epsilon}((p, q), \bar{A}_\epsilon(p, q)) \geq 1$.

This completes the proof of the theorem in the 3 dimensional case after smoothing the metric and $\bar{A}_\epsilon$. The higher dimensional cases are handled by spherical suspension. Specifically given a metric g on the n sphere S^n we construct a (singular) warped product metric $g' = \cos^2(t)g + dt^2$ on the $n+1$ sphere $S^{n+1} = (S^n \times [-\frac{\pi}{2}, \frac{\pi}{2}])/\sim$, where $(x, \frac{\pi}{2}) \sim (y, \frac{\pi}{2})$ and $(x, -\frac{\pi}{2}) \sim (y, -\frac{\pi}{2})$ for all $x, y \in S^n$. The antipodal map $A_{n+1} : S^{n+1} \to S^{n+1}$ is just $A_{n+1}(x, t) = (A_n(x), -t)$. It is easy to check that if g has $d(x, A_n(x)) \geq \pi$ and $Vol(g) = V$ then $D(x, A_{n+1}(x)) \geq \pi$ and $Vol(g') = Vol(g)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^n(t)dt = \frac{\omega_{n+1}}{\omega_n} Vol(g)$. So spherically suspending (and smoothing) a sequence of examples that shows $h(n) = 0$ gives us a sequence showing that $h(n+1) = 0$.

References

[B77] M. Berger, *Volume et rayon d'injectivité dans les variétés riemanniennes de dimension 3*, Osaka J. Math., **14** (1977), 191- 200.

[Cr80] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Scient. Ec. Norm. Sup., 4e serie, t.**13** (1980), 419-435.

[Cr84] C. Croke, *Curvature Free Volume Estimates*, Inventiones Mathematicae **76** (1984), 515-521.

[Cr02] C. Croke, *the volume and lengths on a three sphere*, Comm. Anal. Geom., **10** (2002) no. 3, 467–474.

[CK03] Croke, C.; Katz, M.: Universal volume bounds in Riemannian manifolds, *Surveys in Differential Geometry* **V**III, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3- 5, 2002, edited by S.T. Yau (Somerville, MA: International Press, 2003.) pp. 109 - 137. See `arXiv:math.DG/0302248`

[Gr83] Gromov, M.: Filling Riemannian manifolds, *J. Diff. Geom.* **1**8 (1983), 1-147.

[I98] S. Ivanov *Gromov-Hausdorff Convergence and volumes of manifolds*, St. Petersburg Math. J., **9**(1998) No.5, 945-959

[Pu52] Pu, P.M.: Some inequalities in certain nonorientable Riemannian manifolds, *Pacific J. Math.* **2** (1952), 55–71.

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