Scattering rigidity versus lens rigidity

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Scattering relation

Definition 1 (Scattering relation)

Let M be a Riemannian manifold with boundary ∂M. Suppose that X is a unit tangent vector based at a point on the boundary and that X is pointing inwards. Then X is the initial tangent vector of a geodesic. If the geodesic leaves the manifold at some time, call the ending tangent vector $\alpha_M(X)$, the map α_M is called the *scattering relation* of M.

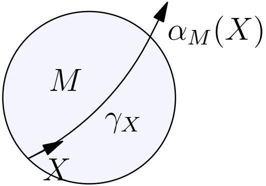


Figure : X and $\alpha(X)$ are the initial and ending tangent vector of a geodesic respectively.

Scattering rigidity and lens rigidity

Definition 2 (Scattering data and lens data)

Suppose that the boundary of two Riemannian manifolds M and N are isometric, then there is an isometry φ between the boundary of ΩM and the boundary of ΩN. If $\varphi \circ \alpha_M = \alpha_N \circ \varphi$, then M and N are said to have the same *scattering data*. If we also have $\ell(\gamma_X) = \ell(\gamma_{\varphi(X)})$ for all X, then M and N are said to have the same *lens data*.

Definition 3 (Scattering rigidity and lens rigidity)

M is *scattering rigid* (resp. *lens rigid*) if the space M and the metric on M is determined by its scattering data (resp. lens data) up to an isometry which leaves the boundary fixed.

M and N have the same scattering data —(if M is scattering rigid)→ M and N are isometric

Example 1: Same scattering data but different lens data (C. Croke)

1. The scattering data does not determine the lens data in general.
2. The scattering data does not determine the topology of a manifold.

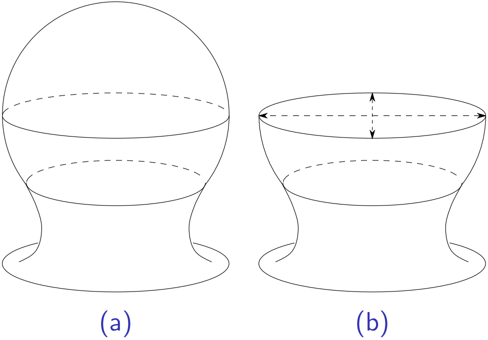
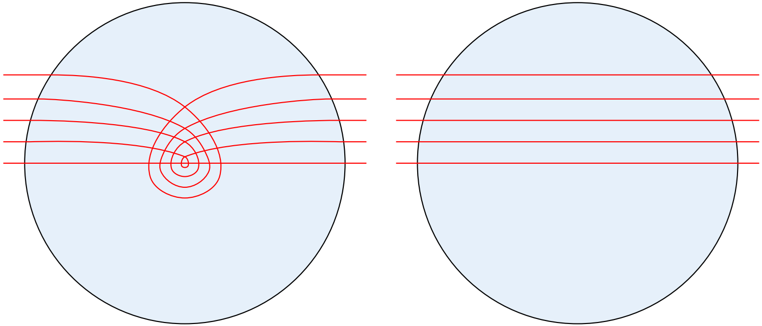


Figure : 5a and 5b have the same scattering data but different lens data.

Example 2: The Eaton lens

3. Local scattering data does not determine the convexity of the boundary.



(a) Geodesics on the Eaton lens (b) Geodesics on the flat disk

Figure : The Eaton lens has the same scattering data as the flat disk if you ignore the geodesics hitting the center (sigularity), but its boundary is totally geodesic.

Previous results

A large class of manifolds are known to be lens rigid:

1. Compact simple Riemannian surfaces with boundary (L. Pestov–G. Uhlmann, 2005)
2. Compact subdomains of $\mathbb{R}^n$ with flat metrics (M. Gromov, 1983) or metrics close to that (D. Burago–S. Ivanov, 2010)
3. Almost hyperbolic metrics (D. Burago–S. Ivanov, 2010)
4. Compact subdomains of symmetric spaces of negative curvature (G. Besson–G. Courtois–S. Gallot, 1995)
5. $D^n \times \mathbb{S}^1$ when $n > 1$ (C. Croke, 2011) and when $n = 1$ (C. Croke–P. Herroros, 2011)
6. ...

Very little is known for scattering rigidity:

1. $D^n \times \mathbb{S}^1$ is scattering rigid when $n > 1$. (C. Croke, 2011)

SGM manifolds

Definition 4

A Riemannian manifold with boundary is said to be *strong geodesic minimizing* (SGM) if it has no conjugate points and no trapped geodesics.

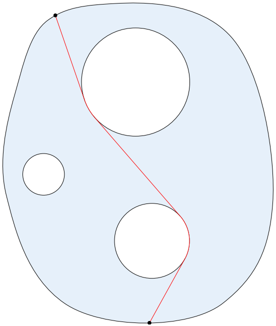
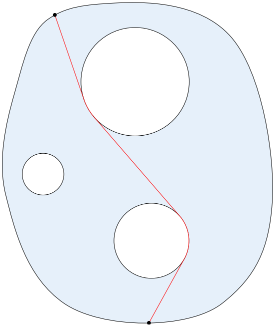


Figure : One any SGM manifold, there is a unique geodesic between any pair of points on the boundary in a given homotopy class of such curves, and this geodesic is length minimizing in its homotopy class.

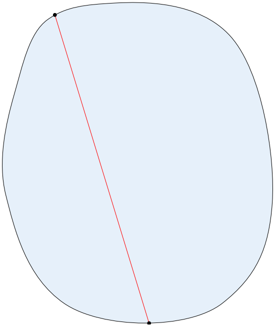
Simple manifolds

Definition 5

A Riemannian manifold with boundary is said to be *simple* if its boundary is strictly convex, there is a unique geodesic between any pair of points on the boundary and there are no conjugate points.



(a) An SGM manifold



(b) A simple manfold (also SGM)

The main result

Conjecture 6 (R. Michel, 1981)

Compact simple Riemannian manifolds are lens (boundary) rigid.

Theorem 7 (L. Pestov–G. Uhlmann, 2005)

Compact simple Riemannian surfaces are lens (boundary) rigid.

Theorem 8 (W.)

A compact SGM surface is scattering rigid if and only if it is lens rigid.

Corollary 9 (W.)

Compact simple Riemannian surfaces are scattering rigid.

No closed geodesics

Lemma 10

Suppose M and N have the same scattering data. If M is SGM, then N also have no trapped geodesics.

Proof.

Proof by contradiction.

1. Each closed geodesic lifts to a non-trivial knot (its projectivized Legendrian lift) in the *projectivized unit tangent bundle $P\Omega N$ of N^2*. $(P\Omega N = \Omega N/\{(x,\xi) \sim (x,-\xi)\}.)$
2. The family of the closed geodescis tangent to the boundary component can be lifted to a compressible torus in $P\Omega N$ and each closed geodesic will be lifted to the boundary of an embedded (compressing) disk, and thus be an unknot.

Step 1 works for any N. The SGM condition is used in step 2. □

Space of geodesics

Definition 11

Let Γ_M be the set of smooth geodesics $[0,1] \to M$ which satisfy the geodesic equation with end points are on ∂M. Let $\tilde{\Gamma}_M$ be the set geodesics $[0,1] \to M$ with end points are on ∂M.

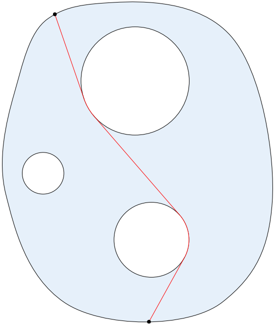


Figure : The red geodesic is in $\tilde{\Gamma}_M$ but not in Γ_M.

Lemma 12

Suppose that M and N have the same scattering data. In particular, there is an isometry $h : \partial M \to \partial N$ and an isometry $\varphi : \partial\Omega M \to \partial\Omega N$. Then, there is a natural continuous map $\Phi : \tilde{\Gamma}_M \to \tilde{\Gamma}_N$ which commute with h after reparametrizing the geodesics.

Proof.

1. For any X based at a boundary point and pointing inwards, define $\Phi(\gamma_X) := \gamma_{\varphi(X)}$. This defines Φ on Γ_M°, the interior of Γ_M.
2. Extend Φ to $\overline{\Gamma_M^\circ}$ by taking limits.
3. If γ is a geodesic running along ∂M, define $\Phi(\gamma) := h \circ \gamma$.
4. Extend Φ to $\tilde{\Gamma}_M$ but putting $\Phi(\alpha + \beta) = \Phi(\alpha) + \Phi(\beta)$.

Note that we need to use Lemma 10 in step 2 and step 3 to guarantee that Φ is well-defined. □

Proof of the main theorem

Theorem 8 (W.)

A compact SGM surface is scattering rigid if and only if it is lens rigid.

Proof.

1. Define $e : \tilde{\Gamma}_M \to \mathbb{R}$ as $e(\gamma) = \ell(\Phi(\gamma)) - \ell(\gamma)$.
2. For any $\gamma \in \tilde{\Gamma}_M$, construct two sequence of geodesics $\alpha_i, \beta_i \in \tilde{\Gamma}_M$ such that

$$e(\gamma) - e(\gamma(1)) = e(\alpha_0) - e(\beta_0) = e(\alpha_1) - e(\beta_1) \\ = \cdots = e(\alpha_n) - e(\beta_n) = e(\gamma(0)) - e(-\gamma) \quad (1)$$

   where $-\gamma(t) = \gamma(1-t)$ and where $\gamma(1)$ and $\gamma(0)$ are constant curves.
3. $e(\gamma) = e(-\gamma)$ and $e(\gamma(1)) = e(\gamma(0)) = 0$, so (1) implies that $e(\gamma) = 0$. Thus M and N also have the same lens data.

□

α_i and β_i

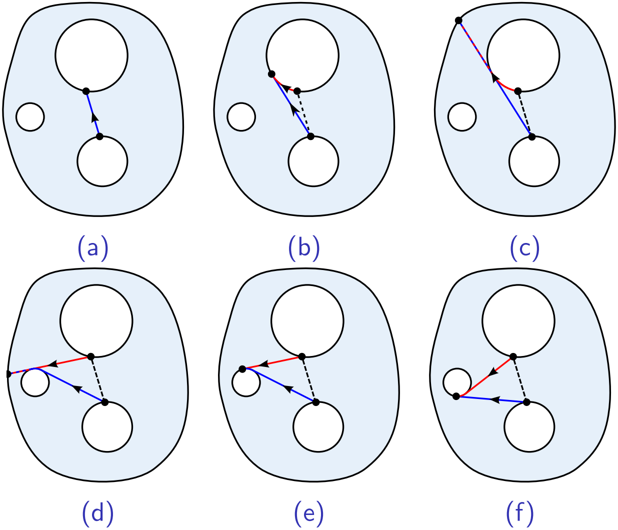


Figure : α_i is in blue and β_i is in red.

α_i and β_i (Continued)

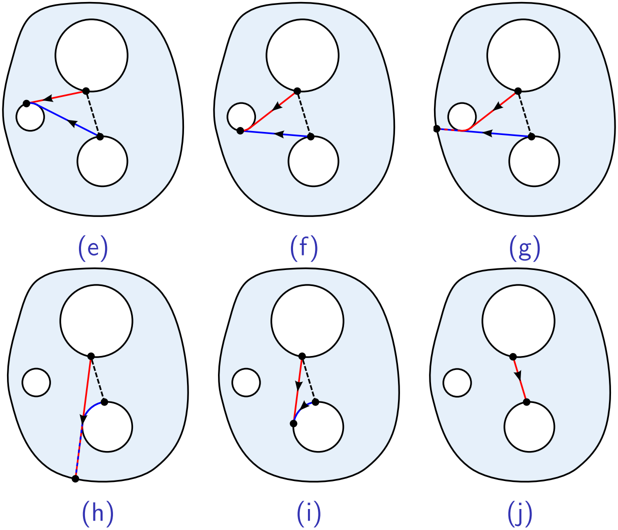


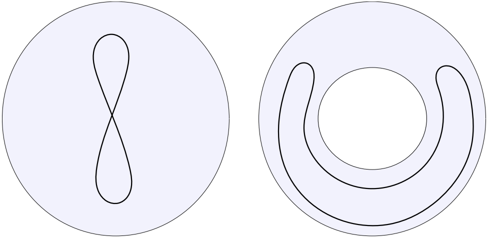
Figure : α_i is in blue and β_i is in red.

Legendrian lift

Definition 13

The *Legendrian lift* of a smoothly immerse closed curve γ on a Riemannian surface M^2 is a smoothly immersed closed curve $\tilde{\gamma} : \mathbb{S} \to \Omega M$ defined as

$$\tilde{\gamma}(t) = \left(\gamma(t), \frac{\gamma'(t)}{|\gamma'(t)|}\right).$$



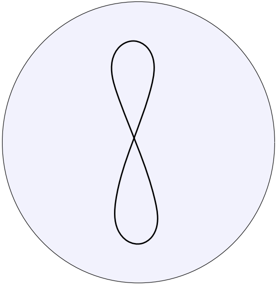
(a) $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ (b) $\tilde{\gamma} : \mathbb{S}^1 \to \Omega\mathbb{R}^2$

Figure : The Legendrian lift of a figure eight plane curve is an unknot.

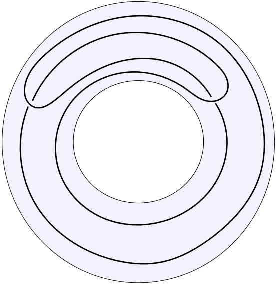
Projectivized Legendrian lift

Definition 14

Let $P : \Omega M \to P\Omega M$ be the quotient map on the unit tangent bundle which identifies the opposite vectors based at the same point. For any smoothly immersed closed curve $\gamma : \mathbb{S}^1 \to M^2$, $P \circ \tilde{\gamma}$ is called the *projectivized Legendrian lift* of γ.



(a) $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$

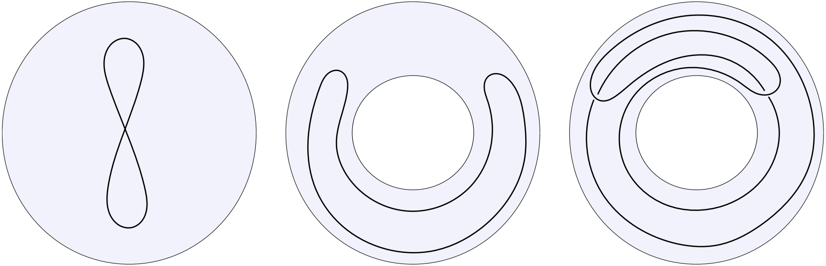


(b) $P \circ \tilde{\gamma} : \mathbb{S}^1 \to P\Omega\mathbb{R}^2$

Figure : The projectivized Legendrian lift of a figure eight plane curve is knotted.

Some observations

1. When $\gamma : \mathbb{S}^1 \to M^2$ has no direct self-tangencies, its Legendrian lift $\tilde{\gamma}$ is a knot embedded in the unit tangent bundle ΩM.
2. When $\gamma : \mathbb{S}^1 \to M^2$ has no self-tangencies of any type, its projectivized Legendrian lift $P \circ \tilde{\gamma}$ is a knot embedded in the projectivized unit tangent bundle $P\Omega M$.
3. The Legendrian lift of the figure eight plane curve is an unknot, while its projectivized Legendrian lift is knotted.



(a) $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ (b) $\tilde{\gamma} : \mathbb{S}^1 \to \Omega\mathbb{R}^2$ (c) $P \circ \tilde{\gamma} : \mathbb{S}^1 \to P\Omega\mathbb{R}^2$

Projectivized Legendrian lifts are non-trivial

Theorem 15 (W.)

The projectivized Legendrian lift of any smoothly immersed closed curve $\gamma : \mathbb{S}^1 \to M^2$ without self-tangencies is a non-trivial knot.

Note that Theorem 15 can also be viewed as a negative result saying that the unknot in the projectivized unit tangent bundle can not be realized by projectivized Legendrian lifts, as opposed to Legendrian lifts.

Theorem 16 (S.Chmutov–V.Goryunov–H.Murakami, 2000)

Every knot type in $\Omega\mathbb{R}^2$ is realized by the Legendrian lift of an immersed plane curve.

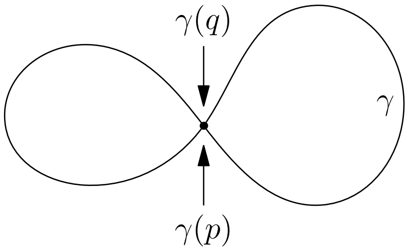
Crossings

We shall define a family of knot invariants for contractible knots in the projectivized unit tangent bundle $P\Omega N$ and use these invariants to prove Theorem 15.

Let $\beta : \mathbb{S}^1 \to P\Omega M$ be a contractible smooth knot in the projectivized unit tangent bundle $P\Omega M$, whose projection to the surface M^2 is a smoothly immersed curve $\gamma : \mathbb{S}^1 \to M^2$ without self-tangencies.

Definition 17

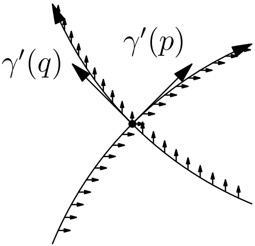
β has a *crossing* at $(p, q) \in \mathbb{S}^1 \times \mathbb{S}^1$ if $p \neq q$ and $\gamma(p) = \gamma(q)$.



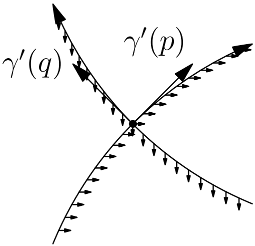
Since $\beta : \mathbb{S}^1 \to P\Omega M$ is contractible, we can lift β to $\hat{\beta} : \mathbb{S}^1 \to \Omega M$, a knot embedded in the unit tangent bundle.

Definition 18

A crossing of β at (p, q) is *positive* if the two pairs of vectors $(\hat{\beta}(p), \hat{\beta}(q))$ and $(\gamma'(p), \gamma'(q))$ are of the same orientation.



(a) A positive crossing



(b) A negative crossing

Figure : Each little arrow means a point on $\hat{\beta}$

Smoothing a crossing

Notice that $\hat{\beta}(p)$ and $\hat{\beta}(q)$ are two unit vectors with the same base point $x = \pi(\beta(p))$ and they are neither opposite to each other nor the same. Hence there is a unique shortest curve $\hat{\beta}_{(p,q)}$ in $\pi^{-1}(x)$ connecting $\hat{\beta}(p)$ and $\hat{\beta}(q)$.

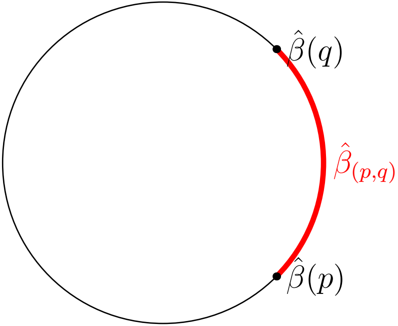


Figure : $\hat{\beta}(p)$ and $\hat{\beta}(q)$ are in the same fiber (a circle) of the unit tangent bundle ΩM. $\hat{\beta}_{(p,q)}$ is the shortest curve connecting them.

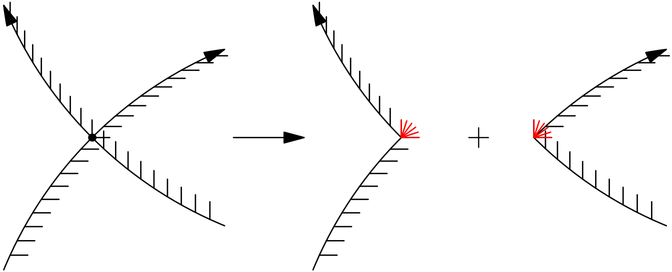


Figure : Smoothing a crossing. Here each little black bar means a point on $\hat{\beta}$ and each little red bar means a point on $\hat{\beta}_{(p,q)}$.

Definition 19

Separate $\hat{\beta}$ into two arcs by cutting at $\hat{\beta}(p)$ and $\hat{\beta}(q)$. Pick one arc and glue it to $\hat{\beta}_{(p,q)}$, obtaining a closed curve $\hat{\beta}'$. Denote the unoriented free homotopy class of $P \circ \hat{\beta}'$ as $g_{(p,q)}$, which will be called the *type* of the crossing of β at (p, q).

The knot invariants

Definition 20

For each **nontrivial** unoriented free homotopy class g of closed curves in the projectivized unit tangent bundle $P\Omega M$, define

$$W_g(\beta) = \#\{\text{positive crossings of } \beta \text{ of type } g\} - \#\{\text{negative crossings of } \beta \text{ of type } g\}$$

Theorem 21

W_g can be extended to all the contractible knots embedded in $P\Omega M$ as a knot invariant.

Proof of Theorem 21.

Check "Reidemeister moves". □

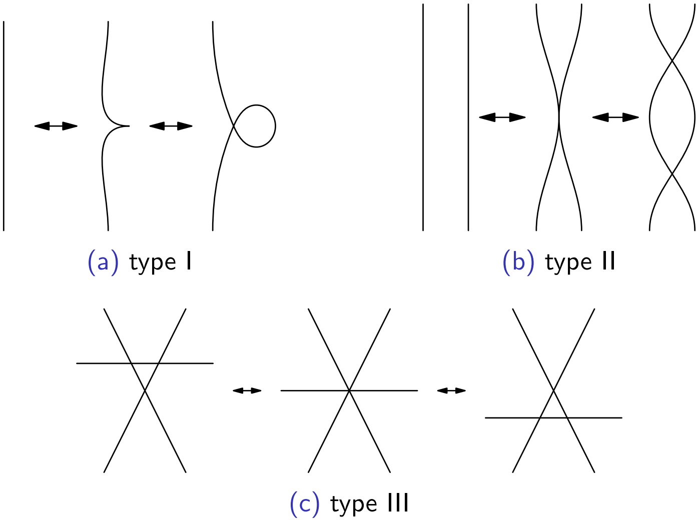


Figure : "Reidemeister moves"

Theorem 15 (W.)

The projectivized Legendrian lift of any smoothly immersed closed curve $\gamma : \mathbb{S}^1 \to M$ without self-tangencies is a non-trivial knot.

Proof.

Let $\beta = P \circ \tilde{\gamma}$ be the projectivized Legendrian lift of γ.

1. If β is not contractible, then β is a non-trivial knot.
2. If β is contractible, then $W_g(\beta) > 0$ for some g, while $W_g(\text{unknot}) = 0$ for any g.
   - (i) Every crossing of β is positive.
   - (ii) β has at least one crossing.
   - (iii) β has at least one crossing of a non-trivial type g.

□

Summary

Theorem 8 (W.)

A compact SGM surface is scattering rigid if and only if it is lens rigid.

Corollary 9 (W.)

Compact simple Riemannian surfaces are scattering rigid.

A crucial step in proving Theorem 8 is to understand closed geodesics on surfaces, which can be studied via knot theory using the projectivized Legendrian lifts.

Theorem 15 (W.)

The projectivized Legendrian lift of any smoothly immersed closed curve $\gamma : \mathbb{S}^1 \to M^2$ without self-tangencies is a non-trivial knot.

Thank you!