

# BOUNDARY CASE OF EQUALITY IN OPTIMAL LOEWNER-TYPE INEQUALITIES

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ABSTRACT. We prove certain optimal systolic inequalities for a closed Riemannian manifold  $(X, g)$ , depending on a pair of parameters,  $n$  and  $b$ . Here  $n$  is the dimension of  $X$ , while  $b$  is its first Betti number. The proof of the inequalities involves constructing Abel-Jacobi maps from  $X$  to its Jacobi torus  $\mathbb{T}^b$ , which are area-decreasing (on  $b$ -dimensional areas), with respect to suitable norms. These norms are the stable norm of  $g$ , the conformally invariant norm, as well as other  $L^p$ -norms. The case of equality is characterized in terms of the criticality of the lattice of deck transformations of  $\mathbb{T}^b$ , while the Abel-Jacobi map is a harmonic Riemannian submersion. That the resulting inequalities are actually nonvacuous follows from an isoperimetric inequality of Federer and Fleming.

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## 1. INTRODUCTION, CONJECTURES, AND SOME RESULTS

In the study of optimal systolic inequalities on a Riemannian manifold  $(X, g)$ , generalizing C. Loewner's inequality, two questions arise:

- (1) Which stable systolic inequalities can be replaced by their conformal analogues?
- (2) Calculate the values of the constants arising in the inequalities.

The definitions of the systolic invariants  $\text{stsys}_1(g)$ ,  $\text{confsys}_1(g)$ , and  $\text{sys}_{\pi_1}(g)$  can be found in the survey [CK03], while the Abel-Jacobi map  $\mathcal{A}_X$ , in [Li69, Gr96], *cf.* [BK04, (4.3)], [IK04]. The first such optimal inequality was proved in [Gr99, pp. 259-260] (*cf.* [CK03, inequality (5.14)]) based on the techniques of D. Burago and the third author [BI94, BI95].

**Theorem 1.1** (M. Gromov). *Let  $X$  be a closed orientable manifold. Let  $n = \dim(X)$  and assume  $n = b_1(X) = \text{cuplength}_{\mathbb{R}}(X)$ . Then every metric  $g$  on  $X$  satisfies the following optimal inequality:*

$$\text{stsys}_1(g)^n \leq \deg(\mathcal{A}_X)^{-1} (\gamma_n)^{\frac{n}{2}} \text{vol}_n(g), \quad (1.1)$$

where  $\gamma_n$  is the Hermite constant of (1.3), while  $\mathcal{A}_X : X \rightarrow \mathbb{T}^n$  is the classifying map of the natural epimorphism  $\pi_1(X) \rightarrow H_1(X, \mathbb{Z})/\text{torsion}$ .

The following conformally invariant generalisation of Gromov's inequality (1.1) is immediate from Proposition 4.3, *cf.* inequality (3.3).

**Theorem 1.2.** *Let  $X$  be a closed orientable manifold. Let  $n = \dim(X)$  and assume  $n = b_1(X) = \text{cuplength}_{\mathbb{R}}(X)$ . Then every metric  $g$  on  $X$  satisfies the following optimal inequality:*

$$\text{confsys}_1(g)^n \leq \deg(\mathcal{A}_X)^{-1} (\gamma_n)^{\frac{n}{2}}. \quad (1.2)$$

In connection with question 1 above, we aim for the following results. We show in section 5 that if the  $L^p$  norm is equal to the comass norm, then the minimizing measurable 1-form is in fact continuous and has

constant pointwise norm. Then we can apply a Voronoi-type argument to prove that in an extremal situation, in fact all cohomology classes are represented by 1-forms of constant norm. This allows us to prove that a manifold  $X$  such that  $\dim(X) = b_1(X) = \text{cuplength}_{\mathbb{R}}(X)$ , which satisfies the boundary case of equality of (Gromov's) inequality relating the stable 1-systole and the volume, must be a critical flat torus (Theorem 1.4). The proof passes via the conformally invariant strengthening of Gromov's inequality, namely Theorem 1.2.

Given a Euclidean lattice  $L \subset \mathbb{R}^n$ , denote by  $\lambda_1(L) > 0$  the least length of a nonzero vector in  $L$ . A lattice  $L$  is called *critical* if it realizes the supremum

$$\sup_{L \subset \mathbb{R}^n} \frac{\lambda_1(L)^n}{\text{vol}(\mathbb{R}^n/L)} = (\gamma_n)^{\frac{n}{2}}, \quad (1.3)$$

where  $\gamma_n$  is the Hermite constant. Equivalently,  $L$  is critical if the  $L$ -centered packing by balls is the densest one in dimension  $n$ . Recall that a critical (or, more generally, extremal) lattice is necessarily perfect and eutactic [Bar57]. The following two theorems are proved in Section 9.

**Theorem 1.3.** *The boundary case of equality in the conformally invariant inequality (1.2), occurs if and only if  $X$  is a torus and the metric  $g$  is conformal to a flat critical metric.*

**Theorem 1.4.** *In the hypotheses of Theorem 1.1, equality in (1.1) is attained precisely when  $X$  is a torus, while  $g$  is a flat metric whose deck transformations form a critical lattice.*

In Section 2, we discuss a notion of degree of the Abel-Jacobi map when  $n > b$ , and prove its positivity under the topological hypothesis of homological nonvanishing. In Section 3, we extend Theorem 1.3 and Theorem 1.4 to a general pair  $n \geq b$ . In Section 4, we define the norms involved, and state a key proposition used in the proof of Theorems 1.3 and 1.4. In Sections 5 and 6, we collect analytical results pertaining to closed forms that minimize an  $L^p$ -norm. This constitutes the analytical backbone of the paper. Sections 7 and 11 contain successive generalisations of a construction of [BI94, BI95]. The remaining sections are devoted to the proof of the results of Section 3.

## 2. NOTION OF DEGREE WHEN DIMENSION EXCEEDS BETTI NUMBER

Let  $(X, g)$  be a closed Riemannian manifold,  $b = b_1(X)$ . We denote by  $\mathcal{A}_X$  the Abel-Jacobi map,

$$\mathcal{A}_X : X \rightarrow H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})_{\mathbb{R}} \simeq \mathbb{T}^b,$$

inducing (the natural) isomorphism in 1-dimensional cohomology. Let  $\overline{X}$  be the maximal free abelian cover of  $X$ , *cf.* Definition 7.2. Denote by  $\overline{\mathcal{A}}_X$  the proper map obtained as the lift of  $\mathcal{A}_X$  to  $\overline{X}$ :

$$\overline{\mathcal{A}}_X : \overline{X} \rightarrow H_1(X, \mathbb{R}) \simeq \mathbb{R}^b.$$

If  $\overline{X}$  is orientable we fix an orientation for  $\overline{X}$ . In the case that  $X$  is orientable, the following definition is due to M. Gromov [Gr83, p. 101].

**Definition 2.1.** Let  $[\overline{F}_X] \in H_{n-b}(\overline{X}, R)$  denote the homology class of the regular fibers  $\overline{\mathcal{A}}_X^{-1}(y)$  of  $\overline{\mathcal{A}}_X$  where we take  $R = \mathbb{Z}$  if  $\overline{X}$  is orientable and  $R = \mathbb{Z}_2$  otherwise. Then the geometric degree

$$\deg(\mathcal{A}_X)$$

of  $\overline{\mathcal{A}}$  is the infimum of the  $(n-b)$ -volumes of all cycles representing  $[\overline{F}_X]$ .

**Remark 2.2.** If  $\overline{X}$  is orientable, then changing the orientation of  $\overline{X}$  will not change  $\deg(\mathcal{A}_X)$ . If  $n = b$  and  $X$  is orientable and connected, then  $\deg(\mathcal{A})$  is (the absolute value of) the topological degree of  $\mathcal{A}_X$ . If  $n > b$  and  $[\overline{F}_X] \neq 0$ , then  $\deg(\mathcal{A}_X)$  depends on the metric on  $X$ .

For our arguments the following obvious consequence of Definition 2.1 will be crucial.

**Lemma 2.3.** *Suppose  $f : X \rightarrow H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})_{\mathbb{R}}$  is homotopic to  $\mathcal{A}_X$  and  $y$  is a regular value of  $f$ . Then we have*

$$\text{vol}_{n-b}(f^{-1}(y)) \geq \deg(\mathcal{A}_X).$$

*Proof.* If  $\bar{y} \in H_1(X, \mathbb{R})$  is a preimage of  $y$ , and if  $\bar{f} : \overline{X} \rightarrow H_1(X, \mathbb{R})$  is a lift of  $f$ , then the covering projection  $\overline{X} \rightarrow X$  maps  $\bar{f}^{-1}(\bar{y})$  isometrically onto  $f^{-1}(y)$ , so that  $\text{vol}_{n-b}(\bar{f}^{-1}(\bar{y})) = \text{vol}_{n-b}(f^{-1}(y))$ . On the other hand,  $\bar{f}$  is homotopic to  $\overline{\mathcal{A}}_X$  by a proper map, and hence  $\bar{f}^{-1}(\bar{y})$  represents  $[\overline{F}_X]$ .  $\square$

All the results discussed below are nonvacuous only if  $\deg(\mathcal{A})$  is positive. So it is important to know when this is the case. First note that this is a topological property.

**Remark 2.4.** If  $[\overline{F}_X] \neq 0$  then  $\deg(\mathcal{A}_X) > 0$ .

This follows from the deformation theorem as stated in [Fe69, 4.2.9], complemented by [Fe69, (4.2.9) $^{\nu}$ ] for the case of  $\mathbb{Z}_2$ -coefficients, see also [FF60], [Wh99], and [Mo95, Chapter 5] for a helpful picture. The deformation theorem implies that for every compact Riemannian manifold  $X$ , there exists  $\epsilon > 0$  such that for every  $q \in \mathbb{N}$  every  $q$ -cycle of  $q$ -volume smaller than  $\epsilon$  can be deformed into the  $(q-1)$ -skeleton of a triangulation of  $X$ . Now, if  $z$  is a  $q$ -cycle in  $\overline{X}$  of  $q$ -volume smaller

than  $\epsilon$ , then the projection of  $z$  to  $X$  can be deformed into the  $(q-1)$ -skeleton. Lifting this deformation to  $\overline{X}$  we conclude that the homology class of  $z$  vanishes regardless of the ring of coefficients.

If  $X$  is the quotient of the 3-dimensional Heisenberg group by its subgroup of integral elements, then the fibers of  $\mathcal{A}_X$  are homologically trivial in  $X$  while  $[\overline{F}_X] \neq 0$ . This explains why one looks at the situation in  $\overline{X}$  in Definition 2.1.

In analogy with the maximal cuplength $_{\mathbb{R}}$  condition in Theorem 1.1, we note that  $[\overline{F}_X] \neq 0$  if  $X$  is orientable and if  $\alpha_1 \cup \dots \cup \alpha_b \neq 0$  for every basis  $\alpha_1, \dots, \alpha_b$  of  $H^1(X, \mathbb{R})$ .

Note that, if the minimizing rectifiable current in  $[\overline{F}_X]$  happens to be nonsingular, then the work [Lê93] produces explicit lower bounds for  $\deg(\mathcal{A})$  in terms of the injectivity radius and an upper bound on the sectional curvature.

### 3. THE THEOREMS

The following two results were proved in [IK04].

**Theorem 3.1.** *Let  $X$  be a closed Riemannian manifold. Let  $n = \dim(X)$  and  $b = b_1(X)$ , and assume  $n \geq b \geq 1$ . Then every metric  $g$  on  $X$  satisfies the inequality*

$$\deg(\mathcal{A}_X) \text{stsys}_1(g)^b \leq (\gamma_b)^{\frac{b}{2}} \text{vol}_n(g). \quad (3.1)$$

**Corollary 3.2.** *Let  $X$  be a closed orientable manifold. Let  $b = b_1(X)$ . Assume that  $\dim(X) = b+1$ , and  $[\overline{F}_X] \neq 0$ . Then every metric  $g$  on  $X$  satisfies the following optimal inequality:*

$$\text{stsys}_1(g)^b \text{sys}\pi_1(g) \leq (\gamma_b)^{\frac{b}{2}} \text{vol}_{b+1}(g). \quad (3.2)$$

The main tool in studying the case of equality of the above inequalities is the following result concerning a kind of an  $L^p$ -systole, cf. Definition 11.1. Let  $p \geq 1$ .

**Definition 3.3.** The norm  $\|\cdot\|_p$  on  $H_1(X; \mathbb{R})$  is the dual norm to the  $L^p$ -norm  $\|\cdot\|_p^*$  on  $H^1(X; \mathbb{R})$ , cf. [BK03].

**Remark 3.4.** In general, if  $h \in H_k(X, \mathbb{R})$ , then the quantity  $\|h\|_{\frac{n}{k}}$  is the supremum of the stable norms of  $h$  with respect to all metrics that are conformal to  $g$  and have unit volume, cf. [Gr83, 7.4.A]. In particular, one has

$$\text{stsys}_k(g) \leq \text{confsys}_k(g) \text{vol}_n(g)^{\frac{k}{n}}. \quad (3.3)$$

Given a lattice  $L$  equipped with a norm  $\|\cdot\|$ , denote by  $\lambda_1(L, \|\cdot\|)$  the least norm of a nonzero vector in  $L$ . The following proposition generalizes Theorem 3.1.

**Proposition 3.5.** *Let  $X$  be a closed orientable manifold. Let  $b = b_1(X)$ . Let  $p > \max\{b, 2\}$ . Normalize the metric  $g$  on  $X$  to unit total volume. Then*

$$\deg(\mathcal{A}_X) \lambda_1(H_1(X, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|_p)^b \leq (\gamma_b)^{\frac{b}{2}}. \quad (3.4)$$

By choosing  $p = n$  in Proposition 3.5, we obtain the following generalisation of Theorem 1.2 and Theorem 3.1.

**Theorem 3.6.** *Let  $X$  be a closed orientable manifold. Let  $n = \dim(X)$  and  $b = b_1(X)$ . Then every metric  $g$  on  $X$  satisfies the inequality*

$$\deg(\mathcal{A}_X) \text{confsys}_1(g)^b \leq (\gamma_b)^{\frac{b}{2}} \text{vol}_n(X)^{n-b}. \quad (3.5)$$

In analogy with our Theorem 1.4, we prove the following.

**Theorem 3.7.** *Let  $X$  be a closed orientable manifold. Assume that  $\dim(X) > b_1(X)$ . Suppose equality is attained in (3.5). Then*

- (1) *all harmonic one-forms on  $X$  have pointwise constant norm;*
- (2)  *$X$  admits a harmonic Riemannian submersion onto a flat  $b$ -torus whose deck group is a critical lattice in  $\mathbb{R}^b$ ;*
- (3) *the submersion is given by the Abel-Jacobi map defined via harmonic one-forms;*
- (4) *the fibers of the submersion are minimal surfaces of constant  $(n - b)$ -dimensional area.*

Note that Riemannian manifolds for which the Abel-Jacobi map is a harmonic Riemannian submersion onto a flat torus are studied in [BK04].

As a consequence of Theorem 3.7, we obtain the following, see some related results in [NV04].

**Corollary 3.8.** *In the hypotheses of Corollary 3.2, equality in (3.2) is attained precisely when  $X$  is a two-step nilpotent manifold with 1-dimensional center, while  $g$  is the metric of a Riemannian submersion with geodesic fibers of constant length.*

The case  $n = b + 2$  is studied in [IK04] and the companion paper [BCIK04], which explores the closely related filling area conjecture.

#### 4. STABLE NORMS AND CONFORMAL NORMS

Let  $X$  and  $Y$  be compact smooth manifolds. Let  $\varphi : X \rightarrow Y$  be a continuous map inducing an epimorphism in one-dimensional real homology. Given a Riemannian metric on  $X$  (more generally, the structure of a locally simply connected length space), one defines the *relative*

stable norm  $\| \cdot \|_{\text{st}/\varphi}$  on  $H_1(Y; \mathbb{R})$  by setting

$$\| \alpha \|_{\text{st}/\varphi} = \inf \{ \| \beta \|_{\text{st}} \mid \beta \in H_1(X; \mathbb{R}), \varphi_*(\beta) = \alpha \}, \quad (4.1)$$

where  $\| \cdot \|_{\text{st}}$  is the ordinary (“absolute”) stable norm of  $X$ . The stable norm itself may be thought of as the relative stable norm defined by the Abel-Jacobi map to the torus  $H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})_{\mathbb{R}}$ .

Let  $X^n$  be a compact Riemannian manifold,  $V^b$  a vector space, and  $\Gamma$  a lattice in  $V$ . We will identify  $V$  and  $H_1(V/\Gamma; \mathbb{R})$ . Let  $\varphi : X \rightarrow V/\Gamma$  be a continuous map inducing an epimorphism of the fundamental groups. Note that there is a natural isomorphism  $\Gamma \simeq \pi_1(V/\Gamma) \simeq \pi_1(X)/\ker(\varphi_*)$ . The following proposition was proved in [IK04].

Denote by  $\| \cdot \|_E$  the Euclidean norm on  $V$ , which is defined by the John ellipsoid, *cf.* [Jo48], of the relative stable norm  $\| \cdot \|_{\text{st}/\varphi}$ .

**Proposition 4.1.** *There exists a Lipschitz map  $X \rightarrow (V/\Gamma, \| \cdot \|_E)$  which is homotopic to  $\varphi$  and non-expanding on all  $b$ -dimensional areas, where  $b = \dim V$ .*

The following statement may be known by convex set theorists. Anyway, there is a proof in [BI94].

**Lemma 4.2.** *Let  $(V^b, \| \cdot \|)$  be a Banach space. Let  $\| \cdot \|_E$  be the Euclidean norm determined by the John ellipsoid of the unit ball of the norm  $\| \cdot \|$ . Then there exists a decomposition of  $\| \cdot \|_E^2$  into rank-1 quadratic forms:*

$$\| \cdot \|_E^2 = \sum_{i=1}^N \lambda_i L_i^2$$

such that  $N \leq \frac{b(b+1)}{2}$ ,  $\lambda_i > 0$  for all  $i$ ,  $\sum \lambda_i = b$ , and  $L_i : V \rightarrow \mathbb{R}$  are linear functions with  $\| L_i \|_* = 1$  where  $\| \cdot \|_*$  is the dual norm to  $\| \cdot \|$ .

Given a Riemannian manifold  $X^n$ , one defines a conformally invariant norm  $\| \cdot \|_{\text{conf}}$  on  $H_1(X; \mathbb{R})$  as the dual norm to the  $L^n$ -norm  $\| \cdot \|_n^*$  on  $H^1(X; \mathbb{R})$ , *cf.* Remark 3.4. If  $\text{vol}(X) = 1$ , then we have an inequality  $\| \cdot \|_n^* \leq \| \cdot \|_*$ , where  $\| \cdot \|_*$  is the comass norm in cohomology, dual to the stable norm in homology. Therefore dually, we have the inequality  $\| \cdot \|_{\text{conf}} \geq \| \cdot \|_{\text{st}}$ , *cf.* inequality (3.3). In particular, the volume form defined by the John ellipsoid of the conformal norm is greater than or equal to that of the stable norm.

For a continuous map  $\varphi : X \rightarrow Y$  inducing an epimorphism in one-dimensional homology, one defines a relative conformal norm  $\| \cdot \|_{\text{conf}/\varphi}$  on  $H_1(Y; \mathbb{R})$  by setting

$$\| \alpha \|_{\text{conf}/\varphi} = \inf \{ \| \beta \|_{\text{conf}} \mid \beta \in H_1(X; \mathbb{R}), \varphi_*(\beta) = \alpha \}. \quad (4.2)$$

We present an “integral” version of Proposition 4.1 for conformal norms.

**Proposition 4.3.** *Let  $X^n$  be a compact Riemannian manifold,  $V^n$  a vector space, and let  $\Gamma$  be a lattice in  $V$ . Let  $\varphi : X \rightarrow V/\Gamma$  be a continuous map inducing an epimorphism of the fundamental groups. Let  $\|\cdot\|_E$  denote the Euclidean norm on  $V$  defined by the John ellipsoid of the relative conformal norm  $\|\cdot\|_{\text{conf}/\varphi}$ . Then there exists a  $C^1$  map  $f : X \rightarrow (V/\Gamma, \|\cdot\|_E)$  which is homotopic to  $\varphi$  and satisfies*

$$\int_X \text{Jac}(f) \leq 1.$$

*In particular, if  $\varphi$  has nonzero degree (or nonzero absolute degree), then we have the following upper bound for the volume of the torus  $V/\Gamma$ :*

$$\text{vol}(V/\Gamma, \|\cdot\|_E) \leq \frac{1}{\text{deg}(\varphi)}.$$

An adaptation of the main construction [BI94, BI95] involved in the proof will be described in Section 7. We will then use it to prove Proposition 4.3 in Section 8.

## 5. EXISTENCE OF $L^p$ -MINIMIZERS IN COHOMOLOGY CLASSES

In this section, we discuss the existence of a (weakly closed)  $L^p$ -form that minimizes the  $L^p$ -norm,  $1 < p < \infty$ , in a given cohomology class, used in the proof of Proposition 4.3 in Section 8.

Although this is known, we could not find a good reference. The proof consists in a straightforward application of the direct method of the calculus of variations. Actually C. Hamburger [Ha92] even proved Hölder continuity of the minimizers.

Note that the existence of a continuous 1-form minimizing the co-mass, and defining a geodesic lamination, is proved in [FaS04, Theorem 1.7].

If  $E \rightarrow X$  is a Euclidean vector bundle over a compact, oriented Riemannian manifold  $X$ , and if  $1 \leq p < \infty$ , let  $L^p(E)$  denote the vector space of  $L^p$ -sections  $s$  of  $E$  endowed with the Banach norm

$$|s|_p = \left( \int_X |s(x)|^p d\text{vol}(x) \right)^{\frac{1}{p}}. \quad (5.1)$$

If  $p = \infty$ , let  $L^\infty(E)$  denote the essentially bounded sections of  $E$  and let  $|s|_\infty$  denote the essential supremum of the function  $x \in X \rightarrow |s(x)|$ .

In analogy with the case of functions  $X \rightarrow \mathbb{R}$ , we have:

**Lemma 5.1.** *If  $1 \leq p < \infty$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then the natural map*

$$J_p : L^p(E) \rightarrow L^q(E)^*, (J_p(s))(s') := \int_X \langle s, s' \rangle d\text{vol}$$



for  $s \in L^p(E)$ ,  $s' \in L^q(E)$  is an isometric isomorphism between  $L^p(E)$  and the dual space  $L^q(E)^*$  of the Banach space  $L^q(E)$ . For  $1 < p < \infty$  the Banach space  $L^p(E)$  is reflexive.

*Proof.* If we add a vector bundle  $E'$  to  $E$  such that  $E \oplus E' \rightarrow X$  is trivial, we can easily reduce the claim to the function case. For this case see *e.g.* [Al99, 4.12 and 6.10].  $\square$

In the sequel, the vector bundle  $E$  will be the bundle  $\Lambda^k X$  of alternating  $k$ -forms,  $0 \leq k \leq \dim X =: n$ . The Euclidean structure on  $\Lambda^k X$  will be the one induced by the Riemannian metric on  $X$ . For  $k \in \{0, 1, n-1, n\}$  the corresponding Euclidean norm on the fibers of  $\Lambda^k X$  coincides with the comass norm, *cf.* [Fe69, 1.8.1]. We assume that the total volume of  $X$  is normalized to one.

For a cohomology class  $\alpha \in H^k(X, \mathbb{R})$ , we let  $\alpha_{dR}$  denote the set of smooth closed  $k$ -forms  $w \in \Omega^k X$  that represent  $\alpha$ .

Then

$$|\alpha|_p^* := \inf\{|w|_p \mid w \in \alpha_{dR}\}$$

defines a norm on the finite-dimensional vector space  $H^k(X, \mathbb{R})$ . To see that  $|\alpha|_p^* > 0$  if  $\alpha \neq 0$ , note that there exists  $\beta \in H^{n-k}(X, \mathbb{R})$  such that the cup product  $\alpha \cup \beta \in H^n(X, \mathbb{R})$  is non-zero. This implies that

$$\left| \int_X w \wedge \pi \right| = |(\alpha \cup \beta)([X])| > 0$$

whenever  $w \in \alpha_{dR}$  and  $\pi \in \beta_{dR}$ . Because of the pointwise inequality  $|w \wedge \pi| \leq \binom{n}{k}^{\frac{1}{2}} |w| |\pi|$ , *cf.* [Fe69, 1.7.5] we obtain

$$\int_X |w| |\pi| d\text{vol} \geq \binom{n}{k}^{-\frac{1}{2}} |(\alpha \cup \beta)([X])|.$$

Hence the Hölder inequality implies

$$|\alpha|_p^* |\beta|_q^* \geq \binom{n}{k}^{-\frac{1}{2}} |(\alpha \cup \beta)([X])| > 0.$$

We let  $\bar{\alpha}_{dR}^p \subseteq L^p(\Lambda^k X)$  denote the closure of  $\alpha_{dR}$  with respect to the  $L^p$ -norm. Note that

$$|\alpha|_p^* = \inf \left\{ |w|_p \mid w \in \bar{\alpha}_{dR}^p \right\} \quad (5.2)$$

Moreover, if  $\beta \in H^{n-k}(X, \mathbb{R})$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $w \in \bar{\alpha}_{dR}^p$  and  $\pi \in \bar{\beta}_{dR}^q$ , then

$$\int_X w \wedge \pi = [\alpha \cup \beta](X). \quad (5.3)$$

Since the cup product  $\cup : H^k(X, \mathbb{R}) \times H^{n-k}(X, \mathbb{R}) \rightarrow \mathbb{R}$  is non-degenerate this implies:

**Lemma 5.2.** *Every smooth closed form in  $\bar{\alpha}_{dR}^p$  represents  $\alpha$ .*

Next we prove that  $\bar{\alpha}_{dR}^p$  contains a unique  $L^p$ -form with minimal  $p$ -norm if  $1 < p < \infty$ .

**Proposition 5.3.** *Let  $1 < p < \infty$  and  $k \in \{1, \dots, \dim X - 1\}$ . For every  $\alpha \in H^k(X, \mathbb{R})$  there exists a unique  $w \in \bar{\alpha}_{dR}^p$  such that*

$$|w|_p = |\alpha|_p^*.$$

*Proof.* Let  $(w_i)_{i \in \mathbb{N}}$  be a sequence in  $\alpha_{dR}$  such that  $\lim_{i \rightarrow \infty} |w_i|_p = |\alpha|_p^*$ . Since  $L^p(\Lambda^k X)$  is a reflexive Banach space, cf. Lemma 5.1, we can assume that the sequence  $(w_i)_{i \in \mathbb{N}}$  converges weakly to some  $w \in L^p(\Lambda^k X)$ , cf. [Al99, 6.9]. Since  $\bar{\alpha}_{dR}^p$  is a closed affine subspace of  $L^p(\Lambda^k X)$ , the Hahn-Banach theorem implies that  $\bar{\alpha}_{dR}^p$  is also weakly sequentially closed, cf. [Al99, 6.12]. This implies that  $w \in \bar{\alpha}_{dR}^p$ . Moreover, the weak convergence of  $(w_i)_{i \in \mathbb{N}}$  to  $w$  implies that

$$|w|_p \leq \liminf_{i \rightarrow \infty} |w_i|_p = |\alpha|_p^*.$$

Using equality (5.2) and  $w \in \bar{\alpha}_{dR}^p$ , we conclude that  $|w|_p = |\alpha|_p^*$ . The uniqueness of the minimizer  $w$  follows from the fact that the equality  $|w + \tilde{w}|_p = |w|_p + |\tilde{w}|_p$  can hold only if  $w$  and  $\tilde{w}$  are linearly dependent in  $L^p(\Lambda^k X)$ .  $\square$

## 6. EXISTENCE OF HARMONIC FORMS WITH CONSTANT NORM

We continue to assume that the volume of  $X$  is normalized to one. Then, as a direct consequence of the Hölder inequality, the function  $p \in [1, \infty] \rightarrow |\alpha|_p^*$  is (weakly) monotonically increasing for every class  $\alpha \in H^k(X, \mathbb{R})$ ,  $0 < k < n = \dim X$ .

**Proposition 6.1.** *Let  $\alpha \in H^k(X, \mathbb{R})$ ,  $0 < k < n$ . Assume there exist reals  $p < p'$  in  $[1, \infty]$  such that  $|\alpha|_p^* = |\alpha|_{p'}^*$ . Then the following holds:*

- (1) *the function  $q \in [1, \infty] \rightarrow |\alpha|_q^*$  is constant;*
- (2) *the harmonic representative  $\omega$  of  $\alpha$  has constant (pointwise) norm;*
- (3) *we have  $|\omega|_q = |\alpha|_q^*$  for all  $q \in [1, \infty]$ .*

*Conversely, if  $p \in [1, \infty)$  and if there exists a form  $\omega \in \bar{\alpha}_{dR}^p$  such that  $|\omega|_p = |\alpha|_p^*$  and  $\omega$  has constant norm almost everywhere, then  $|\omega|_p = |\omega|_q = |\alpha|_q^*$  for all  $q \in [1, \infty]$ . In particular,  $\omega$  is harmonic.*

*Proof.* Let  $p < p'$  in  $[1, \infty]$  satisfy  $|\alpha|_p^* = |\alpha|_{p'}^*$ . Since  $q \mapsto |\alpha|_q^*$  is monotonic, we may assume that  $1 < p < p' < \infty$ . According to Proposition 5.3, there exists  $\omega \in \bar{\alpha}_{dR}^{p'}$  such that  $|\omega|_{p'} = |\alpha|_{p'}^*$ . First we show that  $\omega$  has constant norm. Since  $\bar{\alpha}_{dR}^{p'} \subseteq \bar{\alpha}_{dR}^p$ , we see that

$$|\alpha|_p^* \leq |\omega|_p \leq |\omega|_{p'} = |\alpha|_{p'}^*.$$

Hence our assumption  $|\alpha|_p^* = |\alpha|_{p'}^*$  implies  $|\omega|_p = |\omega|_{p'}$ . Therefore the function  $x \in X \rightarrow |\omega_x|$  is almost everywhere constant.

Next, we prove that  $\omega$  is indeed harmonic. If  $p \leq 2$ , this follows directly from

$$|\alpha|_p^* = |\omega|_p = |\omega|_2 \geq |\alpha|_2^* \geq |\alpha|_p^*.$$

For general  $p$ , we will now show that  $\omega$  is weakly closed and coclosed, i.e. that

$$\int_X \langle \omega, d^* \pi \rangle d \text{vol} = 0 \text{ for all } \pi \in \Omega^{k+1} X, \quad (6.1)$$

and

$$\int_X \langle \omega, d\sigma \rangle d \text{vol} = 0 \text{ for all } \sigma \in \Omega^{k-1} X. \quad (6.2)$$

To prove (6.1), consider a fixed  $\pi \in \Omega^{k+1} X$  and the linear functional

$$\tilde{\omega} \in L^p(\Lambda^k X) \rightarrow \int_X \langle \tilde{\omega}, d^* \pi \rangle d \text{vol}$$

on  $L^p(\Lambda^k X)$ . This functional vanishes on  $\alpha_{dR}$  and hence, by continuity, on  $\omega \in \bar{\alpha}_{dR}^p$ .

Equation (6.2) is a consequence of the minimality of  $|\omega|_p$  in  $\bar{\alpha}_{dR}^p$  and of the constancy of  $|\omega_x|$ . Note that we may assume that  $\alpha \neq 0$  and hence  $|\omega_x| = r > 0$  for almost all  $x \in X$ . For every  $\sigma \in \Omega^{k-1} X$  we have

$$\begin{aligned} 0 &= \left. \frac{d}{ds} \right|_{s=0} \left( \int_X |\omega + s d\sigma|^p d \text{vol} \right) = p \int_X |\omega|^{p-2} \langle \omega, d\sigma \rangle d \text{vol} \\ &= pr^{p-2} \int_X \langle \omega, d\sigma \rangle d \text{vol}. \end{aligned}$$

Since  $r \neq 0$  this implies (6.2). Finally, we consider the Euclidean bundle  $\Lambda^* X = \bigoplus_{k=0}^n \Lambda^k X$  and the selfadjoint, first order linear differential operator  $d + d^*$  acting on sections of  $\Lambda^* X$ . It is wellknown that  $d + d^*$  is elliptic. Hence the standard regularity theory for linear elliptic operators implies that  $\omega$  is smooth and  $(d + d^*)(\omega) = 0$  in the strong sense. Since  $(d + d^*)^2 = \Delta$ , we conclude that  $\omega$  is harmonic. From the fact that  $\omega$  is a harmonic form of constant norm, we can easily conclude that  $|\alpha|_q^*$  does not depend on  $q$ . This is proved in [BK04, Proposition 8.1] for the case  $k \in \{1, n-1\}$ , and the proof carries over to the general case without any change.

Finally, we assume that  $p \in [1, \infty)$ , and that the form  $\omega \in \overline{\alpha}_{dR}^p$  has constant norm almost everywhere and satisfies  $|\omega|_p = |\alpha|_p^*$ . For every  $p' \in (p, \infty]$ , we have  $\omega \in \overline{\alpha}_{dR}^{p'}$  and

$$|\alpha|_{p'}^* \leq |\omega|_{p'} = |\omega|_p = |\alpha|_p^* \leq |\alpha|_{p'}^*.$$

Hence the first part of the proof shows that  $|\alpha|_q^*$  does not depend on  $q$ . Since  $|\omega|_q$  is independent of  $q$  as well, we obtain

$$|\alpha|_p^* = |\omega|_p = |\omega|_q = |\alpha|_q^*$$

for all  $q \in [1, \infty]$ . □

## 7. THE BI CONSTRUCTION ADAPTED TO CONFORMAL NORMS

We now continue with the notation of Section 4. Consider the norm  $\| \cdot \|_{\text{conf}/\varphi}$  in  $V$ . We apply Lemma 4.2 to the norm  $\| \cdot \|_{\text{conf}/\varphi}$ . This yields a decomposition

$$\| \cdot \|_E^2 = \sum_{i=1}^N \lambda_i L_i^2,$$

where  $\lambda_i > 0$ ,  $\sum \lambda_i = n$ ,  $L_i \in V^*$  and  $\|L_i\|_{\text{conf}/\varphi}^* = 1$ . Then a linear map  $L : V \rightarrow \mathbb{R}^N$  defined by

$$L(x) = (\lambda_1^{1/2} L_1(x), \lambda_2^{1/2} L_2(x), \dots, \lambda_N^{1/2} L_N(x)) \quad (7.1)$$

is an isometry from  $(V, \| \cdot \|_E)$  onto a subspace  $L(V)$  of  $\mathbb{R}^N$ , equipped with the restriction of the standard coordinate metric of  $\mathbb{R}^N$ .

**Proposition 7.1.** *For every  $L_i$  there exists a closed 1-form  $\omega_i$  on  $X$  from the pullback cohomology class  $\omega_i \in \varphi^*(L_i)$  and*

$$\|\omega_i\|_n = \|L_i\|_{\text{conf}/\varphi}^* = 1.$$

*Proof.* The proof results from Proposition 5.3. Briefly, the issue is that the minimizing form  $\omega_i$  will *a priori* only be an  $L^n$ -form, *i.e.* the  $L^n$ -limit of a sequence of smooth forms in the pullback class  $\varphi^*(L_i)$ . Then, it is a non-trivial result from nonlinear PDE that  $\omega_i$  is Hölder. This can be found in [To84], and in [Ha92] for the case of forms of arbitrary degree. □

**Definition 7.2.** Denote by  $\overline{X}$  the covering space of  $X$  defined by the subgroup  $\ker(\varphi_*) \subset \pi_1(X)$ . We denote the action of  $\Gamma = \pi_1(X)/\ker(\varphi_*)$  on  $\overline{X}$  by  $(v, x) \mapsto x + v$ .

**Corollary 7.3.** *There exists a function*

$$f_i : \overline{X} \rightarrow \mathbb{R} \text{ such that } df_i = \tilde{\omega}_i, \quad (7.2)$$

where  $\tilde{\omega}_i$  is the lift of  $\omega_i$ . Furthermore,  $f_i$  satisfies the relation

$$f_i(x + v) = f_i(x) + L_i(v) \quad (7.3)$$

for all  $x \in \overline{X}$  and  $v \in \Gamma$ .

*Proof.* The proof of Proposition 7.1 implies the existence of such  $f_i$  with Hölder continuous first derivatives.  $\square$

Now we use the functions  $f_i : \overline{X} \rightarrow \mathbb{R}$  of formula (7.2) to define a map  $F : \overline{X} \rightarrow \mathbb{R}^N$  as follows:

$$F(x) = (\lambda_1^{1/2} f_1(x), \lambda_2^{1/2} f_2(x), \dots, \lambda_N^{1/2} f_N(x)). \quad (7.4)$$

Observe that both the map  $L$  of (7.1) and  $F$  of (7.4) are  $\Gamma$ -equivariant with respect to the following action of  $\Gamma$  on  $\mathbb{R}^N$ :

$$\Gamma \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad (v, x) \mapsto x + L(v).$$

Now let  $\text{Pr}_{L(V)} : \mathbb{R}^N \rightarrow L(V)$  be the orthogonal projection to the image  $L(V)$ . Then the composition

$$L^{-1} \circ \text{Pr}_{L(V)} \circ F : \overline{X} \rightarrow V \quad (7.5)$$

is a  $\Gamma$ -equivariant map covering a map  $f : X \rightarrow V/\Gamma$ . We will exploit it in Section 8.

## 8. PROOF OF PROPOSITION 4.3

The proof is similar to the one in [IK04], with some crucial differences. Therefore we present the modified proof here. The main difference is that the functions  $f_i$  in formula (7.2) were defined differently in [IK04]. Consider the norm  $\|\cdot\|_{\text{conf}/\varphi}$ . We apply the construction of Section 7 to this norm. Since the projection is nonexpanding and the map  $L$  is an isometry, it suffices to prove the proposition for the map  $F$  of formula (7.4).

Let  $A = dF : T_x X \rightarrow \mathbb{R}^N$ . Then

$$\text{trace}(A^* A) = \sum \lambda_i |df_i|^2.$$

By the inequality of geometric and arithmetic means, we have

$$\begin{aligned} \text{Jac}(F)(x) &= \det(A^* A)^{1/2} \\ &\leq \left(\frac{1}{n} \text{trace}(A^* A)\right)^{n/2}. \end{aligned}$$

Thus, the map  $F : \bar{X} \rightarrow \mathbb{R}^N$  satisfies

$$\text{Jac}(F) \leq \left( \frac{1}{n} \sum \lambda_i |df_i|^2 \right)^{n/2}. \quad (8.1)$$

Therefore,

$$\begin{aligned} \text{Jac}(F) &\leq \left( \sum \frac{\lambda_i}{n} |\tilde{\omega}_i|^2 \right)^{n/2} \\ &\leq \sum \frac{\lambda_i}{n} |\tilde{\omega}_i|^n. \end{aligned}$$

The last inequality follows from Jensen's inequality applied to the function  $t \mapsto t^{n/2}$ , when  $n \geq 2$ . In the case  $n = 1$ , there is only one term in the ellipsoid decomposition, so Jensen is not needed.

Integrating over  $X$ , and taking into account the fact that  $\int_X |\omega_i|^n = 1$ , yields the desired inequality for the area of the corresponding map  $X \rightarrow V/\Gamma$ .

## 9. PROOF OF THEOREM 1.3 AND THEOREM 1.4

*Proof of Theorem 1.3.* Note that equality in (8.1) is possible only if  $F$  is conformal. The discussion of equality in (8.1) leads to the fact that equality in (1.2) implies that the map  $f$  is conformal at all points where the differential does not vanish. If  $n \geq 3$ , then by [Fer77], the map  $f$  is indeed a smooth local (conformal) diffeomorphism, hence a covering. Here we are assuming that the metric on  $X$  is smooth. Hence  $X$  is indeed a torus and  $f$  is a conformal diffeomorphism.

If  $n = 2$ , then we can lift  $f$  to a holomorphic map between the universal covers. Since  $f$  is not constant, we see that both have to be the complex plane. Since the Klein bottle does not qualify as  $X$ , we see that  $X$  is a 2-torus, while  $f$  is a nonconstant holomorphic map between 2-tori. So its lift to the universal covers is affine, and hence  $f$  is a covering in this case, as well.  $\square$

*Proof of Theorem 1.4.* We apply Proposition 6.1 together with Theorem 1.2 (which is immediate from Proposition 4.3) and an argument based on the perfection of a critical lattice similar to the proof of [BK03, Proposition 10.5]. In more detail, let us normalize the metric to unit volume to fix ideas. The equality of the stable norm and the conformally invariant norm for each of the integral elements realizing  $\lambda_1(H_1(X, \mathbb{Z})_{\mathbb{R}}, \|\cdot\|)$  proves that the harmonic 1-form in each of those integral classes has pointwise constant norm. Next, perfection implies that in fact *every* harmonic 1-form on  $X$  has pointwise constant norm. It follows that the standard Abel-Jacobi map defined by those forms, is a Riemannian submersion onto a flat torus, without any ramification.

Since the induced homomorphism of fundamental groups is surjective, the degree of the submersion is 1, proving the theorem.  $\square$

### 10. CASE $n \geq b$ AND $L^p$ NORMS IN HOMOLOGY

In this section, we generalize the techniques of Section 7.

**Definition 10.1.** Let  $X^n$  and  $Y^b$  be Riemannian manifolds,  $n \geq b$ , and  $f: X \rightarrow Y$  a smooth map. For every  $x \in X$ , we consider the derivative  $d_x f$ , and define

$$\text{Jac}^\perp f(x) = \sup_{\sigma} \frac{\text{vol}_b(d_x f(\sigma))}{\text{vol}_b(\sigma)}$$

where  $\sigma$  ranges over all  $b$ -dimensional parallelotopes in  $T_x X$ .

Obviously, if  $x$  is a regular point of  $f$ , then the above supremum is attained at  $\sigma$  lying in the orthogonal complement of the fiber  $f^{-1}(f(x))$ , and  $\text{Jac}^\perp f(x) = 0$  if  $x$  is not a regular point. We will use the following coarea formula. Every smooth map  $f: X^n \rightarrow Y^b$  satisfies the identity

$$\int_X \text{Jac}^\perp f(x) d \text{vol}_n(x) = \int_Y \text{vol}_{n-b}(f^{-1}(y)) d \text{vol}_b(y). \quad (10.1)$$

We present a version of Proposition 4.1 and Proposition 4.3, which allows us to prove inequality (3.5) for the conformal systole, as well as to handle the case of equality when  $n \geq b$ .

**Proposition 10.2.** *Let  $X^n$  be a closed Riemannian manifold of unit volume, and  $V^b$  a vector space,  $n \geq b$ . Let  $p > \max\{b, 2\}$ . Let  $\Gamma$  be lattice in  $V$ , and let  $\varphi: X \rightarrow V/\Gamma$  be a continuous map inducing an epimorphism of the fundamental groups. Consider the associated  $L^p$ -norm  $\|\cdot\|_p$  on  $V$ , cf. Definition 3.3. Denote by  $\|\cdot\|_E$  the Euclidean norm on  $V$ , defined by the John ellipsoid of  $\|\cdot\|_p$ . Then the homotopy class of  $\varphi$  contains a smooth map  $f: X \rightarrow (V/\Gamma, \|\cdot\|_E)$  such that*

$$\int_X \text{Jac}^\perp f \leq 1, \quad (10.2)$$

*and, furthermore, if the equality is attained, then  $f$  is a harmonic Riemannian submersion.*

In section 11, we will generalize the BI construction of Section 7, and use it in Section 12 to prove the proposition.

11. THE BI CONSTRUCTION IN THE CASE  $n \geq b$ 

Applying Lemma 4.2 to the norm  $\|\cdot\|_p$  yields a decomposition

$$\|\cdot\|_E^2 = \sum_{i=1}^N \lambda_i L_i^2,$$

where  $\lambda_i > 0$ ,  $\sum \lambda_i = b$ ,  $L_i \in V^*$  and  $\|L_i\|_p^* = 1$ . Then a linear map  $L : V \rightarrow \mathbb{R}^N$  defined by

$$L(x) = (\lambda_1^{1/2} L_1(x), \lambda_2^{1/2} L_2(x), \dots, \lambda_N^{1/2} L_N(x))$$

is an isometry from  $(V, \|\cdot\|_E)$  onto a subspace  $L(V)$  of  $\mathbb{R}^N$ , equipped with the restriction of the standard coordinate metric of  $\mathbb{R}^N$ .

We identify the dual space  $V^*$  of  $V$  with  $H^1(V/\Gamma; \mathbb{R})$ . Then, we have a pull-back map  $\varphi^* : V^* \rightarrow H^1(X; \mathbb{R})$ . For every  $\alpha \in V^*$ , we have

$$\|\alpha\|_p^* = \|\varphi^*(\alpha)\|_p^* = \inf \{ |\omega|_p \mid \omega \in \varphi^*(\alpha)_{\text{dR}} \}.$$

**Definition 11.1.** Fix a real  $p > \max\{b, 2\}$  and let  $\omega_i$ ,  $i = 1, \dots, N$  denote the  $L^p$ -minimizer in the cohomology class  $\varphi^*(L_i) \in H^1(X; \mathbb{R})$ .

Since  $\text{vol}(X) = 1$ , we have  $\omega_i = \|L_i\|_p^* = 1$ . The rest of the construction is the same as in Section 7. In particular, we define a map

$$F(x) = (\lambda_1^{1/2} f_1(x), \lambda_2^{1/2} f_2(x), \dots, \lambda_N^{1/2} f_N(x)). \quad (11.1)$$

In conclusion, we obtain a map  $\tilde{f} : \bar{X} \rightarrow V$  defined by

$$\tilde{f} = L^{-1} \circ \text{Pr}_{L(V)} \circ F \quad (11.2)$$

where  $\text{Pr}_{L(V)} : \mathbb{R}^N \rightarrow L(V)$  is the orthogonal projection to  $L(V)$ . Clearly  $\tilde{f}$  is  $\Gamma$ -equivariant, hence it is a lift of a map  $f : X \rightarrow V/\Gamma$ . We show in Section 12 that  $f$  satisfies inequality (10.2).

## 12. PROOF OF PROPOSITION 10.2

Consider the map  $f$  constructed in Section 11. Let  $x \in X$ . If  $x$  is a regular point of  $f$ , we let  $\Sigma_x = (\ker d_x f)^\perp$ . Otherwise let  $\Sigma_x$  be an arbitrary  $d$ -dimensional linear subspace of  $T_x X$ . Then

$$\text{Jac}^\perp f(x) = \text{Jac}((d_x f)|_{\Sigma_x}) \leq \text{Jac}((d_x F)|_{\Sigma_x}) \quad (12.1)$$

since the projection in (11.2) is nonexpanding and  $L$  is an isometry. Let  $A$  be the linear map

$$A = (d_x F)|_{\Sigma_x} : \Sigma_x \rightarrow \mathbb{R}^N. \quad (12.2)$$

Then, by (11.1),

$$\text{trace}(A^* A) = \sum \lambda_i |(d_x f_i)|_{\Sigma_x}|^2 \leq \sum \lambda_i |d_x f_i|^2 = \sum \lambda_i |\omega_i(x)|^2$$



By the inequality of geometric and arithmetic means, we have

$$\text{Jac}((d_x F)|_{\Sigma_x}) = \det(A^* A)^{1/2} \leq \left(\frac{1}{b} \text{trace}(A^* A)\right)^{b/2}. \quad (12.3)$$

Thus, the map  $f$  satisfies

$$\text{Jac}^\perp f \leq \left(\sum \frac{\lambda_i}{b} |\omega_i|^2\right)^{b/2} \leq \left(\sum \frac{\lambda_i}{b} |\omega_i|^p\right)^{b/p}. \quad (12.4)$$

The last inequality follows from Jensen's inequality applied to the function  $t \mapsto t^{p/2}$ ; recall that  $p > 2$  and  $\sum \lambda_i = b$ . Integrating over  $X$  yields

$$\int_X \text{Jac}^\perp f \leq \int_X \left(\sum \frac{\lambda_i}{b} |\omega_i|^p\right)^{b/p} \leq \left(\int_X \sum \frac{\lambda_i}{b} |\omega_i|^p\right)^{b/p} \quad (12.5)$$

since  $b/p < 1$  and  $\text{vol}(X) = 1$ . Since  $|\omega_i|_p \leq 1$ , we have

$$\int_X \sum \frac{\lambda_i}{b} |\omega_i|^p = \sum \frac{\lambda_i}{b} \int_X |\omega_i|^p \leq \sum \frac{\lambda_i}{b} = 1, \quad (12.6)$$

and the desired inequality (10.2) follows.

In the case of equality in (10.2), observe that all functions under integrals are continuous, hence all the inequalities throughout the argument turn to equalities.

**Lemma 12.1.** *Equality in (10.2) implies that  $|\omega_i| = 1$  everywhere.*

*Proof.* Equality in (12.4) implies that  $\text{Jac}^\perp f = 1$  everywhere. Equality in (12.3) is attained precisely when the map  $A$  of formula (12.2) is a conformal linear map. Then equality in (12.1) implies that  $(df)|_{\Sigma_x}$  is conformal, too. Thus  $f$  is a ( $C^1$ ) Riemannian submersion.

Next, equality in (12.4) implies that  $|\omega_i| = |\omega_j|$  for all  $i, j$ . The equality in (12.5) implies that the function

$$\sum_i \frac{\lambda_i}{b} |\omega_i|^p$$

is constant. These together imply that  $|\omega_i|$  is constant. (The fact that they are minimizers is not used here.)

Finally, since  $|\omega_i| = 1$  everywhere and  $\omega_i$  is an  $L^p$ -minimizer, Proposition 6.1 implies that  $\omega_i$  is harmonic for every  $i$ . Hence  $f$  is harmonic (and, in particular, smooth).  $\square$

### 13. PROOF OF PROPOSITION 3.5 AND THEOREM 3.7

*Proof of Proposition 3.5.* We apply Proposition 10.2 to the Abel-Jacobi map  $\mathcal{A} : X \rightarrow \mathbb{T}^b = H_1(X; \mathbb{R})/H_1(X; \mathbb{Z})_{\mathbb{R}}$ . We obtain a map  $f$  which is homotopic to  $\mathcal{A}$  and satisfies inequality (10.2), where  $\mathbb{T}^b$  is equipped with the flat Euclidean metric defined by the John ellipsoid of the unit

ball of the the norm  $\| \cdot \|_p$ , where the case  $p = \infty$  corresponds to the stable norm. Then, by the coarea formula (10.1),

$$\int_X \text{Jac}^\perp f(x) d \text{vol}_n(x) = \int_{\mathbb{T}^b} \text{vol}_{n-b}(f^{-1}(y)) d \text{vol}_b(y).$$

Observe that if  $y \in \mathbb{T}^b$  is a regular value for  $f$ , then

$$\text{vol}_{n-b}(f^{-1}(y)) \geq \deg(\mathcal{A}), \quad (13.1)$$

as in Definition 2.1. Hence we have

$$\deg(\mathcal{A}) \text{vol}_b(\mathbb{T}^b, \| \cdot \|_E) \leq \text{vol}_n(X).$$

by inequality (10.2). The proposition now results from the definition of the Hermite constant (1.3).  $\square$

*Proof of Theorem 3.7.* The theorem is immediate from Proposition 3.5 combined with Proposition 6.1, by the argument of the proof of Theorem 1.4 in Section 9. Alternatively, one could argue as follows. In the case of equality, the equalities in (10.2) and (13.1) are attained. This implies the inequality (3.1), and the theorem follows from Lemma 12.1 characterizing the equality case in Proposition 10.2. Note that equality in (13.1) implies that  $f^{-1}(y)$  is minimal.  $\square$

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