Chapter 6

The Radon transform

In Chapter 3 we introduced the Radon transform and discussed its simpler features. After reviewing its definition we obtain several properties of the Radon transform analogous to properties of the Fourier transform. We next prove the Central Slice Theorem which establishes a close connection between the Fourier transform and the Radon transform. Using it, the inversion formula for the Radon transform is deduced from the inversion formula for the Fourier transform. In imaging this formula is known as the *filtered back-projection formula.* After analyzing the exact formula we consider methods for approximately inverting the Radon transform which are relevant in medical imaging.

6.1 The Radon transform

See: A.2.3, A.6.

In section 1.2.1 we identified $\mathbb{R} \times S^1$ with the space of oriented lines in $\mathbb{R}^2$. The pair $(t, \boldsymbol{\omega})$ corresponds to the line

$$l_{t,\boldsymbol{\omega}} = \{\mathbf{x} \,:\, \langle \boldsymbol{\omega}, \mathbf{x} \rangle = t\} = \{t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}} \,:\, s \in \mathbb{R}\}.$$

Here $\hat{\boldsymbol{\omega}}$ is the unit vector perpendicular to $\boldsymbol{\omega}$ with the orientation determined by

$$\det(\boldsymbol{\omega}\hat{\boldsymbol{\omega}}) > 0.$$

The variable t is called the *affine parameter*, it is the oriented distance of the line $l_{t,\boldsymbol{\omega}}$ to the origin.

Representing the point $\boldsymbol{\omega} \in S^1$ as

$$\boldsymbol{\omega}(\theta) = (\cos\theta, \sin\theta)$$

allows an identification of $\mathbb{R} \times S^1$ with $\mathbb{R} \times [0, 2\pi)$. With this identification $d\theta$ can be used as a line element in the S^1-direction. This is often denoted by $d\boldsymbol{\omega}$ in the sequel. The integral of a function h over $S^1 \times \mathbb{R}$ is given by

$$\int_0^{2\pi} \int_{-\infty}^{\infty} h(t, \boldsymbol{\omega}(\theta)) dt d\theta,$$

which is often denoted

$$\int_0^{2\pi}\int_{-\infty}^{\infty} h(t,\boldsymbol{\omega})dtd\boldsymbol{\omega}.$$

Definition 6.1.1. The set $L^2(\mathbb{R}\times S^1)$ consists of locally integrable functions for which the square-integral,

$$\|h\|^2_{L^2(\mathbb{R}\times S^1)} = \int_0^{2\pi}\int_{-\infty}^{\infty} |h(t,\boldsymbol{\omega}(\theta))|^2 dtd\theta \tag{6.1}$$

is finite.

A function h on $\mathbb{R}\times S^1$ is continuous if $h(t,\theta) \stackrel{d}{=} h(t,\boldsymbol{\omega}(\theta))$ is 2π-periodic in θ and continuous as a function on $\mathbb{R}\times[0,2\pi]$. Similarly h is differentiable if it is 2π-periodic and differentiable on $\mathbb{R}\times[0,2\pi]$ and $\partial_\theta h$ is also 2π-periodic. Higher orders of differentiability have similar definitions.

Recall that the Radon transform of f at $(t,\boldsymbol{\omega})$ is defined by the integral

$$\mathrm{R}f(t,\boldsymbol{\omega}) = \int_{-\infty}^{\infty} f(t\boldsymbol{\omega}+s\hat{\boldsymbol{\omega}})ds.$$

For the moment we restrict our attention to piecewise continuous functions with bounded support. Because $l_{t,\boldsymbol{\omega}}$ and $l_{-t,-\boldsymbol{\omega}}$ are the same line, the Radon transform is an even function

$$\mathrm{R}f(-t,-\boldsymbol{\omega}) = \mathrm{R}f(t,\boldsymbol{\omega}). \tag{6.2}$$

The Radon transform has several properties analogous to those established for the Fourier transform in the previous chapter. Suppose that f and g are functions with bounded supported. There is a simple formula relating $\mathrm{R}(f*g)$ to $\mathrm{R}f$ and $\mathrm{R}g$.

Proposition 6.1.1. *Let f and g be piecewise continuous functions with bounded support then*

$$\mathrm{R}[f*g](t,\boldsymbol{\omega}) = \int_{-\infty}^{\infty} \mathrm{R}f(s,\boldsymbol{\omega})\,\mathrm{R}g(t-s,\boldsymbol{\omega})ds. \tag{6.3}$$

Remark 6.1.1. Colloquially one says that the Radon transform converts convolution in the plane to convolution in the affine parameter.

Proof. The proof is a calculation. Fix a direction $\boldsymbol{\omega}$, coordinates (s,t) for the plane are defined by the assignment

$$(s,t)\mapsto s\hat{\boldsymbol{\omega}}+t\boldsymbol{\omega}.$$

This is an orthogonal change of variables so the area element on $\mathbb{R}^2$ is given by $dsdt$. In these variables the convolution of f and g becomes

$$f*g(s\hat{\boldsymbol{\omega}}+t\boldsymbol{\omega}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a\hat{\boldsymbol{\omega}}+b\boldsymbol{\omega})g((s-a)\hat{\boldsymbol{\omega}}+(t-b)\boldsymbol{\omega})dadb.$$

The Radon transform of $f * g$ is computing by switching the order of the integrations:

$$\begin{aligned}
\mathrm{R}(f * g)(\tau, \boldsymbol{\omega}) &= \int_{-\infty}^{\infty} f * g(\tau\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})ds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a\hat{\boldsymbol{\omega}} + b\boldsymbol{\omega})g((s-a)\hat{\boldsymbol{\omega}} + (\tau - b)\boldsymbol{\omega})dadbds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a\hat{\boldsymbol{\omega}} + b\boldsymbol{\omega})g((s-a)\hat{\boldsymbol{\omega}} + (\tau - b)\boldsymbol{\omega})dsdadb \\
&= \int_{-\infty}^{\infty} \mathrm{R}f(b, \boldsymbol{\omega})\,\mathrm{R}g(\tau - b, \boldsymbol{\omega})db.
\end{aligned} \tag{6.4}$$

In the second to last line we interchanged the s-integration with the a and b integrations. □

Remark 6.1.2. The smoothness of a function with bounded support is reflected in the decay properties of its Fourier transform. From Proposition 6.1.1 it follows that the smoothness of a function of bounded support is also reflected in the smoothness of its Radon transform in the affine parameter. To see this suppose that f is a continuous function of bounded support and φ is a radially symmetric function, with bounded support and k-continuous derivatives. The convolution $f * \varphi$ has bounded support and k-continuous derivatives. The Radon transform of φ is only a function of t; the Radon transform of the convolution,

$$\mathrm{R}(f * \varphi)(t, \boldsymbol{\omega}) = \int_{-\infty}^{\infty} \mathrm{R}f(\tau, \boldsymbol{\omega})\,\mathrm{R}\varphi(t - \tau)d\tau,$$

has the same smoothness in t as $\mathrm{R}\varphi$. Regularity of f is also reflected in smoothness of $\mathrm{R}f$ in the angular variable, though it is more difficult to see explicitly, see exercise 6.1.6.

For $\mathbf{v}$ a vector in $\mathbb{R}^2$ the translate of f by $\mathbf{v}$ is the function $f_{\mathbf{v}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{v})$. There is a simple relation between the Radon transform of f and that of $f_{\mathbf{v}}$.

Proposition 6.1.2. *Let f be a piecewise continuous function with bounded support then*

$$\mathrm{R}f_{\mathbf{v}}(t, \boldsymbol{\omega}) = \mathrm{R}f(t - \langle \boldsymbol{\omega}, \mathbf{v}\rangle, \boldsymbol{\omega}). \tag{6.5}$$

Using this formula we can relate the Radon transform of f to that of its partial derivatives.

Lemma 6.1.1. *If f is a function with bounded support and continuous first partial derivatives, then $\mathrm{R}f(t, \boldsymbol{\omega})$ is differentiable in t and*

$$\mathrm{R}\partial_x f(t, \boldsymbol{\omega}) = -\omega_1 \partial_t \,\mathrm{R}f(t, \boldsymbol{\omega}), \quad \mathrm{R}\partial_y f(t, \boldsymbol{\omega}) = -\omega_2 \partial_t \,\mathrm{R}f(t, \boldsymbol{\omega}). \tag{6.6}$$

Proof. We consider only the x-derivative, the proof for the y-derivative is identical. Let $\mathbf{e}_1 = (1, 0)$, the x-partial derivatives of f is defined by

$$\partial_x f(\mathbf{x}) = \lim_{h \to 0} \frac{f_{h\mathbf{e}_1}(\mathbf{x}) - f(\mathbf{x})}{h}.$$

From (6.5) and the linearity of the Radon transform we conclude that

$$\mathrm{R}\left[\frac{f_{h\mathbf{e}_1} - f}{h}\right](t,\boldsymbol{\omega}) = \frac{\mathrm{R}f(t - h\omega_1,\boldsymbol{\omega}) - \mathrm{R}f(t,\boldsymbol{\omega})}{h}.$$

The lemma follows by allowing h to tend to zero. □

This result extends, by induction to higher partial derivatives.

Proposition 6.1.3. *Suppose that f has bounded support and continuous partial derivatives of order k, then $\mathrm{R}f(t,\boldsymbol{\omega})$ is k-times differentiable in t and, for non-negative integers i, j with $i + j \leq k$, we have the formula*

$$\mathrm{R}\left[\partial_x^i \partial_y^j f\right](t,\boldsymbol{\omega}) = (-1)^{i+j}\omega_1^i \omega_2^j \partial_t^{i+j}\,\mathrm{R}f(t,\boldsymbol{\omega}). \tag{6.7}$$

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be an rigid rotation of the plane, that is A is a linear map such that

$$\langle A\mathbf{v}, A\mathbf{w}\rangle = \langle \mathbf{v},\mathbf{w}\rangle \text{ for all } \mathbf{v},\mathbf{w} \in \mathbb{R}^2.$$

If f is a piecewise continuous function with bounded support then

$$f_A(\mathbf{x}) = f(A\mathbf{x})$$

is as well. The Radon transform of f_A is related to that of f in a simple way.

Proposition 6.1.4. *Let A be an rigid rotation of $\mathbb{R}^2$ and f a piecewise continuous function with bounded support. Then*

$$\mathrm{R}f_A(t,\boldsymbol{\omega}) = \mathrm{R}f(t, A\boldsymbol{\omega}). \tag{6.8}$$

Proof. The result follows from the fact that $\langle A\boldsymbol{\omega}, A\hat{\boldsymbol{\omega}}\rangle = \langle \boldsymbol{\omega},\hat{\boldsymbol{\omega}}\rangle = 0$ and therefore

$$\begin{aligned}\mathrm{R}f_A(t,\boldsymbol{\omega}) &= \int_{-\infty}^{\infty} f(tA\boldsymbol{\omega} + sA\hat{\boldsymbol{\omega}})ds \\ &= \mathrm{R}f(t, A\boldsymbol{\omega}).\end{aligned} \tag{6.9}$$

□

Thus far we have only considered the Radon transform for piecewise continuous functions with bounded supported. As discussed in Chapter 3, this transform extends to sufficiently regular functions with enough decay at infinity. A function belongs to the *natural domain* of the Radon transform if the restriction of f to every line $l_{t,\boldsymbol{\omega}}$ is an absolutely integrable function. If for example, f is a piecewise continuous function, satisfying an estimate of the form

$$|f(\mathbf{x})| \leq \frac{M}{(1 + \|\mathbf{x}\|)^{1+\epsilon}},$$

for an $\epsilon > 0$, then f belongs to the natural domain of the Radon transform. The results in this section extend to functions in the natural domain of R. The proofs in this case are left to the reader. Using functional analytic methods the domain of the Radon transform can be further extended, allowing functions with both less regularity and slower decay. An example of such an extension was already presented in section 3.4.3. We return to this in section 6.6.

Exercises

Exercise 6.1.1. Prove formula (6.5). The argument is similar to that used in the proof of (6.3).

Exercise 6.1.2. Give the details of the argument in the proof of Lemma 6.1.1 showing that $\mathrm{R}f(t,\boldsymbol{\omega})$ is differentiable in the t-variable.

Exercise 6.1.3. Show how to derive formula (6.7) from (6.6).

Exercise 6.1.4. The Laplace operator Δ is defined by $\Delta f = -(\partial_x^2 f + \partial_y^2 f)$. Find a formula for $\mathrm{R}[\Delta f]$ in terms of $\mathrm{R}f$.

Exercise 6.1.5. Suppose that $A : \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary linear transformation how is $\mathrm{R}f_A$ related to $\mathrm{R}f$?

Exercise 6.1.6. Let A_θ denote the rotation through the angle θ. Setting $\boldsymbol{\omega}(\theta) = (\cos\theta, \sin\theta)$, let $\mathrm{R}f(t,\theta) = \mathrm{R}f(t,\boldsymbol{\omega}(\theta))$ so that

$$\mathrm{R}f_{A_\phi}(t,\theta) = \mathrm{R}f(t,\theta+\phi).$$

Using these formulæ show that

$$\mathrm{R}[(y\partial_x - x\partial_y)f]\,(t,\theta) = (\partial_\theta\,\mathrm{R})f(t,\theta).$$

6.2 Inversion of the Radon Transform

Now we are ready to use the Fourier transform to invert the Radon transform.

6.2.1 The Central Slice Theorem

The Fourier transform and Radon transform are connected in a very simple way. In medical imaging this relationship is called the *Central Slice Theorem.*

Theorem 6.2.1 (Central slice theorem). *Let f be an absolutely integrable function in the natural domain of* R. *For any real number r and unit vector $\boldsymbol{\omega}$ we have the identity*

$$\int\limits_{-\infty}^{\infty} \mathrm{R}f(\boldsymbol{\omega},t)e^{-itr}dt = \hat{f}(r\boldsymbol{\omega}). \tag{6.10}$$

Proof. Using the definition of the Radon transform we comput the integral on the left:

$$\int\limits_{-\infty}^{\infty} \mathrm{R}f(\boldsymbol{\omega},t)e^{-itr}dt = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(t\boldsymbol{\omega}+s\hat{\boldsymbol{\omega}})e^{-itr}dsdt. \tag{6.11}$$

This integral is absolutely convergent and therefore we may make the change of variables, $\mathbf{x} = t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}}$. Checking that the Jacobian determinant is 1 and noting that

$$t = \langle \mathbf{x}, \boldsymbol{\omega}\rangle,$$

the above integral therefore becomes

$$\begin{aligned}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f(t\boldsymbol{\omega}+s\hat{\boldsymbol{\omega}})e^{-itr}dsdt &= \int\limits_{\mathbb{R}^2} f(\mathbf{x})e^{-i\langle\mathbf{x},\boldsymbol{\omega}\rangle r}d\mathbf{x} \\ &= \hat{f}(r\boldsymbol{\omega})\end{aligned} \tag{6.12}$$

This completes the proof of the central slice theorem. □

For a given vector $\boldsymbol{\xi} = (\xi_1, \xi_2)$ the inner product, $\langle \mathbf{x}, \boldsymbol{\xi} \rangle$ is constant along any line perpendicular to the *direction* of $\boldsymbol{\xi}$. The Central Slice Theorem interprets the computation of the Fourier transform at $\boldsymbol{\xi}$ as a two step process:

(1). First we integrate the function along lines perpendicular to $\boldsymbol{\xi}$, this gives us a function of the affine parameter alone.

(2). Compute the *1-dimensional* Fourier transform of this function of the affine parameter.

To understand this better we consider an example. Let $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ and $(t, \boldsymbol{\omega}) = (x, \mathbf{e}_1)$. Since $\hat{\mathbf{e}}_1 = \mathbf{e}_2$, the Radon transform at $(x, \mathbf{e}_1)$ is given by

$$\begin{aligned} \mathrm{R}f(x, \mathbf{e}_1) &= \int_{-\infty}^{\infty} f(x\mathbf{e}_1 + y\mathbf{e}_2) dy \\ &= \int_{-\infty}^{\infty} f(x, y) dy. \end{aligned}$$

The Fourier transform of $\mathrm{R}f(x, \mathbf{e}_1)$ is

$$\int_{-\infty}^{\infty} \mathrm{R}f(x, \mathbf{e}_1) e^{-irx} dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-irx} dy dx.$$

As $\langle r\mathbf{e}_1, (x, y) \rangle = rx$ this is the definition of $\hat{f}(r\mathbf{e}_1)$.

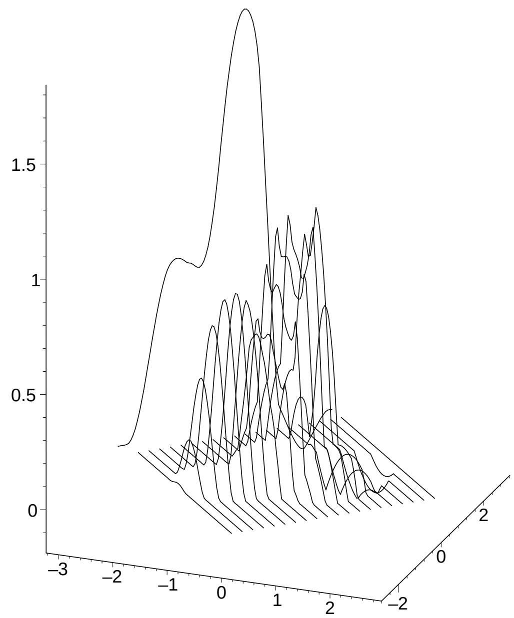
The operations in the central slice theorem are depicted in figure 6.1. On the left we have a function, f of 2-variables depicted as slices along lines in a family, $\{\langle \mathbf{x}, \boldsymbol{\omega} \rangle = t\}$. Beyond the graph of f, the integrals of these functions of a single variable are plotted. This, of course, is just the Radon transform $\mathrm{R}f(t, \boldsymbol{\omega})$. To the right and below are the real and imaginary parts of the Fourier transform, in t, of $\mathrm{R}f(t, \boldsymbol{\omega})$.

To simplify the formulæ which follow, we introduce notation for the 1-dimensional Fourier transform, in the affine parameter, of a function $h(t, \boldsymbol{\omega})$ defined on $\mathbb{R} \times S^1$:

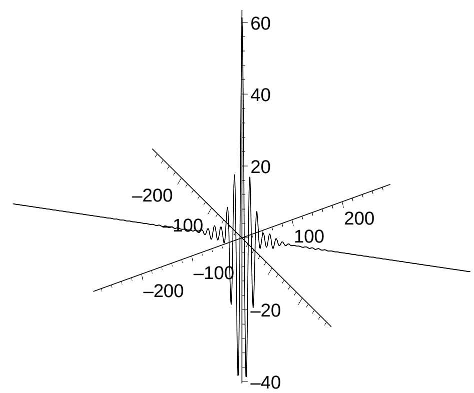
$$\widetilde{h}(r, \boldsymbol{\omega}) \stackrel{d}{=} \int_{-\infty}^{\infty} h(t, \boldsymbol{\omega}) e^{-itr} dt. \tag{6.13}$$

If $h(t, \boldsymbol{\omega})$ belongs to $L^2(\mathbb{R})$ for a fixed $\boldsymbol{\omega}$ then the one dimensional Parseval formula implies that

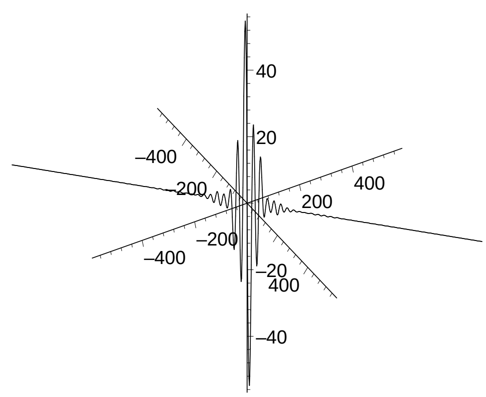
$$\int_{-\infty}^{\infty} |h(t, \boldsymbol{\omega})|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\widetilde{h}(r, \boldsymbol{\omega})|^2 dr. \tag{6.14}$$



(a) The slices of a function, f along the family of lines $\langle \mathbf{x}, \boldsymbol{\omega} \rangle = t$ and $\mathrm{R}f(t, \boldsymbol{\omega})$.



(b) The real part of $\hat{f}$ in the direction $\boldsymbol{\omega}$.



(c) The imaginary part of $\hat{f}$ in the direction $\boldsymbol{\omega}$.

Figure 6.1: According to the Central Slice Theorem the 2d-Fourier transform, $\hat{f}(r\boldsymbol{\omega})$ is the 1d-Fourier transform of $\mathrm{R}f(t, \boldsymbol{\omega})$.

The Parseval formula for the $2d$-Fourier transform and the central slice theorem give a Parseval formula for the Radon transform.

Theorem 6.2.2 (Parseval Formula for the Radon Transform). *Suppose that f is in* *the natural domain of the Radon transform and is square integrable then*

$$\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{[2\pi]^2} \int_0^{\pi} \int_{-\infty}^{\infty} |\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega}. \tag{6.15}$$

Proof. We begin by assuming that f is also absolutely integrable. The central slice theorem applies to show that

$$\begin{aligned}\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} &= \frac{1}{[2\pi]^2} \int_0^{2\pi} \int_0^{\infty} |\hat{f}(r\boldsymbol{\omega})|^2 r dr d\boldsymbol{\omega} \\ &= \frac{1}{[2\pi]^2} \int_0^{\pi} \int_{-\infty}^{\infty} |\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega}.\end{aligned} \tag{6.16}$$

In the last line we use the fact that the evenness of $\mathrm{R}f$ implies that

$$\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega}) = \widetilde{\mathrm{R}f}(-r,-\boldsymbol{\omega}). \tag{6.17}$$

This proves (6.15) with the additional assumption. To remove this assumption we need to approximate f by absolutely integrable functions. Let φ be a non-negative, infinitely differentiable, radial function with support in the disk of radius 1 with total integral one. As usual, for $\epsilon > 0$ set $\varphi_\epsilon(\mathbf{x}) = \epsilon^{-2}\varphi(\epsilon^{-1}\mathbf{x})$. A smooth function with bounded support approximating f is given by

$$f_\epsilon = \left[\chi_{[0,\epsilon^{-1}]}(r) f\right] * \varphi_\epsilon \tag{6.18}$$

For $\epsilon > 0$ these functions satisfy the hypotheses of both Theorem 6.2.2 and the central slice theorem; the argument above therefore applies to f_ϵ. The proof is completed by showing that

$$\frac{1}{[2\pi]^2} \int_0^{\pi} \int_{-\infty}^{\infty} |\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega} = \lim_{\epsilon \downarrow 0} \frac{1}{[2\pi]^2} \int_0^{\pi} \int_{-\infty}^{\infty} |\widetilde{\mathrm{R}f_\epsilon}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega}, \tag{6.19}$$

and that, as ϵ goes to 0, f_ϵ converges in $L^2(\mathbb{R}^2)$ to f. These claims are left as exercises for the reader. □

Remark 6.2.1. * Formula (6.15) has two interesting consequences for the map $f \mapsto \mathrm{R}f$ as a map between L^2-spaces. It shows that R does **not** have an extension as a continuous mapping from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R} \times S^1)$ and that R^{-1} also cannot be a continuous map from $L^2(\mathbb{R} \times S^1)$ to $L^2(\mathbb{R}^2)$. These assertions follow from Corollary A.6.1 and the observation that

$$\|h\|^2_{L^2(\mathbb{R}\times S^1)} = \frac{1}{2\pi} \int_0^{2\pi} \int_{-\infty}^{\infty} |\widetilde{h}(r,\boldsymbol{\omega})|^2 dr d\boldsymbol{\omega}.$$

Because $|r|$ varies between zero and infinity in (6.15) we see that there cannot exist constants M or M' so that either estimate,

$$\|\mathrm{R}f\|_{L^2(\mathbb{R}\times S^1)} \leq M\|f\|_{L^2(\mathbb{R}^2)} \text{ or } \|\mathrm{R}f\|_{L^2(\mathbb{R}\times S^1)} \geq M'\|f\|_{L^2(\mathbb{R}^2)}$$

is valid for f in a dense subset of $L^2(\mathbb{R}^2)$.

To express the Parseval formula as an integral over the space of oriented lines we define a "half derivative" operator

$$D_{\frac{1}{2}}\,\mathrm{R}f(t,\boldsymbol{\omega}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|r|^{\frac{1}{2}}e^{irt}dr.$$

The Parseval formula can then be re-written

$$\int\limits_{\mathbb{R}^2} |f|^2 dxdy = \frac{1}{2\pi}\int\limits_0^{\pi}\int\limits_{-\infty}^{\infty} |D_{\frac{1}{2}}\,\mathrm{R}f(t,\boldsymbol{\omega})|^2 dtd\boldsymbol{\omega}. \tag{6.20}$$

This implies that in order for a function on the space of lines to be the Radon transform of a square integrable function it must have a "half-derivative" in the affine parameter. Unlike the Fourier transform, the Radon transform is not defined on all of $L^2(\mathbb{R}^2)$.

Exercises

Exercise 6.2.1. If $f \in L^2(\mathbb{R}^2)$ and f_ϵ is defined in (6.18) show that $\underset{\epsilon\downarrow 0}{LIM} f_\epsilon = f$. Hint: Use Proposition 5.2.2 to handle the convolution. Do not forget the $\chi_{[0,\epsilon^{-1}]}$-term!

Exercise 6.2.2. If f is in the natural domain of R and f_ϵ is defined in (6.18) prove (6.19). Hint: Use Proposition 6.1.1 and the Plancherel formula.

6.2.2 The Radon Inversion Formula

The central slice theorem and the inversion formula for the Fourier transform, (4.79) give an inversion formula for the Radon transform.

Theorem 6.2.3 (Radon inversion formula). *If f is an absolutely integrable function in the natural domain of the Radon transform and $\hat{f}$ is absolutely integrable then*

$$f(\mathbf{x}) = \frac{1}{[2\pi]^2}\int\limits_0^{\pi}\int\limits_{-\infty}^{\infty} e^{ir\langle\mathbf{x},\boldsymbol{\omega}\rangle}\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|r|drd\boldsymbol{\omega} \tag{6.21}$$

Proof. Because $\mathrm{R}f$ is an even function, it follows that its Fourier transform satisfies

$$\widetilde{\mathrm{R}f}(t,\boldsymbol{\omega}) = \widetilde{\mathrm{R}f}(-t,-\boldsymbol{\omega}). \tag{6.22}$$

As f and $\hat{f}$ are absolutely integrable Theorem 4.5.1 applies to show that

$$f(\mathbf{x}) = \frac{1}{[2\pi]^2}\int\limits_{\mathbb{R}^2}\hat{f}(\boldsymbol{\xi})e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle}d\boldsymbol{\xi}.$$

Re-expressing the Fourier inversion formula using polar coordinates gives

$$\begin{aligned}
f(\mathbf{x}) &= \frac{1}{[2\pi]^2}\int\limits_{\mathbb{R}^2} e^{i\langle\mathbf{x},\boldsymbol{\xi}\rangle}\hat{f}(\boldsymbol{\xi})d\boldsymbol{\xi}\\
&= \frac{1}{[2\pi]^2}\int\limits_0^{2\pi}\int\limits_0^{\infty} e^{ir\langle\mathbf{x},\boldsymbol{\omega}\rangle}\hat{f}(r\boldsymbol{\omega})rdrd\boldsymbol{\omega}\\
&= \frac{1}{[2\pi]^2}\int\limits_0^{2\pi}\int\limits_0^{\infty} e^{ir\langle\mathbf{x},\boldsymbol{\omega}\rangle}\widetilde{\mathrm{R}f}(r\boldsymbol{\omega})rdrd\boldsymbol{\omega}
\end{aligned}$$

The central slice theorem is used in the last line. Using the relation (6.22) we can re-write this as

$$f(x,y) = \frac{1}{[2\pi]^2} \int\limits_0^{\pi} \int\limits_{-\infty}^{\infty} e^{ir\langle (x,y),\boldsymbol{\omega}\rangle} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|r| dr d\boldsymbol{\omega}. \tag{6.23}$$

□

Remark 6.2.2. As was the case with the Fourier transform, the inversion formula for the Radon transform holds under weaker hypotheses than those stated in Theorem 6.2.3. Under these hypotheses all the integrals involved are absolutely convergent and therefore do not require any further interpretation. In imaging applications the data are usually piecewise continuous, vanishing outside a bounded set. As we know from our study of the Fourier transform, this does not imply that $\hat{f}$ is absolutely integrable and so the Fourier inversion formula requires a careful interpretation in this case. Such data are square integrable and therefore it follows from the results in section 4.5.3 that

$$f = \underset{\rho\to\infty}{LIM} \frac{1}{[2\pi]^2} \int\limits_0^{\pi} \int\limits_{-\rho}^{\rho} e^{ir\langle \mathbf{x},\boldsymbol{\omega}\rangle} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|r| dr d\boldsymbol{\omega}. \tag{6.24}$$

In most cases of interest, at a point $\mathbf{x}$, where f is continuous, the integral

$$\frac{1}{[2\pi]^2} \int\limits_{-\infty}^{\infty} \int\limits_0^{\pi} e^{ir\langle \mathbf{x},\boldsymbol{\omega}\rangle} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})|r| d\boldsymbol{\omega} dr$$

exists as an improper Riemann integrals and equals $f(\mathbf{x})$. Additional care is required in manipulating these expressions.

Remark 6.2.3. Formula (6.21) allows the determination of f from its Radon transform. This formula completes a highly idealized, mathematical model for X-ray CT-imaging:

- We consider a two-dimensional slice of a three-dimensional object, the physical parameter of interest is the attenuation coefficient f of the two dimensional slice. According to Beer's law, the intensity $I_{(t,\boldsymbol{\omega})}$ of X-rays (of a given energy) traveling along a line, $l_{t,\boldsymbol{\omega}}$ is attenuated according the differential equation:

  $$\frac{dI_{(t,\boldsymbol{\omega})}}{ds} = -fI_{(t,\boldsymbol{\omega})}.$$

  Here s is arclength along the line.

- By comparing the intensity of an incident beam of X-rays to that emitted we "measure" the Radon transform of f

  $$\mathrm{R}f(t,\boldsymbol{\omega}) = -\log\left[\frac{I_{o,(t,\boldsymbol{\omega})}}{I_{i,(t,\boldsymbol{\omega})}}\right].$$

- Using formula (6.21) the attenuation coefficient f is reconstructed from the measurements $\mathrm{R}f$.

The most obvious flaw in this model is that, in practice $\mathrm{R}f(t,\boldsymbol{\omega})$ can only be measured for a finite set of pairs $(t,\boldsymbol{\omega})$. Nonetheless formula (6.21) provides a good starting point for the development of more practical algorithms.

6.2.3 Filtered Back-projection

The inversion formula for the Radon transform can be understood as a two step process:

(1). The radial integral is interpreted as a *filter* applied to the Radon transform. The filter acts only in the affine parameter, the output of the filter is denoted by

$$\mathcal{G}\,\mathrm{R}f(t,\boldsymbol{\omega}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})e^{irt}|r|dr. \tag{6.25}$$

(2). The angular integral is then interpreted as the back-projection of the *filtered* Radon transform. The function f is expressed as

$$f(x,y) = \frac{1}{2\pi}\int\limits_{0}^{\pi}(\mathcal{G}\,\mathrm{R})f(\langle(x,y),\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{6.26}$$

For this reason the Radon inversion formula is often called the *filtered back-projection formula.*

Back-projection is both conceptually and computationally simple, whereas the filtering step requires a more careful analysis. If we were to omit the $|r|$ factor then it would follow from the 1-dimensional Fourier inversion formula applied to $\widetilde{\mathrm{R}f}$ that f would be given by

$$\begin{aligned} f(\mathbf{x}) =& \frac{1}{[2\pi]^2}\int\limits_{0}^{\pi}\int\limits_{-\infty}^{\infty} e^{ir\langle\mathbf{x},\boldsymbol{\omega}\rangle}\hat{f}(r\boldsymbol{\omega})drd\boldsymbol{\omega} \\ =& \frac{1}{2\pi}\int\limits_{0}^{\pi}\mathrm{R}f(\langle\mathbf{x},\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega} \end{aligned}$$

Note that the line in the family $\{l_{t,\boldsymbol{\omega}}\,|\,t\in(-\infty,\infty)\}$ passing through the point $\mathbf{x}$ is the one with affine parameter $t=\langle\mathbf{x},\boldsymbol{\omega}\rangle$. The value at $\mathbf{x}$ obtained this way is half the average of the Radon transform of f over all lines passing through this point. This is the back-projection formula introduced in section 3.4.2. By comparison with the true inversion formula (6.21) it is now clear why the back-projection formula cannot be correct. In the true formula the low frequency components are suppressed by $|r|$ whereas the high frequency components are amplified.

The actual filter is comprised of two operations. Recall that the Fourier transform of the derivative of a function g is equal to the Fourier transform of g multiplied by $i\xi$: $\widehat{\partial_t g}(\xi)=(i\xi)\hat{g}(\xi)$. If, in the inversion formula (6.21), we had r instead of $|r|$ then the formula would give

$$\frac{1}{2\pi i}\int\limits_{0}^{\pi}\partial_t\,\mathrm{R}f(\langle(x,y),\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega};$$

This is the back-projection of the t-derivative of $\mathrm{R}f$. If f is real valued then this function is purely imaginary! Because differentiation is a *local operation* this is a relatively easy formula to understand. The subtlety in (6.21) therefore stems from the fact that $|r|$ appears and not r itself.

To account for the difference between r and $|r|$ we define another operation on functions of a single variable which is called the *Hilbert transform.* The *signum* function is defined by

$$\operatorname{sgn}(r) = \begin{cases} 1 \text{ if} & r > 0, \\ -1 \text{ if} & r \leq 0. \end{cases}$$

Definition 6.2.1. Suppose that g is an L^2-function defined on $\mathbb{R}$, the Hilbert transform of g is defined by

$$\mathcal{H}g = \mathcal{F}^{-1}(\operatorname{sgn}\hat{g}).$$

If $\hat{g}$ is also absolutely integrable then

$$\mathcal{H}g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(r)\operatorname{sgn}(r)e^{itr}dr. \tag{6.27}$$

The Hilbert transform of g is the function whose Fourier transform is $\operatorname{sgn}\hat{g}$. For any given point t_0, the computation of $\mathcal{H}g(t_0)$ requires a knowledge of $g(t)$ for *all* values of t. Unlike differentiation, the Hilbert transform is not a *local* operation. Conceptually, the Hilbert transform is the most difficult part of the Radon inversion formula. On the other hand, because the Hilbert transform has a very simple expression in terms of the Fourier transform it is easy to efficiently implement.

We compute a couple of examples of Hilbert transforms.

Example 6.2.1. Let

$$f(x) = \frac{\sin(x)}{\pi x},$$

its Fourier transform is

$$\hat{f}(\xi) = \chi_{[-1,1]}(\xi) = \begin{cases} 1 \text{ if} & |\xi| \leq 1, \\ 0 \text{ if} & |\xi| > 1. \end{cases}$$

The Hilbert transform of f is expressed as a Fourier integral by

$$\begin{aligned} \mathcal{H}\left(\frac{\sin(x)}{\pi x}\right) &= \frac{1}{2\pi}\left[\int_0^1 e^{ix\xi}dx - \int_{-1}^0 e^{ix\xi}dx\right] \\ &= i\frac{1-\cos(x)}{\pi x}. \end{aligned} \tag{6.28}$$

This pair of functions is graphed in figure 6.2(a).

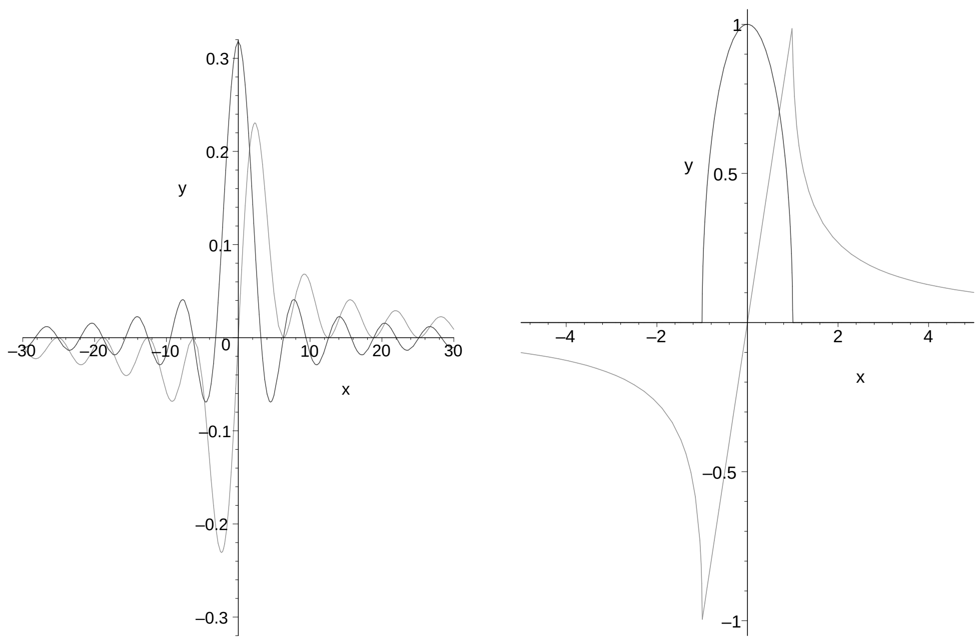
Example 6.2.2. The next example is of interest in medical imaging. It is difficult to do this example by a direct calculation. A method to do this calculation, using functions of a complex variable is explained in the final section of this chapter. Let

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

The Hilbert transform of f is given by

$$\mathcal{H}(f) = \begin{cases} ix & \text{for } |x| < 1, \\ i(x + \sqrt{x^2-1}) & \text{for } x < -1, \\ i(x - \sqrt{x^2-1}) & \text{for } x > 1. \end{cases} \tag{6.29}$$

Notice the very different character of $\mathcal{H}f(x)$ for $|x| < 1$ and $|x| > 1$. For $|x| < 1$, $\mathcal{H}f(x)$ is a smooth function with a bounded derivative. Approaching ± 1 from the set $|x| > 1$, the derivative of $\mathcal{H}f(x)$ blows up. This pair of functions is graphed in figure 6.2(b).



(a) $\frac{\sin(x)}{\pi x}$ and its Hilbert transform.

(b) $\chi_{[-1,1]}\sqrt{1-x^2}$ and its Hilbert transform.

Figure 6.2: Hilbert transform pairs.

From the differentiation formula for the Fourier transform we conclude that

$$\widetilde{\partial_t \mathrm{R} f}(r) = ir\widetilde{\mathrm{R} f}(r).$$

The Hilbert transform of $\partial_t \mathrm{R} f$ is given by

$$\begin{aligned} \mathcal{H}(\partial_t \mathrm{R} f)(t, \boldsymbol{\omega}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{\partial_t \mathrm{R} f}(r, \boldsymbol{\omega}) \operatorname{sgn}(r) e^{itr} dr \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} i|r| \widetilde{\mathrm{R} f}(r, \boldsymbol{\omega}) e^{irt} dr. \end{aligned}$$

Since $\operatorname{sgn}(r)r = |r|$ we can identify the filtration step in (6.26):

$$\mathcal{G}\,\mathrm{R}f(t,\boldsymbol{\omega}) = \frac{1}{i}\mathcal{H}(\partial_t\,\mathrm{R}f)(t,\boldsymbol{\omega}); \tag{6.30}$$

putting this into (6.26) we obtain

$$f(\mathbf{x}) = \frac{1}{2\pi i}\int_0^{\pi} \mathcal{H}(\partial_t\,\mathrm{R}f)(\langle\mathbf{x},\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{6.31}$$

The function f is reconstructed by backprojecting the Hilbert transform of $\frac{1}{i}\partial_t\,\mathrm{R}f$.

Remark 6.2.4. The Fourier transform of the function

$$F = \frac{1}{2}(f + \mathcal{H}f)$$

vanishes for $\xi < 0$ and therefore F has an analytic extension to the upper half plane, see Theorem 4.4.4. This explains why the Hilbert transform is intimately connected to the theory of analytic functions. Using the Fourier representation, it is easy to see that $\hat{F}(\xi) = \chi_{[0,\infty)}(\xi)\hat{f}(\xi)$ and therefore, if $y > 0$ then

$$F(x+iy) = \frac{1}{2\pi}\int_0^{\infty} \hat{f}(\xi)e^{-y\xi}e^{ix\xi}d\xi$$

is an absolutely convergent integral. The function $F(x)$ is the boundary value of a analytic function. A basic theorem in function theory states that an analytic function cannot vanish on an open interval, see [52]. This shows that if f has bounded support then $\mathcal{H}f$ cannot. For more on the connection between the Hilbert transform and analytic function theory see section 6.8.

This observation has important implications in image reconstruction. Formula (6.31) expresses f as the back-projection of $-i\mathcal{H}\partial_t\,\mathrm{R}f$. If f has bounded support then so does $\partial_t\,\mathrm{R}f$ and therefore $-i\mathcal{H}\partial_t\,\mathrm{R}f$ *does not*. If $\mathbf{x}$ lies outside the support of f, then this means that the integrand in (6.31) is, generally speaking, not zero. The integral vanishes due to subtle cancelations between the positive and negative parts of $-i\mathcal{H}(\partial_t\,\mathrm{R}f)(\langle\mathbf{x},\boldsymbol{\omega}\rangle,\boldsymbol{\omega})$. We return to this question in section 10.6.3.

Exercises

Exercise 6.2.3. Suppose that f is a differentiable function with bounded supported. Show that $\mathcal{H}(\partial_t f) = \partial_t(\mathcal{H}f)$.

Exercise 6.2.4. * Use the previous exercise and a limiting argument to show that $\mathcal{H}(\partial_t f) = \partial_t(\mathcal{H}f)$ functions in $L^2(\mathbb{R})$ which have an L^2-derivative.

Exercise 6.2.5. * Use the Schwarz reflection principle to prove the statement that if $F(x+iy)$ is an analytic function in $y > 0$ such that, for $a < x < b$,

$$\lim_{y\downarrow 0} F(x+iy) = 0$$

then $F \equiv 0$.

6.2.4 Inverting the Radon transform, two examples

Before continuing our analysis of R^{-1} we compute the inverse of the Radon transform in two examples.

Example 6.2.3. In the first example $f = \chi_{D_1}$, characteristic function on the unit disk. Using the rotational symmetry, we check that

$$\mathrm{R}f(t,\boldsymbol{\omega}) = \begin{cases} 2\sqrt{1-t^2} & |t| \leq 1 \\ 0 & |t| > 1. \end{cases} \tag{6.32}$$

Note that $\mathrm{R}f$ satisfies

$$\limsup_{h,t} \left| \frac{\mathrm{R}f(t+h) - \mathrm{R}f(t)}{\sqrt{|h|}} \right| < \infty.$$

In other words $\mathrm{R}f$ is a Hölder-$\frac{1}{2}$ function of t.

To apply the filtered back-projection formula we need to compute either $\partial_t \mathcal{H}\,\mathrm{R}f$ or $\mathcal{H}\partial_t \mathrm{R}f$. It is instructive to do both. In section 6.8 it is shown that

$$\frac{1}{i}\mathcal{H}\,\mathrm{R}f(t,\boldsymbol{\omega}) = \begin{cases} 2t & \text{for } |t| < 1, \\ 2(t+\sqrt{t^2-1}) & \text{for } t < -1, \\ 2(t-\sqrt{t^2-1}) & \text{for } t > 1. \end{cases} \tag{6.33}$$

Even though this function is not differentiable at $t = \pm 1$, it does have an absolutely integrable, weak derivative given by

$$\frac{1}{i}\partial_t \mathcal{H}\,\mathrm{R}f(t,\boldsymbol{\omega}) = \begin{cases} 2 - \frac{2|t|}{\sqrt{t^2-1}} & \text{for } |t| \geq 1 \\ 2 & \text{for } |t| < 1. \end{cases} \tag{6.34}$$

On the other hand we could first compute the weak derivative of $\mathrm{R}f$:

$$\partial_t \mathrm{R}f(t,\boldsymbol{\omega}) = \begin{cases} \frac{-2t}{\sqrt{1-t^2}} & |t| < 1 \\ 0 & |t| > 1. \end{cases}$$

Unfortunately this function does not belong to $L^2(\mathbb{R})$. Thus far we have only defined the Hilbert transform for L^2-functions. It is also possible to define the Hilbert transform of a function in $L^p(\mathbb{R})$ for any $1 < p \leq 2$, see [40]. As

$$\int |\partial_t \mathrm{R}f|^p < \infty \text{ for } p < 2$$

the Hilbert transform of $\partial_t \mathrm{R}f$ is still defined and can be computed using the complex variable method described in section 6.8. It is given by the formula

$$\frac{1}{i}\mathcal{H}(\partial_t \mathrm{R}f)(t) = \begin{cases} 2 - \frac{2|t|}{\sqrt{t^2-1}} & \text{for } |t| \geq 1 \\ 2 & \text{for } |t| < 1. \end{cases} \tag{6.35}$$

Now we do the back-projection step. If $\mathbf{x}$ is inside the unit disc then

$$|\langle \mathbf{x}, \boldsymbol{\omega} \rangle| \leq 1.$$

At such points, the inverse of the Radon transform is quite easy to compute:

$$\frac{1}{2\pi i}\int_0^{\pi} \mathcal{H}(\partial_t \mathrm{R} f)(\langle \mathbf{x}, \boldsymbol{\omega}\rangle, \boldsymbol{\omega}) d\boldsymbol{\omega} = \frac{1}{2\pi}\int_0^{\pi} 2 d\boldsymbol{\omega} = 1.$$

This is precisely the value of f for $\|\mathbf{x}\| \leq 1$. On the other hand, if $\|\mathbf{x}\| > 1$, then the needed calculation is more complicated. Since f is radially symmetric it suffices to consider $f(x,0)$. If $x > 1$ then there is an angle $0 < \theta_x < \frac{\pi}{2}$ so that $x \cos\theta_x = 1$, the inversion formula can be written

$$f(x,0) = \frac{1}{2\pi}\left[4\int_0^{\theta_x} d\theta - 2\int_{\theta_x}^{\pi-\theta_x}\left(1 - \frac{|x\cos\theta|}{\sqrt{x^2\cos^2\theta - 1}}\right) d\theta\right].$$

This is a much more complicated formula. From the point of view of computation it is notable that the Radon inversion formula now involves an *divergent* integrand. It is of course absolutely integrable, but this divergence leads to significant *numerical* difficulties.

The important lesson of this example is the qualitative difference in the filtered back-projection formula between points inside and outside the unit disk. This fact has significant consequences in medical imaging, see section 10.6.3.

Example 6.2.4. Our next example is a bit smoother than the characteristic function of the disk. Let $r = \sqrt{x^2 + y^2}$ and define g by

$$g(x,y) = \begin{cases} 1 - r^2 & |r| < 1, \\ 0 & |r| \geq 1. \end{cases}$$

Again using the rotational symmetry, we obtain

$$\mathrm{R} g(t, \boldsymbol{\omega}) = \begin{cases} \frac{4}{3}(1 - t^2)^{3/2} & |t| \leq 1, \\ 0 & |t| > 1. \end{cases}$$

This function $\mathrm{R}g$ is classically differentiable, the derivative of $\mathrm{R}g$ is

$$\partial_t \mathrm{R} g(t, \boldsymbol{\omega}) = \begin{cases} -4t(1 - t^2)^{1/2} & |t| \leq 1, \\ 0 & |t| > 1. \end{cases}$$

It satisfies

$$\limsup_{h,t}\left|\frac{\partial_t \mathrm{R} g(t+h) - \partial_t \mathrm{R} g(t)}{|h|^{\frac{1}{2}}}\right| < \infty.$$

This time $\partial_t \mathrm{R} g$ is a Hölder-$\frac{1}{2}$ function. This is a “half” a derivative smoother than g itself. It is a general fact that the Radon transform has better regularity in the affine parameter than the original function by half a derivative. The Hilbert transform of $\partial_t \mathrm{R} g$ is

$$\frac{1}{i}\mathcal{H}(\partial_t \mathrm{R} g)(t) = \begin{cases} 2 - 4t^2 & |t| \leq 1, \\ 4[\ 4|t|(t^2 - 1)^{1/2} - (2t^2 - 1)\ ] & |t| > 1. \end{cases}$$

Once again we see that the back-projection formula for points inside the unit disk is, numerically a bit simpler than for points outside. While $\sqrt{t^2 - 1}$ is continuous, it is not differentiable at $t = \pm 1$. This makes the numerical integration in the back-projection step more difficult for points outside the disk.

Exercises

Exercise 6.2.6. Prove that (6.34) gives the weak derivative of $\mathcal{H}\mathrm{R}f$ defined in (6.33).

Exercise 6.2.7. Use Simpson's rule to numerically integrate $\sqrt{1-t^2}$ from 0 to 1. Determine how the accuracy of the result depends on the mesh size and compare it to the accuracy when instead, $1-t^2$ is integrated.

Exercise 6.2.8. * Give an algorithm to numerically integrate the function $\frac{1}{\sqrt{1-t^2}}$ from -1 to 1. Provide an estimate for the accuracy of your method.

Exercise 6.2.9. * Generalize the method in the previous exercise to functions of the form $\frac{f}{\sqrt{1-t^2}}$ where f is differentiable on an interval containing $[-1,1]$.

6.2.5 Back-projection*

See: A.2.5.

The operation of back-projection has a nice mathematical interpretation. If $(X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \langle\cdot,\cdot\rangle_Y)$ are inner product spaces and $A : X \to Y$ is a linear map recall that the *adjoint* of A, $A^* : Y \to X$ is defined by the relations

$$\langle A\mathbf{x}, \mathbf{y}\rangle_Y = \langle \mathbf{x}, A^*\mathbf{y}\rangle_Y \text{ for all } \mathbf{x} \in X \text{ and } \mathbf{y} \in Y.$$

If we use the L^2-inner product for functions on $\mathbb{R}^2$ and the inner product for functions on $\mathbb{R} \times S^1$ compatible with the L^2-norm defined in (6.1),

$$\langle h, k\rangle_{\mathbb{R}\times S^1} = \int_0^{2\pi}\int_{-\infty}^{\infty} h(t,\boldsymbol{\omega})k(t,\boldsymbol{\omega})dtd\boldsymbol{\omega}$$

then back-projection is $[4\pi]^{-1}$ times the formal adjoint of the Radon transform. It is only a formal adjoint because, as noted above, the Radon transform does not extend to define a continuous map from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R} \times S^1)$. The proof is a simple calculation; for the sake of simplicity assume that f is a function of bounded support on $\mathbb{R}^2$ and h is a function of bounded support on $\mathbb{R} \times S^1$:

$$\begin{aligned}\langle \mathrm{R}f, h\rangle_{\mathbb{R}\times S^1} &= \int_0^{2\pi}\int_{-\infty}^{\infty} \mathrm{R}f(t,\boldsymbol{\omega})h(t,\boldsymbol{\omega})dtd\boldsymbol{\omega} \\ &= \int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t\boldsymbol{\omega}+s\hat{\boldsymbol{\omega}})h(t,\boldsymbol{\omega})dsdtd\boldsymbol{\omega}\end{aligned} \tag{6.36}$$

Let $\mathbf{x} = t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}}$ so that

$$t = \langle \mathbf{x}, \boldsymbol{\omega}\rangle,$$

interchanging the $\boldsymbol{\omega}$- and the $\mathbf{x}$-integrals we obtain

$$\begin{aligned}\langle \mathrm{R}f, h\rangle_{\mathbb{R}\times S^1} &= \int\limits_{\mathbb{R}^2}\int\limits_0^{2\pi} f(\mathbf{x})h(\langle \mathbf{x},\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega}d\mathbf{x} \\ &= \langle f, \mathrm{R}^* h\rangle_{\mathbb{R}^2}.\end{aligned} \tag{6.37}$$

This verifies the assertion that back-projection is $[4\pi]^{-1}$ times the formal adjoint of the Radon transform. The fact that $\mathrm{R}^* \neq \mathrm{R}^{-1}$ is reflection of the fact that R is not a unitary transformation from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}\times S^1)$.

Using the identification of back-projection with the adjoint, along with the Parseval formula, (4.5.2) we can derive an interesting relationship between $\widehat{\mathrm{R}^*\mathrm{R}f}$ and $\hat{f}$.

Proposition 6.2.1. *Suppose that f is an absolutely integrable and square integrable function in the natural domain of the Radon transform then*

$$\frac{r}{4\pi}\widehat{\mathrm{R}^*\mathrm{R}f}(r\boldsymbol{\omega}) = \hat{f}(r\boldsymbol{\omega}). \tag{6.38}$$

Proof. The proof of this proposition uses the basic principle that, in an inner product space, $(X, \langle\cdot,\cdot\rangle_X)$, an element $\mathbf{x}$ is zero if and only if $\langle \mathbf{x},\mathbf{y}\rangle = 0$ for all $\mathbf{y}$ belonging to a dense subset of X. Let f and g be two functions satisfying the hypotheses of the proposition. From the definition of the adjoint it follows that

$$\langle \mathrm{R}f, \mathrm{R}g\rangle_{\mathbb{R}\times S^1} = \langle f, \mathrm{R}^*\mathrm{R}g\rangle_{\mathbb{R}^2}. \tag{6.39}$$

Using the Parseval formula we get the relations

$$\begin{aligned}\langle f, \mathrm{R}^*\mathrm{R}g\rangle_{\mathbb{R}^2} &= \frac{1}{[2\pi]^2}\langle \hat{f}, \widehat{\mathrm{R}^*\mathrm{R}g}\rangle_{\mathbb{R}^2} \\ &= \frac{1}{[2\pi]^2}\int\limits_0^{2\pi}\int\limits_0^{\infty} \hat{f}(r\boldsymbol{\omega})\overline{\widehat{\mathrm{R}^*\mathrm{R}g}}(r\boldsymbol{\omega})rdrd\boldsymbol{\omega},\end{aligned} \tag{6.40}$$

and

$$\begin{aligned}\langle \mathrm{R}f, \mathrm{R}g\rangle_{\mathbb{R}\times S^1} &= \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_{-\infty}^{\infty} \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})\overline{\widetilde{\mathrm{R}g}}(r,\boldsymbol{\omega})drd\boldsymbol{\omega} \\ &= \frac{1}{\pi}\int\limits_0^{2\pi}\int\limits_0^{\infty} \hat{f}(r\boldsymbol{\omega})\overline{\hat{g}(r\boldsymbol{\omega})}drd\boldsymbol{\omega}.\end{aligned} \tag{6.41}$$

In the last line we use the central slice theorem and the evenness of the Radon transform. Since these formulæ hold for all f and g with bounded support, a dense subset of L^2, it follows that

$$\frac{r}{4\pi}\widehat{\mathrm{R}^*\mathrm{R}g}(r\boldsymbol{\omega}) = \hat{g}(r\boldsymbol{\omega}). \tag{6.42}$$

□

Proposition 6.2.1 leads to an alternate formula for R^{-1}. In this approach, the back-projection is done first. Then a filter is applied to the function $\mathrm{R}^*\mathrm{R}f$ which is defined on $\mathbb{R}^2$. If f is a piecewise continuous function of bounded support then Proposition 6.2.1 states that

$$\hat{f}(r\boldsymbol{\omega}) = \frac{r}{4\pi}\widehat{\mathrm{R}^*\mathrm{R}f}(r\boldsymbol{\omega}).$$

If $\hat{f}$ is absolutely integrable then the Fourier inversion formula therefore implies that

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{[2\pi]^2} \int_0^{2\pi} \int_0^{\infty} \frac{r}{4\pi} \widehat{\mathrm{R}^* \mathrm{R} f}(r\boldsymbol{\omega}) e^{i\langle r\boldsymbol{\omega}, \mathbf{x}\rangle} r dr d\boldsymbol{\omega} \\ &= \frac{1}{[2\pi]^2} \int_{\mathbb{R}^2} \frac{\|\boldsymbol{\xi}\|}{4\pi} \widehat{\mathrm{R}^* \mathrm{R} f}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi}, (x,y)\rangle} d\boldsymbol{\xi}. \end{aligned} \tag{6.43}$$

The Laplace operator on $\mathbb{R}^2$ is defined as the second order differential operator

$$\Delta f = -(\partial_x^2 f + \partial_y^2 f).$$

As a constant coefficient differential operator it can be expressed in terms of the Fourier transform by

$$\Delta f(\mathbf{x}) = \frac{1}{[2\pi]^2} \int_{\mathbb{R}^2} \|\boldsymbol{\xi}\|^2 \hat{f}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi}, \mathbf{x}\rangle} d\boldsymbol{\xi}.$$

This formula motivates a definition for the non-negative powers of the Laplace operator. For $s \geq 0$ and f, a smooth function with bounded support define

$$\Delta^s f(\mathbf{x}) = \frac{1}{[2\pi]^2} \int_{\mathbb{R}^2} \|\boldsymbol{\xi}\|^{2s} \hat{f}(\boldsymbol{\xi}) e^{i\langle \boldsymbol{\xi}, \mathbf{x}\rangle} d\boldsymbol{\xi}. \tag{6.44}$$

Using the Parseval formula this operation can be extended to all functions in $L^2(\mathbb{R}^2)$ such that $\|\boldsymbol{\xi}\|^s \hat{f}(\boldsymbol{\xi})$ is square integrable. With this definition for Δ^s we can re-write (6.43) as

$$4\pi f(\mathbf{x}) = [\Delta^{\frac{1}{2}} \mathrm{R}^*(\mathrm{R} f)](\mathbf{x}). \tag{6.45}$$

Remark 6.2.5. Note that $\Delta \mathrm{R}^*(\mathrm{R} f) = 4\pi \Delta^{\frac{1}{2}} f$. This gives an expression for $\Delta^{\frac{1}{2}} f$ which, given $\mathrm{R} f$, can be computed using entirely elementary operations, i.e. back-projection and differentiation. The functions f and $\Delta^{\frac{1}{2}} f$ have the same singularities. As edges are discontinuities, this formula gives a straightforward way to find the edges in an image described by a density function f. I thank Gunther Uhlmann for this observation.

Remark 6.2.6. Thus far we have produced a left inverse for the Radon transform. If f is a function in the plane satisfying appropriate regularity and decay hypotheses then, for example,

$$f = (\Delta^{\frac{1}{2}} \mathrm{R}^*) \mathrm{R} f.$$

We have **not** said that if h is an even function on $\mathbb{R} \times S^1$ then

$$h = \mathrm{R}(\Delta^{\frac{1}{2}} \mathrm{R}^*) h.$$

That is we have not shown that $(\Delta^{\frac{1}{2}} \mathrm{R}^*)$ is also a right inverse for R. Under some mild hypotheses on h this is in fact true. The proof of this statement involves characterizing the range of the Radon transform and is beyond the scope of this book. Treatments of this problem can found in [48], [44] and [50].

Exercises

Exercise 6.2.10. Let g be a continuous function with bounded support on $\mathbb{R} \times S^1$. Show that there is a constant C so that

$$|\,\mathrm{R}^* g(\mathbf{x})| \leq \frac{C}{1 + \|\mathbf{x}\|}.$$

Show that if g is a non-negative function which is not identically zero then there is also a constant $C' > 0$ so that

$$|\,\mathrm{R}^* g(\mathbf{x})| \geq \frac{C'}{1 + \|\mathbf{x}\|}.$$

Exercise 6.2.11. Explain how we arrived at the limits of integration in the second line of (6.41).

Exercise 6.2.12. Using the definition, (6.44) show that

(1). If s is a positive integer then the two definitions of Δ^s agree.

(2). For s and t non-negative numbers, show that

$$\Delta^s \Delta^t = \Delta^{s+t}. \tag{6.46}$$

(3). Conclude from the previous part that

$$\Delta\,\mathrm{R}^*\,\mathrm{R} f = \Delta^{\frac{1}{2}} f.$$

6.3 The Hilbert transform

See: A.5.6, A.6.

To implement the inversion formula for the Radon transform one needs to perform the filter operation, in this section we further analyze the Hilbert transform. As above, $\mathcal{F}^{-1}$ denotes the inverse of the Fourier transform. The Hilbert transform is defined by

$$\mathcal{H} f = \mathcal{F}^{-1}(\hat{f}(\xi)\,\mathrm{sgn}(\xi)) \Rightarrow \widehat{\mathcal{H} f}(\xi) = \mathrm{sgn}(\xi)\hat{f}(\xi).$$

In general the Fourier transform of a convolution is the product of their Fourier transforms, that is

$$\mathcal{F}^{-1}(\hat{f}\hat{g}) = f * g.$$

Hence, if there existed a nice function h such that $\hat{h}(\xi) = \mathrm{sgn}(\xi)$, then the Hilbert transform would be just $h * f$. Unfortunately the signum function is not the Fourier transform of a nice function because it does not go to zero, in any sense, as $|\xi| \to \infty$. Approximating $\mathrm{sgn}(\xi)$ by a function which decays at infinity gives approximations to the Hilbert transform expressible as convolutions with nice functions.

Modify the signum function by setting

$$\widehat{h_\epsilon}(\xi) := \mathrm{sgn}(\xi) e^{-\epsilon|\xi|} \text{ for } \epsilon > 0.$$

The inverse Fourier transform of $\widehat{h_\epsilon}$ is

$$h_\epsilon = \frac{i}{\pi}\frac{t}{t^2+\epsilon^2}.$$

This function behaves like $1/t$ as t goes to infinity which is not fast enough for integrability but at least it goes to zero and has no singularities. Most of the functions encountered in medical imaging have bounded support and therefore the integrals $h_\epsilon * f$ converge absolutely. For each $\epsilon > 0$ define an approximate Hilbert transform

$$\mathcal{H}_\epsilon f = \mathcal{F}^{-1}(\hat{f}\widehat{h_\epsilon}) = f * h_\epsilon.$$

Letting ϵ go to 0 we see that h_ϵ converges pointwise to $i[t\pi]^{-1}$. Formally this seems to imply that

$$\mathcal{H}f(t) = \frac{i}{\pi}\int\limits_{-\infty}^{\infty}\frac{f(s)ds}{t-s}.$$

Because $1/|t|$ is not integrable in any neighborhood of 0, this expression is not an absolutely convergent integral. In this instance, the correct interpretation for this formula is as a Cauchy Principal Value:

$$\mathcal{H}f(t) = \frac{i}{\pi}\,\text{P. V.}(f * \frac{1}{t}) = \frac{i}{\pi}\lim_{\epsilon\to 0}\left[\int\limits_{-\infty}^{-\epsilon} + \int\limits_{\epsilon}^{\infty}\frac{f(t-s)}{s}ds\right]. \tag{6.47}$$

This limit is finite, at least if f has bounded support and is once differentiable. Since the function $1/s$ is odd and the interval on which we are integrating is symmetric, we have

$$\left(\int\limits_{-R}^{-\epsilon} + \int\limits_{\epsilon}^{R}\right)\frac{ds}{s} = 0.$$

We can multiply this by $f(t)$ and still get zero:

$$\left(\int\limits_{-R}^{-\epsilon} + \int\limits_{\epsilon}^{R}\right)f(t)\frac{ds}{s} = 0.$$

If we assume that the support of $f(t)$ is contained in $[-\frac{R}{2}, \frac{R}{2}]$ then subtracting this from the above integral we obtain

$$\mathcal{H}f(t) = \frac{i}{\pi}\lim_{\epsilon\to 0}\left(\int\limits_{-R}^{-\epsilon} + \int\limits_{\epsilon}^{R}\right)\frac{f(t-s)-f(t)}{s}ds. \tag{6.48}$$

If f is once differentiable then the integrand in (6.48) remains bounded as ϵ goes to 0. If, for some $\alpha > 0$, f satisfies the α-Hölder-condition,

$$\frac{|f(t)-f(s)|}{|t-s|^\alpha} \le M, \tag{6.49}$$

then $\mathcal{H}f$ is given by the absolutely convergent integral

$$\mathcal{H}f(t) = \frac{i}{\pi}\int\limits_{-R}^{R}\frac{f(t-s)-f(t)}{s}ds. \tag{6.50}$$

By this process, we have replaced the limit in (6.47) by an absolutely convergent integral. The cancelation due to the symmetric interval used in the definition of the principal value is critical to obtain this result.

There are other ways to regularize convolution with $1/t$. For example, we could add an imaginary number to the denominator to make it non-vanishing,

$$\lim_{\epsilon\downarrow 0}\frac{i}{\pi}\int\limits_{-R}^{R}\frac{f(t-s)}{s\pm i\epsilon}ds.$$

A computation shows that

$$\frac{i}{\pi}\frac{1}{2}\left(\frac{1}{s+i\epsilon}+\frac{1}{s-i\epsilon}\right)=\frac{i}{\pi}\frac{s}{s^2+\epsilon^2}=h_\epsilon(s).$$

This shows that the average of the two regularizations, $(s\pm i\epsilon)^{-1}$ results in the same approximation as before. The difference of these two regularizations is

$$\frac{i}{\pi}\cdot\frac{1}{2}\left(\frac{1}{s+i\epsilon}-\frac{1}{s-i\epsilon}\right)=\frac{1}{\pi}\frac{\epsilon}{s^2+\epsilon^2}$$

which does not tend to zero as ϵ tends to zero. As an example we "test" the characteristic function of the interval $\chi_{[-1,1]}$ by evaluating the limit at $t=0$,

$$\lim_{\epsilon\downarrow 0}\int\limits_{-\infty}^{\infty}\chi_{[-1,1]}(-s)\frac{\epsilon}{s^2+\epsilon^2}ds=\lim_{\epsilon\downarrow 0}\int\limits_{-1}^{1}\frac{\epsilon}{s^2+\epsilon^2}ds=\lim_{\epsilon\downarrow 0}\int\limits_{-1/\epsilon}^{1/\epsilon}\frac{dt}{t^2+1}=\pi.$$

So we see that in general

$$\lim_{\epsilon\downarrow 0}\frac{i}{\pi}\int\limits_{-\infty}^{\infty}\frac{f(t-s)ds}{s\pm i\epsilon}\neq\mathcal{H}f(t).$$

The lesson is that care must be exercised in choosing a regularization for convolution with $1/t$. Different regularizations lead to different results.

Things are less delicate using the Fourier representation.

Theorem 6.3.1. *Suppose that $\phi_\epsilon(\xi)$ is a uniformly bounded family of functions which converges pointwise to $\operatorname{sgn}(\xi)$ as $\epsilon\to 0$. If f and $\hat{f}$ are square integrable then the Hilbert transform of f is given by the limit*

$$\mathcal{H}f(t)=\underset{\epsilon\downarrow 0}{LIM}\,\mathcal{F}^{-1}(\phi_\epsilon\hat{f}).$$

Proof. The Parseval formula shows that

$$\|\mathcal{H}f - \mathcal{F}^{-1}(\phi_\epsilon f)\|_{L^2} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |(\operatorname{sgn}(\xi) - \phi_\epsilon(\xi))\hat{f}(\xi)|^2 t d\xi.$$

As ϕ_ϵ is uniformly bounded and converges to sgn, the conclusion follows from the Lebesgue dominated convergence theorem. □

Remark 6.3.1. If f is sufficiently smooth, so that $\hat{f}$ decays then $\mathcal{H}f(t)$ is given by the pointwise limit

$$\mathcal{H}f(t) = \lim_{\epsilon\downarrow 0}\int\limits_{-\infty}^{\infty} \phi_\epsilon(\xi)\hat{f}(\xi)e^{it\xi}\frac{d\xi}{2\pi}.$$

The function $\hat{h}_\epsilon$ satisfies the hypotheses of the theorem. Another important example of a regularization is given by ϕ_ϵ defined by

$$\phi_\epsilon(\xi) = \begin{cases} -1 & -\epsilon^{-1} \leq \xi \leq 0, \\ 1 & 0 < \xi \leq \epsilon^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Computing the inverse Fourier transform of ϕ_ϵ we obtain a different sequence of kernels which approximately compute the Hilbert transform.

$$\mathcal{F}^{-1}(\phi_\epsilon) = \frac{\cos(\epsilon^{-1}x) - 1}{\pi x}.$$

Remark 6.3.2. This discussion shows that there are several different philosophies for approximating the Hilbert transform and therefore the Radon inversion formula. On the one hand we can use the convolution formula for $\mathcal{H}$ and directly approximate $\text{P. V.}(f * \frac{1}{t})$. On the other hand we can use the Fourier integral representation and instead approximate $\operatorname{sgn}(\xi)$ as described in Theorem 6.3.1. For sufficiently smooth functions with bounded support we could use (6.50). Mathematically these approaches are equivalent; computationally they can lead to vastly different results. In most real applications the Fourier representation is used because it is more efficient and does not involve regularizing a divergent expression. Instead an integral over $\mathbb{R}$ is replaced by an integral over a finite interval.

Exercises

Exercise 6.3.1. Suppose that f and g are continuous functions with bounded support. Show that

$$\mathcal{H}(f * g) = (\mathcal{H}f) * g = f * (\mathcal{H}g).$$

Exercise 6.3.2. Ordinarily one might not want to use an approximation like $\mathcal{F}^{-1}(\phi_L)$ because ϕ_L has discontinuities at $\pm L$. Why is this less of an issue for this case?

Exercise 6.3.3. Suppose that f has bounded support and satisfies an α-Hölder condition for an $0 < \alpha \leq 1$. Show that

$$\lim_{\epsilon\downarrow 0} h_\epsilon * f = \frac{i}{\pi}\text{P. V.}(f * \frac{1}{t}).$$

Exercise 6.3.4. Below are linear operators defined in terms of the Fourier transform. Re-express these operators in terms of differentiations and the Hilbert transform. For example, if Af is defined by

$$Af(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \xi\hat{f}(\xi)e^{ix\xi}d\xi$$

then the answer to this question is

$$Af(x) = -i\partial_x f(x).$$

Do not worry about convergence.

(1).

$$A_1 f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |\xi|^3\hat{f}(\xi)e^{ix\xi}d\xi$$

(2).

$$A_2 f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} (\xi^4 + |\xi| + 1)\hat{f}(\xi)e^{ix\xi}d\xi$$

(3). In this problem take note of the lower limit of integration.

$$A_3 f(x) = \frac{1}{2\pi}\int\limits_{0}^{\infty} \hat{f}(\xi)e^{ix\xi}d\xi$$

Exercise 6.3.5. If $f \in L^2(\mathbb{R})$ then show that

$$\int\limits_{-\infty}^{\infty} |f(x)|^2 dx = \int\limits_{-\infty}^{\infty} |\mathcal{H}f(x)|^2 dx.$$

Exercise 6.3.6. This exercise addresses the "spectral theory" of the Hilbert transform.

(1). Which real numbers are eigenvalues of the Hilbert transform? That is, for which real numbers λ does there exist a function f_λ in $L^2(\mathbb{R})$ so that

$$\mathcal{H}f = \lambda f?$$

Hint: Use the Fourier transform.

(2). Can you describe the eigenspaces? That is if λ is an eigenvalue of $\mathcal{H}$ describe the set of all functions in $L^2(\mathbb{R})$ which satisfy

$$\mathcal{H}f = \lambda f.$$

(3). Show that $\mathcal{H} \circ \mathcal{H}f = \mathcal{H}(\mathcal{H}(f)) = f$ for any $f \in L^2(\mathbb{R})$.

6.3.1 Mapping properties of the Hilbert transform*

See: A.5.1.

The Hilbert transform has very good mapping properties with respect to most function spaces. Using the Parseval formula one easily establishes the L^2-result.

Proposition 6.3.1. *If $f \in L^2(\mathbb{R})$ then $\mathcal{H}f \in L^2(\mathbb{R})$ and in fact*

$$\|f\|_{L^2} = \|\mathcal{H}f\|_{L^2}.$$

The Hilbert transform also has good mapping properties on other L^p-spaces as well as Hölder spaces, though the proofs of these results requires more advanced techniques.

Proposition 6.3.2. *For each $1 < p < \infty$ the Hilbert transform extends to define a bounded map $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$.*

Proposition 6.3.3. *Suppose that f is α-Hölder continuous for an $\alpha \in (0, 1)$ and vanishes outside a bounded interval then $\mathcal{H}f$ is also α-Hölder continuous.*

Notice that the case of $\alpha = 1$ is excluded in this proposition. The result is false in this case. There exist differentiable functions f such that $\mathcal{H}f$ is not even 1-Hölder continuous. Proofs of these propositions can be found in [66].

Exercise 6.3.7. By using formula (6.50), which is valid for a Hölder continuous function vanishing outside a bounded interval, prove Proposition 6.3.3.

6.4 Approximate inverses for the Radon transform

To exactly invert the Radon transform we need to compute the Hilbert transform of a derivative. The measured data is a function, g_m on the space of lines. Measured data is rarely differentiable and the exact Radon inverse entails the computation of $\partial_t g_m$. Indeed the Parseval formula, (6.15) implies that unless g_m has a half an L^2-derivative then it is not the Radon transform of an L^2-function. Thus it is important to investigate how to approximate the inverse of the Radon transform in a way that is usable with realistic data. The various approaches to approximating the Hilbert transform lead to different approaches to approximating the Radon inverse. Because the approximate inverses involve some sort of smoothing, they are often called *regularized inverses*.

Recall that a convolution has the following useful properties with respect to derivatives:

$$\partial_x(f * g) = \partial_x f * g = f * \partial_x g.$$

Using formula (6.31) we get an approximate inverse for the Radon transform

$$\begin{aligned} f(\mathbf{x}) &\approx \frac{1}{2\pi i}\int_0^\pi \mathcal{H}_\epsilon(\partial_t \mathrm{R} f)(\langle \mathbf{x}, \boldsymbol{\omega}\rangle, \boldsymbol{\omega}) d\boldsymbol{\omega} \\ &= \frac{1}{2\pi i}\int_0^\pi h_\epsilon * (\partial_t \mathrm{R} f)(\langle \mathbf{x}, \boldsymbol{\omega}\rangle, \boldsymbol{\omega}) d\boldsymbol{\omega} \end{aligned} \tag{6.51}$$

Using the formula for h_ϵ and the fact that $f * \partial_t g = \partial_t f * g$ we get

$$\begin{aligned} f(\mathbf{x}) \approx & \frac{1}{2\pi i}\int\limits_0^{\pi}\int\limits_{-\infty}^{\infty} \mathrm{R}f(s,\boldsymbol{\omega})\partial_t h_\epsilon(\langle \mathbf{x},\boldsymbol{\omega}\rangle - s)ds \\ = & \frac{1}{2\pi^2}\int\limits_0^{\pi}\int\limits_{-\infty}^{\infty}\left[\mathrm{R}f(s,\boldsymbol{\omega})\frac{\epsilon^2-(t-s)^2}{(\epsilon^2+(t-s)^2)^2}ds\bigg|_{t=\langle \mathbf{x},\boldsymbol{\omega}\rangle}\right]d\boldsymbol{\omega}. \end{aligned} \tag{6.52}$$

The expression in (6.52) has an important practical advantage: we have moved the t-derivative from the potentially noisy measurement $\mathrm{R}f$ over to the smooth, exactly known function h_ϵ. This means that we do not have to approximate the derivatives of $\mathrm{R}f$.

In most applications convolution operators, such as derivatives and the Hilbert transform are computed using the Fourier representation. Theorem 6.3.1 suggests approximating the filtering step, (6.25) in the exact inversion formula by cutting off the high frequency components. Let $\hat{\psi}(r)$ be a bounded, even function, satisfying the conditions:

$$\begin{aligned} &\hat{\psi}(0) = 1, \\ &\hat{\psi}(r) = 0 \text{ for } |r| > W. \end{aligned} \tag{6.53}$$

For l a function on $\mathbb{R}\times S^1$ define

$$\mathcal{G}_\psi(l)(t,\boldsymbol{\omega}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\tilde{l}(r,\boldsymbol{\omega})e^{irt}\hat{\psi}(r)|r|dr, \tag{6.54}$$

and

$$\mathrm{R}_\psi^{-1} l(\mathbf{x}) = \frac{1}{2\pi}\int\limits_0^{\pi}\mathcal{G}_\psi(l)(\langle \mathbf{x},\boldsymbol{\omega}\rangle,\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{6.55}$$

For notational convenience let

$$f_\psi = \mathrm{R}_\psi^{-1}\circ \mathrm{R}f.$$

How is $\mathrm{R}_\psi^{-1} f$ related to f? The answer to this question is surprisingly simple. The starting point for our analysis is Proposition 6.1.1 which says that if f and g are functions on $\mathbb{R}^2$ then

$$\mathrm{R}(f*g)(t,\boldsymbol{\omega}) = \int\limits_{-\infty}^{\infty}\mathrm{R}f(t-\tau,\boldsymbol{\omega})\,\mathrm{R}g(\tau,\boldsymbol{\omega})d\tau.$$

Using the convolution theorem for the Fourier transform we see that

$$\widetilde{\mathrm{R}f*g}(r,\boldsymbol{\omega}) = \widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})\widetilde{\mathrm{R}g}(r,\boldsymbol{\omega}).$$

Suppose now that g is a radial function so that $\mathrm{R}g$ is independent of $\boldsymbol{\omega}$. The filtered back-projection formula for $f*g$ reads

$$f*g(\mathbf{x}) = \frac{1}{4\pi^2}\int\limits_0^{\pi}\int\limits_{-\infty}^{\infty}\widetilde{\mathrm{R}f}(r,\boldsymbol{\omega})\widetilde{\mathrm{R}g}(r)e^{ir\langle \mathbf{x},\boldsymbol{\omega}\rangle}|r|drd\boldsymbol{\omega}. \tag{6.56}$$

Comparing (6.56) with the definition of f_ψ we see that, if we can find a radial function k_ψ, defined on $\mathbb{R}^2$, so that

$$\mathrm{R}(k_\psi)(t,\boldsymbol{\omega}) = \psi(t),$$

then

$$f_\psi(\mathbf{x}) = k_\psi * f(\mathbf{x}) \tag{6.57}$$

The existence of such a function is a consequence of the results in section 3.5. Because $\hat{\psi}$ has bounded support, $\psi(t)$ is an infinitely differentiable function, with all derivatives bounded. To apply Proposition 3.5.1 we need to know that $\psi(t)$ and $\psi'(t)$ are absolutely integrable. This translates into a requirement that $\hat{\psi}$ is sufficiently continuous. In this case, the function k_ψ is given by the formula

$$k_\psi(\rho) = -\frac{1}{\pi}\int_\rho^\infty \frac{\psi'(t)dt}{\sqrt{t^2-\rho^2}} \tag{6.58}$$

This completes the proof of the following proposition.

Proposition 6.4.1. *Suppose that $\hat{\psi}$ satisfies the conditions in* (6.53) *and ψ is absolutely integrable then*

$$f_\psi(\mathbf{x}) = k_\psi * f(\mathbf{x})$$

where k_ψ is given by (6.58).

Remark 6.4.1. Replacing f by f_ψ produces a somewhat blured image. Increasing the support of $\hat{\psi}$ leads, in general to a more sharply peaked ψ and therefore a more sharply peaked k_ψ. This reduces the blurring but also reduces the suppression of noise in the data. This discussion is adapted from [69].

6.4.1 Addendum*

See: A.4.6.

The analysis in the previous section is unsatisfactory in one particular: we explicitly exclude the possibility that $\hat{\psi}_W(r) = \chi_{[-W,W]}(r)$. The problem is that $\psi_W(t) = \sin(Wt)/(\pi t)$ is not absolutely integrable and so the general inversion result for radial functions does not apply. In this special case the integral defining k_ψ is a convergent, improper integral, which can be computed exactly.

We use the formula

$$\int_1^\infty \frac{\sin(xt)dt}{\sqrt{t^2-1}} = \frac{\pi}{2}J_0(x),$$

here J_0 is a Bessel function, see [49]. Putting this into the inversion formula and using the fact that $J_0' = -J_1$ we obtain

$$k_W(x) = \frac{W}{2\pi x}J_1(Wx).$$

The power series for $J_1(x)$ about $x = 0$ is

$$J_1(x) = \frac{x}{2}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{2^{2k}k!(k+1)!}$$

from which it follows easily that $k_W(x)$ is a smooth function of x^2. The standard asymptotic expansion for $J_1(x)$ as $|x|$ tends to infinity implies that

$$|k_W(x)| \leq \frac{C}{(1+|x|)^{\frac{3}{2}}}$$

and therefore the integrals defining $\mathrm{R}k_W$ converge absolutely. As the Radon transform is linear, we can extend the result of the previous section to allow functions of the form

$$\hat{\psi}(r) = \chi_{[-W,W]}(r) + \hat{\psi}_c(r)$$

where $\psi_c = \mathcal{F}^{-1}\hat{\psi}_c$ satisfies the hypotheses of Proposition 3.5.1. In this case

$$f_\psi = (k_W + k_{\psi_c}) * f. \tag{6.59}$$

Exercise

Exercise 6.4.1. Justify the computations for the function $\hat{\psi} = \chi_{[-W,W]}$ leading up to formula (6.59).

6.5 The Radon transform on data with bounded support*

In medical imaging the data under consideration usually has bounded support. The Radon transform of a function with bounded support satisfies an infinite set of *moment* conditions. From the point of view of measurements these can be viewed as consistency conditions. Mathematically this is a part of the problem of characterizing the *range* of the Radon transform on data with bounded support. The general problem of describing the range of the Radon transform is well beyond the scope of this text. The interested reader is refered to [44], [48] or [50].

Suppose that f is a function which vanishes outside the disk of radius R. As observed above this implies that $\mathrm{R}f(t,\boldsymbol{\omega}) = 0$ if $|t| > R$. For a non-negative integer, n consider the integral,

$$M_n(f)(\boldsymbol{\omega}) = \int\limits_{\mathbb{R}^2} f(\mathbf{x})[\langle \mathbf{x}, \boldsymbol{\omega}\rangle]^n d\mathbf{x}. \tag{6.60}$$

If f has bounded support, then these integrals are well defined for any $n \in \mathbb{N} \cup \{0\}$. On the other hand, if f does not vanish outside a disk of finite radius then, for sufficiently large n, these integral may not make sense.

Changing coordinates with $\mathbf{x} = t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}}$ we can rewrite this integral in terms of $\mathrm{R}f$,

$$\begin{aligned} M_n(f)(\boldsymbol{\omega}) &= \int\limits_{\mathbb{R}^2} f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})t^n ds dt \\ &= \int\limits_{-\infty}^{\infty} \mathrm{R}f(t,\boldsymbol{\omega})t^n dt. \end{aligned} \tag{6.61}$$

The function $M_n(f)(\boldsymbol{\omega})$ is called the n^{th} *moment* of the Radon transform of f. If $\mathrm{R}f(t,\boldsymbol{\omega})$ vanishes for $|t| > R$ then this integral is well defined for all n. In example 3.4.7 we showed that there are functions, which do **not** have bounded support, for which the Radon transform is defined and vanishes for large enough values of t. If f itself has bounded support then $M_n(f)(\boldsymbol{\omega})$ depends on $\boldsymbol{\omega}$ in a very special way.

It is useful to express $\boldsymbol{\omega}$ as a function of the angle θ,

$$\boldsymbol{\omega}(\theta) = (\cos(\theta), \sin(\theta)).$$

Using the binomial theorem we obtain

$$\begin{aligned}\langle \mathbf{x}, \boldsymbol{\omega}(\theta)\rangle^n &= (x\cos\theta + y\sin\theta)^n \\ &= \sum_{j=0}^{n} \binom{n}{j} (x\cos\theta)^j (y\sin\theta)^{n-j} \\ &= \sum_{j=0}^{n} \binom{n}{j} \cos^j\theta \sin^{n-j}\theta x^j y^{n-j}.\end{aligned}$$

Putting the sum into formula (6.60) we see that this integral defines a trigonometric polynomial of degree n.

$$\begin{aligned}M_n(f)(\theta) =& \sum_{j=0}^{n} \binom{n}{j} \cos^j\theta \sin^{n-j}\theta \iint_{\mathbb{R}^2} f(x,y)x^j y^{n-j} dxdy \\ =& \sum_{j=0}^{n} a_{nj} \sin^j\theta \cos^{n-j}\theta\end{aligned} \tag{6.62}$$

where

$$a_{nj} = \binom{n}{j} \iint_{\mathbb{R}^2} f(x,y)x^j y^{n-j} dxdy.$$

If f has bounded support then $M_n(f)(\theta)$ is a trigonometric polynomial of degree n. We summarize these computations in a proposition.

Proposition 6.5.1. *Suppose that f is a function with bounded support then*

(1). $\mathrm{R}f(t,\boldsymbol{\omega})$ *has bounded support.*

(2). *For all non-negative integers, n there exist constants $\{a_{n0}, \dots, a_{nn}\}$ such that*

$$\int_{-\infty}^{\infty} \mathrm{R}f(t,\boldsymbol{\omega}(\theta))t^n dt = \sum_{j=0}^{n} a_{nj} \sin^j\theta \cos^{n-j}\theta.$$

The proposition suggests the following question: Suppose that $h(t,\boldsymbol{\omega})$ is a function on $\mathbb{R} \times S^1$ such that

(1). $h(t,\boldsymbol{\omega}) = h(-t,-\boldsymbol{\omega})$,

(2). $h(t,\boldsymbol{\omega}) = 0$ if $|t| > R$, (3). For each non-negative integer n

$$m_n(h)(\theta) = \int_{-\infty}^{\infty} h(t, \boldsymbol{\omega}(\theta)) t^n dt$$

is a trigonometric polynomial of degree n,

(4). $h(t, \boldsymbol{\omega})$ is a sufficiently smooth function of $(t, \boldsymbol{\omega})$.

Does there exist a function f in the domain of the Radon transform, vanishing outside of the disk of radius R such that

$$h = Rf?$$

In other words: does h belong to the range of the Radon transform, acting on smooth functions with bounded support? According to a theorem of Helgason and Ludwig, the answer to this question turns out to be yes, however the proof of this result requires techniques beyond the scope of this text. For a detailed discussion of this question the reader is referred to [50]. More material can be found in [23], [44], [48] or [14].

We model the data measured in CT-imaging as the Radon transform of a piecewise continuous function with bounded support. If we could make measurements for all $(t, \boldsymbol{\omega})$ then it probably would not be the exact Radon transform of such a function. This is because all measurements are corrupted by errors and noise. In particular the patient's movements, both internal (breathing, heart beat, blood circulation, etc.) and external, affect the measurements. The measured data would therefore be inconsistent and may fail to satisfy the moment conditions prescribed above.

6.6 Continuity of the Radon transform and its inverse*

In order for the measurement process in X-ray tomography to be stable the map $f \mapsto \mathrm{R}f$ should be continuous in a reasonable sense. Estimates for the continuity of this map quantify the sensitivity of the output, $\mathrm{R}f$ of a CT-scanner to changes in the input. The *less* continuous the map, the *more* sensitive the measurements are to changes in the input. Estimates for the continuity of inverse, $h \mapsto \mathrm{R}^{-1}h$ quantify the effect of errors in the measured data on the quality of the reconstructed image. Because we actually *measure* the Radon transform, estimates for the continuity of R^{-1} are more important for the problem of image reconstruction. To discuss the continuity properties of either transform we need to select norms for functions in the domain and range. Using the L^2-norms on both, the Parseval formula, (6.15) provides a starting point for this discussion.

The Parseval formula says that if $f \in L^2(\mathbb{R}^2)$ then $D_{\frac{1}{2}} \mathrm{R}f \in L^2(\mathbb{R} \times S^1)$. This estimate has somewhat limited utility, as $|r|$ vanishes at $r = 0$, we cannot conclude that $\mathrm{R}f$ is actually in $L^2(\mathbb{R} \times S^1)$. In medical applications the data has bounded support and in this case additional estimates are available. The implications of the Parseval formula for the inverse transform are somewhat less desirable. It says that in order to control the L^2-norm of the reconstructed image we need to have control on the half-order derivative of the measured data. Due to noise this is, practically speaking, not possible. After discussing the continuity properties of the forward transform for data with bounded support we consider the continuity properties of the *approximate inverse* described in section 6.4.

6.6.1 Bounded support

Functions with bounded support satisfy better L^2-estimates.

Proposition 6.6.1. *Let $f \in L^2(\mathbb{R}^2)$ and suppose that f vanishes outside the disk of radius L then, for each $\boldsymbol{\omega}$, we have the estimate*

$$\int_{-\infty}^{\infty} |\mathrm{R}f(t,\boldsymbol{\omega})|^2 dt \leq 2L\|f\|_{L^2}^2.$$

Proof. The proof of the proposition is a simple application of the Cauchy-Schwarz inequality. Because f vanishes outside the disk of radius L we can express $\mathrm{R}f$ as

$$\mathrm{R}f(t,\boldsymbol{\omega}) = \int_{-L}^{L} f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})ds.$$

Computing the L^2-norm of $\mathrm{R}f$ in the t-variable we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |\mathrm{R}f(t,\boldsymbol{\omega})|^2 dt &= \int_{-L}^{L} \left| \int_{-L}^{L} f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})ds \right|^2 dt \\ &\leq 2L \int_{-L}^{L}\int_{-L}^{L} |f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})|^2 ds dt.\end{aligned} \tag{6.63}$$

In the second line we used the Cauchy-Schwarz inequality. □

The proposition shows that, if f vanishes outside a bounded set, then we control not only the overall L^2-norm of $\mathrm{R}f$ but the L^2-norm in each direction, $\boldsymbol{\omega}$ separately. Using the support properties of f more carefully gives a weighted estimate on the L^2-norm of $\mathrm{R}f$.

Proposition 6.6.2. *Let $f \in L^2(\mathbb{R}^2)$ and suppose that f vanishes outside the disk of radius L then, for each $\boldsymbol{\omega}$, we have the estimate*

$$\int_{-\infty}^{\infty} \frac{|\mathrm{R}f(t,\boldsymbol{\omega})|^2 dt}{\sqrt{L^2 - t^2}} \leq 2\|f\|_{L^2}^2.$$

Proof. To prove this estimate observe that

$$f(x,y) = \chi_{[0,L^2]}(x^2 + y^2) f(x,y).$$

The Cauchy Schwarz inequality therefore implies that, for $|t| \leq L$, we have the estimate

$$\begin{aligned}|\mathrm{R}f(t,\boldsymbol{\omega})|^2 &= \left| \int_{-L}^{L} f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}}) \chi_{[0,L^2]}(s^2 + t^2) ds \right|^2 \\ &\leq 2 \int_{-L}^{L} |f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})|^2 ds \int_{0}^{\sqrt{L^2 - t^2}} ds \\ &= 2\sqrt{L^2 - t^2} \int_{-L}^{L} |f(t\boldsymbol{\omega} + s\hat{\boldsymbol{\omega}})|^2 ds.\end{aligned} \tag{6.64}$$

Thus

$$\int_{-L}^{L} \frac{|\,\mathrm{R}f(t,\boldsymbol{\omega})|^2 dt}{\sqrt{L^2-t^2}} \leq \int_{-L}^{L} \frac{2\sqrt{L^2-t^2}}{\sqrt{L^2-t^2}} \int_{-L}^{L} |f(t\boldsymbol{\omega}+s\hat{\boldsymbol{\omega}})|^2 ds dt$$
$$= 2\|f\|_{L^2}^2. \tag{6.65}$$

□

A function in $f \in L^2(\mathbb{R}^2)$ with support in the disk of radius L can be approximated, in the L^2-norm, by a sequence of smooth functions $< f_n >$. This sequence can also be taken to have support in the disk of radius L. The Radon transforms of these functions satisfy the estimates

$$\int_{-\infty}^{\infty} |\,\mathrm{R}f_n(t,\boldsymbol{\omega})|^2 dt \leq 2L\|f_n\|_{L^2}^2$$

and

$$\frac{1}{[2\pi]^2}\int_{0}^{\pi}\int_{-\infty}^{\infty} |\widetilde{\mathrm{R}f_n}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega} = \|f_n\|_{L^2(\mathbb{R}^2)}^2.$$

In a manner analogous to that used to extend the Fourier transform to L^2-functions we can now extend the Radon transform to L^2-functions with support in a fixed bounded set.

For bounded functions on $\mathbb{R}\times S^1$ vanishing for $|t| > L$ a norm is defined by

$$\|h\|_{2,L}^2 = \sup_{\boldsymbol{\omega}\in S^1}\int_{-L}^{L} |h(t,\boldsymbol{\omega})|^2 dt + \frac{1}{[2\pi]^2}\int_{0}^{\pi}\int_{-\infty}^{\infty} |\tilde{h}(r,\boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega}.$$

The closure of $\mathcal{C}^0([-L,L]\times S^1)$ in this norm is a Hilbert space which can be identified with a subspace of $L^2([-L,L]\times S^1)$. For f as above, $\mathrm{R}f$ is defined as the limit of $\mathrm{R}f_n$ in this norm. Evidently the estimates above hold for $\mathrm{R}f$. On the other hand the elementary formula for $\mathrm{R}f(t,\boldsymbol{\omega})$ may not be meaningful as f may not be absolutely integrable over $l_{t,\boldsymbol{\omega}}$.

While it is well beyond the scope of this text, it is nonetheless, true that a function on $\mathbb{R}\times S^1$ with support in the set $|t| \leq L$ and finite $\|\cdot\|_{2,L}$-norm which satisfies the moment conditions is the generalized Radon transform of function in $L^2(\mathbb{R}^2)$ with support in the disk of radius L. A proof can be found in [23] or [50].

Exercises

Exercise 6.6.1. Suppose that $f \in L^2(\mathbb{R}^2)$ and that f vanishes outside the disk of radius L. Show that $\|\,\mathrm{R}f(\cdot,\boldsymbol{\omega}_1) - \mathrm{R}f(\cdot,\boldsymbol{\omega}_2)\|_{L^2(\mathbb{R})}$ tends to zero as $\boldsymbol{\omega}_1$ approaches $\boldsymbol{\omega}_2$. In other words the map $\boldsymbol{\omega} \mapsto \mathrm{R}f(\cdot,\boldsymbol{\omega})$ is a continuous map from the circle into $L^2(\mathbb{R})$. This shows that, if we measure errors in the L^2-norm then the Radon transform is not excessively sensitive to small changes in the measurement environment.

Exercise 6.6.2. Suppose that $< f_n >$ is a sequence of smooth functions with support in a fixed disk converging to f in $L^2(\mathbb{R}^2)$. For the terms in the approximating sequence, $< \mathrm{R}f_n >$ the moments $\{m_k(\mathrm{R}f_n)\}$ satisfy the conditions in Proposition 6.5.1. Show that for the limiting function, the moments $\{m_k(\mathrm{R}f)\}$ are well defined and also satisfy these conditions.

6.6.2 Estimates for the inverse transform*

The question of more immediate interest is the continuity properties of the *inverse* transform. This is the more important question because we actually measure an approximation, Rf_m to $\mathrm{R}f$. It would appear that to estimate the error in the reconstructed image, we would need to estimate

$$\mathrm{R}^{-1}Rf_m - f = \mathrm{R}^{-1}(Rf_m - \mathrm{R}f). \tag{6.66}$$

There are several problems that immediately arise. The most obvious problem is that Rf_m may not be in the range of the Radon transform. If $Rf_m(t, \boldsymbol{\omega})$ does not have an L^2-half-derivative in the t-direction, that is,

$$\int_0^{2\pi}\int_{-\infty}^{\infty} |\widetilde{Rf_m}(r, \boldsymbol{\omega})|^2 |r| dr d\boldsymbol{\omega} = \infty,$$

then according to the Parseval formula, (6.15) Rf_m is **not** the Radon transform of a function in $L^2(\mathbb{R}^2)$. In order to control the L^2-error,

$$\| \mathrm{R}^{-1}(Rf_m - \mathrm{R}f)\|_{L^2(\mathbb{R}^2)}$$

it is necessary that measurements have such a half derivative and the difference

$$\|D_{\frac{1}{2}}(Rf_m - \mathrm{R}f)\|_{L^2(\mathbb{R}\times S^1)}$$

is small. This means that we need to control the high frequency content of Rf_m; in practice this is not possible. While the mathematical problem of estimating the Radon inverse is quite interesting and important, it has little bearing on the problem of practical image reconstruction. A very nice treatment of the mathematical question is given in [50]. We now turn our attention to understanding the continuity of the *approximate* inverses defined in section 6.4.

An approximate inverse is denoted by R_ψ^{-1}, where ψ is a regularizing function. This is an even function whose Fourier transform satisifies the conditions

$$\begin{aligned} &\hat{\psi}(0) = 1, \\ &\hat{\psi}(r) = 0 \text{ for } |r| > W. \end{aligned} \tag{6.67}$$

It is also assumed that the radial function k_ψ defined in 6.58 is in the domain of the Radon transform and

$$\mathrm{R}k_\psi = \psi.$$

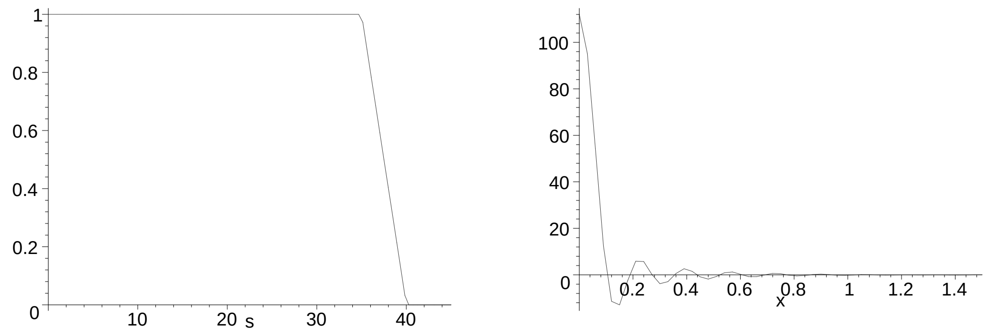
In this case

$$\mathrm{R}_\psi^{-1}\,\mathrm{R}f = k_\psi * f. \tag{6.68}$$

Example 6.6.1. Let $\hat{\psi}$ be the piecewise linear function

$$\hat{\psi}(r) = \begin{cases} 1 & \text{for } |r| < W - C, \\ \frac{W-|r|}{C} & \text{for } W - C \le |r| \le W, \\ 0 & \text{for } |r| > W. \end{cases}$$

Radial graphs of ψ and k_ψ are shown in figure 6.3.



(a) $\hat{\psi}$

(b) k_ψ

Figure 6.3: Graphs of $\hat{\psi}$ and k_ψ with $W = 40, C = 5$.

The reconstructed image is

$$f_\psi = \mathrm{R}_\psi^{-1} R f_m,$$

therefore we need to estimate the difference $f - f_\psi$. As $k_\psi * f = \mathrm{R}_\psi^{-1}\,\mathrm{R} f$ we can rewrite this difference as

$$f - f_\psi = (f - k_\psi * f) + \mathrm{R}_\psi^{-1}(\mathrm{R} f - R f_m). \tag{6.69}$$

The first term on the right hand side is the error caused by using an approximate inverse. It is present even if we have perfect data. Bounds for this term depend in an essential way on the character of the data. If f is assumed to be a continuous function of bounded support then, by taking W very large, the pointwise error,

$$\|f - k_\psi * f\|_\infty = \sup_{\mathbf{x} \in \mathbb{R}^2} |f(\mathbf{x}) - k_\psi * f(\mathbf{x})|$$

can be made as small as desired. It is more realistic to model f is a piecewise continuous function. In this case the difference, $|f(\mathbf{x}) - k_\psi * f(\mathbf{x})|$ can be made small at points where f is continuous. Near points where f has a jump the approximate reconstruction may display an oscillatory artifact. Figure 6.4 is a radial graph of the reconstruction of $\chi_{D_1}(\mathbf{x})$ using the regularizing function graphed in figure 6.3.

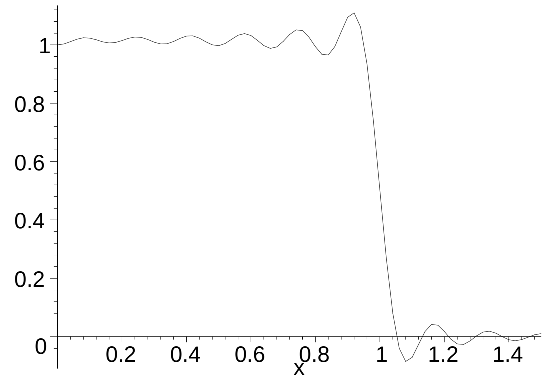


Figure 6.4: Radial graph of $k_\psi * \chi_{D_1}$, with $W = 40, C = 5$.

Robust estimates for the second term are less dependent on the precise nature of f. For h a function on $\mathbb{R} \times S^1$ with bounded support, the approximate inverse is given by

$$\begin{aligned}(\mathrm{R}_\psi^{-1} h)(\mathbf{x}) &= \frac{1}{4\pi^2} \int_0^\pi \int_{-\infty}^{\infty} \widetilde{h}(r, \boldsymbol{\omega}) e^{ir\langle \mathbf{x}, \boldsymbol{\omega}\rangle} \hat{\psi}(r)|r| dr \\ &= \frac{1}{2\pi} \int_0^\pi (g_\psi *_t h)(\langle \mathbf{x}, \boldsymbol{\omega}\rangle, \boldsymbol{\omega}) d\boldsymbol{\omega}.\end{aligned} \tag{6.70}$$

Here $g_\psi = \mathcal{F}^{-1}(\hat{\psi}(r)|r|)$ and $*_t$ indicates convolution in the t-variable.

A simple estimate for the sup-norm of $\mathrm{R}_\psi^{-1} h$ follows from the sup-norm estimate for a convolution

$$\|l * k\|_{L^\infty} \leq \|l\|_{L^\infty} \|k\|_{L^1}.$$

Applying this estimate gives

$$\| \mathrm{R}_\psi^{-1} h\|_{L^\infty} \leq \frac{\|g_\psi\|_{L^\infty}}{2\pi} \int_0^\pi \int_{-\infty}^{\infty} |h(t, \boldsymbol{\omega})| dt d\boldsymbol{\omega} \tag{6.71}$$

If $\hat{\psi}$ is non-negative then

$$|g_\psi(t)| \leq |g_\psi(0)| = \int_{-\infty}^{\infty} |r| \hat{\psi}(r) dr.$$

Assuming that $0 \leq \hat{\psi}(t) \leq M$ and that it vanishes outside the interval $[-W, W]$ leads to the estimate

$$\|g_\psi\|_{L^\infty} \leq MW^2.$$

Combining this with (6.71) gives

$$\| \mathrm{R}_\psi^{-1} h\|_{L^\infty} \leq \frac{MW^2}{2\pi} \|h\|_{L^1(\mathbb{R} \times S^1)}. \tag{6.72}$$

This estimate shows that the sup-norm of the error in the approximate reconstructed image, $\mathrm{R}_\psi^{-1}(\mathrm{R}f - Rf_m)$, can be controlled if the measurement errors can be controlled in the L^1-norm. It also shows that the error increases as W increases.

To summarize, the error in the approximate reconstruction is bounded by

$$|f - f_\psi| \leq |f - k_\psi * f| + \frac{\|g_\psi\|_{L^\infty}}{2\pi} \| \mathrm{R}f - Rf_m\|_{L^1(\mathbb{R}\times S^1)}. \tag{6.73}$$

Recall that

$$\mathcal{F}(k_\psi) = \hat{\psi} \text{ and } \mathcal{F}(g_\psi) = |r|\hat{\psi}.$$

The function k_ψ is rapidly decreasing and sharply peaked if $\hat{\psi}$ is smooth and W is taken large. On the other hand g_ψ cannot decay faster than $O(t^{-2})$. This is a consequence of the fact that $|r|\hat{\psi}(r)$ is singular at $r = 0$.

Exercises

Exercise 6.6.3. Prove that $\|l * k\|_{L^\infty} \leq \|l\|_{L^\infty}\|k\|_{L^1}$.

Exercise 6.6.4. Suppose that $\hat{\psi}(\xi)$ is a smooth function with bounded support such that $\hat{\psi}(0) \neq 0$ and let

$$g_\psi(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \hat{\psi}(\xi)|\xi|e^{it\xi}d\xi.$$

Show that there is a constant $C > 0$ so that the following *lower bound* holds for large enough t :

$$|g_\psi(t)| \geq \frac{C}{1+t^2}. \tag{6.74}$$

Exercise 6.6.5. Use the central slice theorem to give a formula for k_ψ as a Bessel transform of $\hat{\psi}(r)$.

Exercise 6.6.6. Use Hölder's inequality to show that

$$\|l * k\|_{L^\infty} \leq \|l\|_{L^2}\|k\|_{L^2}.$$

Use this estimate to prove that

$$\| \mathrm{R}_\psi^{-1} h\|_{L^\infty} \leq \frac{\|g_\psi\|_{L^2(\mathbb{R})}}{\sqrt{4\pi}}\|h\|_{L^2(\mathbb{R}\times S^1)}.$$

Under the assumptions used above to estimate $\|g_\psi\|_{L^\infty}$ show that

$$\|g_\psi\|_{L^2} \leq \sqrt{\frac{2}{3}}MW^{\frac{3}{2}}.$$

6.7 The higher-dimensional Radon transform*

See: A.2.1, A.2.5.

For the sake of completeness we briefly present the theory of the Radon transform in higher dimensions. The parameterization of the affine hyperplanes in $\mathbb{R}^n$ is quite similar to that used for lines in $\mathbb{R}^2$. Let $\boldsymbol{\omega}$ be a unit vector in $\mathbb{R}^n$, i.e. a point on S^{n-1}, and let $t \in \mathbb{R}$, each affine hyperplane has a representation in the form

$$l_{t,\boldsymbol{\omega}} = \{\mathbf{x} \in \mathbb{R}^n \;:\; \langle \mathbf{x}, \boldsymbol{\omega} \rangle = t\}.$$

As in the two-dimensional case $l_{t,\boldsymbol{\omega}} = l_{-t,-\boldsymbol{\omega}}$ and the choice of vector $\boldsymbol{\omega}$ defines an orientation on the hyperplane.

In order to define the Radon transform it is useful to choose vectors $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ so that

$$\langle \boldsymbol{\omega}, \mathbf{e}_j \rangle = 0 \text{ and } \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij} \text{ for } i, j = 1, \dots, n-1.$$

The n-vectors $\langle \boldsymbol{\omega}, \mathbf{e}_1, \dots, \mathbf{e}_{n-1} \rangle$ are an orthonormal basis for $\mathbb{R}^n$. Define new orthogonal coordinates, $(t, s_1, \dots, s_{n-1})$ on $\mathbb{R}^n$ by setting

$$\mathbf{x} = t\boldsymbol{\omega} + \sum_{j=1}^{n-1} s_j \mathbf{e}_j.$$

The n-dimensional Radon transform is defined by

$$\mathrm{R}f(t, \boldsymbol{\omega}) = \int\limits_{l_{t,\boldsymbol{\omega}}} f d\sigma_{n-1} = \int\limits_{\mathbb{R}^{n-1}} f(t\boldsymbol{\omega} + \sum s_j \mathbf{e}_j) ds_1 \cdots ds_{n-1}.$$

As before the Radon transform is an even function

$$\mathrm{R}f(t, \boldsymbol{\omega}) = \mathrm{R}f(-t, -\boldsymbol{\omega}).$$

With this definition, the n-dimensional analogue of the Central slice theorem is

Theorem 6.7.1 (Central slice theorem). *If f is an absolutely integrable function on $\mathbb{R}^n$ then*

$$\widetilde{\mathrm{R}f}(r, \boldsymbol{\omega}) = \int\limits_{-\infty}^{\infty} \mathrm{R}f(t, \boldsymbol{\omega}) e^{-irt} dt = \hat{f}(r\boldsymbol{\omega}). \tag{6.75}$$

The central slice theorem and the Fourier inversion formula give the Radon inversion formula.

Theorem 6.7.2 (The Radon Inversion Formula). *Suppose that f is a smooth function with bounded support on $\mathbb{R}^n$ then*

$$f(\mathbf{x}) = \frac{1}{2(2\pi)^n} \int\limits_{S^{n-1}} \int\limits_{-\infty}^{\infty} \widetilde{\mathrm{R}f}(r, \boldsymbol{\omega}) r^{n-1} e^{ir\langle \boldsymbol{\omega}, \mathbf{x} \rangle} dr d\boldsymbol{\omega}. \tag{6.76}$$

Remark 6.7.1. This formula holds in much greater generality. Under the hypotheses in the theorem all the integrals converge absolutely and the simplest form of the Fourier inversion formula applies.

This formula takes a very simple form if the dimension is odd, set $n = 2k + 1$. In this case the r-integral in (6.76) can be computed explicitly:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \widetilde{\mathrm{R}f}(r, \boldsymbol{\omega}) r^{n-1} e^{ir\langle \boldsymbol{\omega}, \mathbf{x}\rangle} dr = (-1)^k \partial_t^{2k} \mathrm{R}f(t, \langle \boldsymbol{\omega}, \mathbf{x}\rangle). \tag{6.77}$$

Using this expression in (6.76) we obtain

$$f(\mathbf{x}) = \frac{(-1)^k}{2(2\pi)^{2k}} \int\limits_{S^{n-1}} (\partial_t^{2k} \mathrm{R}f)(\langle \boldsymbol{\omega}, \mathbf{x}\rangle, \boldsymbol{\omega}) d\boldsymbol{\omega}.$$

Thus in odd dimensions the inverse of the Radon transform is differentiation in t followed by back-projection.

The Laplace operator on $\mathbb{R}^n$ is defined by

$$\Delta_{\mathbb{R}^n} f = \sum_{j=1}^{n} \partial_{x_j}^2 f.$$

It is invariant under rotations so it follows that, for the coordinates $(t, s_1, \ldots, s_{n-1})$ introduced above we also have the formula

$$\Delta_{\mathbb{R}^n} f = \partial_t^2 f + \sum_{j=1}^{n-1} \partial_{s_j}^2 f. \tag{6.78}$$

This formula allows us to establish a connection between $\mathrm{R}(\Delta_{\mathbb{R}^n} f)$ and $\mathrm{R}f$.

Proposition 6.7.1. *Suppose that f is a twice differentiable function of bounded support on $\mathbb{R}^n$ then*

$$\mathrm{R}(\Delta_{\mathbb{R}^n} f) = \partial_t^2 \mathrm{R}f. \tag{6.79}$$

We close our discussion by explaining how the Radon transform can be applied to solve the wave equation. Let τ denote the time variable and c the speed of sound. The "wave equation" for a function $u(\mathbf{x}; \tau)$ defined on $\mathbb{R}^n \times \mathbb{R}$ is

$$\partial_\tau^2 u = c^2 \Delta_{\mathbb{R}^n} u.$$

If u satisfies this equation then it follows from the proposition that, for each $\boldsymbol{\omega} \in S^{n-1}$, $\mathrm{R}u(t, \boldsymbol{\omega}; \tau)$ satisfies the equation

$$\partial_\tau^2 Ru = c^2 \partial_t^2 Ru.$$

Here $Ru(t, \boldsymbol{\omega}; \tau)$ is the Radon transform of u in the $\mathbf{x}$-variables with τ the time parameter. In other words, the Radon transform translates the problem of solving the wave equation in n-dimensions into the problem of solving a family of wave equations in 1-dimension.

The one-dimensional wave equation is solved by any function of the form

$$v(t; \tau) = g(ct + \tau) + h(ct - \tau).$$

The initial data is usually $v(t;0)$ and $v_\tau(t;0)$; it is related to g and h by

$$\begin{aligned} g(ct) &= \frac{1}{2}\left[v(t;0) + c\int\limits_{-\infty}^{t} v_\tau(s;0)ds\right], \\ h(ct) &= \frac{1}{2}\left[v(t;0) - c\int\limits_{-\infty}^{t} v_\tau(s;0)ds\right]. \end{aligned} \tag{6.80}$$

If $u(\mathbf{x};0) = u_0(\mathbf{x})$ and $u_\tau(\mathbf{x};0) = u_1(\mathbf{x})$ then we see that

$$\mathrm{R}u(t,\boldsymbol{\omega};\tau) = g(ct+\tau;\boldsymbol{\omega}) + h(ct-\tau;\boldsymbol{\omega})$$

where

$$\begin{aligned} g(ct;\boldsymbol{\omega}) &= \frac{1}{2}\left[\mathrm{R}u_0(t;\boldsymbol{\omega}) + c\int\limits_{-\infty}^{t} \mathrm{R}u_1(s;\boldsymbol{\omega})ds\right], \\ h(ct;\boldsymbol{\omega}) &= \frac{1}{2}\left[\mathrm{R}u_0(t;\boldsymbol{\omega}) - c\int\limits_{-\infty}^{t} \mathrm{R}u_1(s;\boldsymbol{\omega})ds\right]. \end{aligned} \tag{6.81}$$

Using these formulæ along with (6.76) one can obtain an explicit for the solution of the wave equation.

Exercises

Exercise 6.7.1. Prove the central slice theorem.

Exercise 6.7.2. Let $n = 2k+1$ and suppose that f is a function for which

$$\mathrm{R}f(t,\boldsymbol{\omega}) = 0 \text{ if } |t| < R.$$

Prove that $f(\mathbf{x}) = 0$ if $\|\mathbf{x}\| < R$. Is this true in even dimensions?

Exercise 6.7.3. Prove formula (6.78) and formula 6.79.

Exercise 6.7.4. Prove Proposition (6.7.1) . Hint: Integrate by parts.

Exercise 6.7.5. Use the simplified version of the Radon inversion formula available for $n = 3$ to derive an explicit formula for the solution of the wave equation in 3 space dimensions in terms of the initial data $u_0(x)$ and $u_1(x)$.

6.8 The Hilbert transform and complex analysis*

In the earlier part of the chapter we used several explicit Hilbert transforms, here we explain how these computations are done. We restrict to the case of square integrable functions. If $f \in L^2(\mathbb{R})$ with Fourier transform $\hat{f}$ then, as a limit-in-the-mean,

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ix\xi}\hat{f}(\xi)d\xi.$$

Define two L^2-functions

$$f_+(x) = \frac{1}{2\pi}\int\limits_0^\infty e^{ix\xi}\hat{f}(\xi)d\xi,$$
$$f_-(x) = \frac{1}{2\pi}\int\limits_{-\infty}^0 e^{ix\xi}\hat{f}(\xi)d\xi. \tag{6.82}$$

Obviously we have $f = f_+ + f_-$ and $\mathcal{H}f = f_+ - f_-$. This decomposition is useful because the function $f_+(x)$ has an extension as an analytic function in the upper half plane, $H_+ = \{x + iy \ : \ y > 0\}$

$$f_+(x+iy) = \frac{1}{2\pi}\int\limits_0^\infty e^{i(x+iy)\xi}\hat{f}(\xi)d\xi.$$

The Fourier transform of $f_+(x+iy)$ in the x-variable is just $\hat{f}(\xi)\chi[0,\infty)(\xi)e^{-y\xi}$. Since $y\xi > 0$ we see that $f_+(x+iy)$ is in $L^2(\mathbb{R})$ for each $y \geq 0$. A similar analysis shows that f_- has an analytic extension to the lower half plane, $H_- = \{x + iy \ : \ y < 0\}$ such that $f_-(x+iy) \in L^2(\mathbb{R})$ for each $y \leq 0$. Indeed it is not hard to show that this decomposition is unique. The precise statement is the following.

Proposition 6.8.1. *Suppose that $F(x+iy)$ is an analytic function in H_+ such that for $y \geq 0$*

(1).

$$\int\limits_{-\infty}^\infty |F(x+iy)|^2 dx < M,$$

(2).

$$\lim_{y\downarrow 0}\int\limits_{-\infty}^\infty |F(x+iy)|^2 dx = 0$$

then $F \equiv 0$.

Proof. By Theorem 4.4.4, a function satisfying the L^2-boundedness condition has the following property

$$\hat{F}(\cdot + iy) = \hat{f}(\xi)e^{-y\xi}$$

where $\hat{f}(\xi)$ is the Fourier transform $F(x)$. Moreover $\hat{f}(\xi) = 0$ if $\xi < 0$. By the Parseval formula

$$\int\limits_{-\infty}^\infty |F(x+iy)|^2 dx = \int\limits_0^\infty |\hat{f}(\xi)|^2 e^{-2y\xi} d\xi.$$

The second condition implies that $\hat{f}(\xi) = 0$ and therefore $F \equiv 0$. □

If the functions $f_\pm$ can be explicitly determined then $\mathcal{H}f$ can also be computed. If f is a "piece" of an analytic function then this determination is often possible. The following example is typical.

Example 6.8.1. Let

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

The analytic function, $\sqrt{1-z^2}$ has a single valued determination in the complex plane minus the subset of $\mathbb{R}$, $\{x : |x| \geq 1\}$. Denote this function by $F(z)$. Of course $F(x) = f(x)$ for $x \in (-1, 1)$ and the restrictions of F to the upper and lower half planes, $F_\pm$ are analytic. Moreover for $|x| > 1$ we easily compute that

$$\lim_{\epsilon \downarrow 0} F_+(x + i\epsilon) + F_-(x - i\epsilon) = 0.$$

This would solve our problem but for the fact that $F(x + iy)$ is not in L^2 for any $y \neq 0$. To fix this problem we need to add a correction term that reflects the asymptotic behavior of $F(z)$ for large z. Indeed if we set

$$f_\pm(z) = \frac{1}{2}[F_\pm(z) \pm iz]$$

then a simple calculation shows that

$$f_+(x) + f_-(x) = f(x) \text{ for all real } x$$

and that

$$f_\pm(x \pm iy) \simeq \frac{1}{x} \text{ for large } x$$

and therefore $f_\pm(x \pm iy) \in L^2(\mathbb{R})$ for all $y > 0$. This allows us to compute the Hilbert transform of f

$$\mathcal{H}f(x) = f_+(x) - f_-(x) = \begin{cases} ix & \text{for } |x| < 1, \\ i(x + \sqrt{x^2 - 1}) & \text{for } x < -1, \\ i(x - \sqrt{x^2 - 1}) & \text{for } x > 1. \end{cases} \tag{6.83}$$

Exercise

Exercise 6.8.1. Compute the Hilbert transform of $\chi_{[-1,1]}(x)$. A good place to start is with the formula $\mathcal{H}f = \lim_{\epsilon \downarrow 0} h_\epsilon * f$, see section 6.3.