Lecture 7 - F-structures II - Examples and Definitions

August 4, 2011

1 Definitions

We recall the definition of F-structures. An F-structure $\mathcal{G}$ consists of a sheaf (also denoted $\mathcal{G}$) of compact abelian Lie groups (Tori) over a Hausdorff topological space X, with the following additional structure:

I. If $U \subset X$ is an open set, then the group $\mathcal{G}(U)$ has a local action on U, denoted $[\mathcal{G}(U)]$.

II. The assignment of the groups $\mathcal{G}(U)$ to the local actions $[\mathcal{G}(U)]$ commutes with the structure homomorphisms.

III. Given any $x \in X$, there is a saturated neighborhood $V(x) \subset X$ of x, and some finite normal cover $\pi : \widetilde{V}(x) \to V(x)$, that obey the following conditions:

  - If $\widetilde{\mathcal{G}}$ is the lifted sheaf, then if $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{x} \in \widetilde{U}$ where $\widetilde{U} \subset \widetilde{V}(x)$ is an open set, then the structure homomorphism $\widetilde{\mathcal{G}}(\widetilde{V}(x)) \to \widetilde{\mathcal{G}}_{\tilde{x}}$ is an isomorphism.
  - The local action of $\widetilde{\mathcal{G}}(\widetilde{V}(x))$ is a *complete local action*. That is, it is induced by a global action of the torus $\widetilde{\mathcal{G}}(\widetilde{V}(x)) \approx \widetilde{\mathcal{G}}_{\tilde{x}}$ on $\widetilde{V}(x)$.
  - The $V(x)$ can be chosen so that if $x, y \in X$ lie in the closure of the same orbit $\overline{\mathcal{O}}$, then $V(x) = V(y)$.

Because of *III*, we can regard the stalk $\mathcal{G}_x$ at a point x as device that encodes the symmetries of a manifold in some region near a point, at least up to taking finite normal covers.

A $\tilde{\mathfrak{g}}$-structure is a sheaf $\mathcal{G}$ of Lie algebras (not necessarily abelian or compact) that obeys *I*, *II*, and *III*, except that the covering maps $\pi : \widetilde{V}(x) \to V(x)$ need not be finite.

A few additional definitions will be required:

- A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called *pure* if its underlying sheaf is locally constant.

- If $V(x)$ and $\widetilde{V}(x)$ can be chosen independently of x then $\mathcal{G}$ is called an *elementary* $\tilde{\mathfrak{g}}$-structure. Necessarily $V(x) = X$.
- If $\mathcal{G}$ is a $\tilde{\mathfrak{g}}$-structure with sheaf $\mathfrak{g}$ and $\mathfrak{g}' \subset \mathfrak{g}$ is a subsheaf, then (since the action of $\mathfrak{g}$ descends to $\mathfrak{g}'$) $\mathfrak{g}'$ defines a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}'$ called a *substructure*.
- A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called *effective* if its local actions are effective. It can be proved that if $\mathcal{G}$ is an effective $\tilde{\mathfrak{g}}$-structure on a Riemannian manifold, each stalk is a connected Lie group, then the structure homomorphisms of $\mathcal{G}$ are injective.

We define the *rank* of a $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ at x to be $\dim \mathcal{O}_x$, and we say $\mathcal{G}$ has positive rank if $\dim \mathcal{O}_x > 0$ for all x.

Def An *atlas* for an effective $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is a collection $\{(U_\alpha, \mathcal{G}_\alpha)\}$ so that

i. the U_α are connected, saturated (w.r.t. $\mathcal{G}$, not $\mathcal{G}_\alpha$), open sets that form a locally finite covering of X

ii. each $\mathcal{G}_\alpha \subset \mathcal{G}\,|_{U_\alpha}$ is pure

iii. given any x, there is an α with $\mathcal{G}_{\alpha,x} = \mathcal{G}_x$.

A *subatlas* $\mathcal{A}' \subset \mathcal{A}$ is an atlas $\{(U'_\alpha, \mathcal{G}'_\alpha)\}$ so that $U'_\alpha \subset U_\alpha$ and $\mathcal{G}'_\alpha = \mathcal{G}_\alpha\,|_{U'_\alpha}$.

An F-structure is typically defined by specifying an atlas $\{(U_\alpha, \mathcal{G}_\alpha)\}$, determining a global action of $\mathcal{G}_\alpha$ on some cover of U_α, and gluing the stalks of the $\mathcal{G}_\alpha$ on the overlaps in a way dictated by the actions.

A substructure $\mathcal{P} \subseteq \mathcal{G}$ is called a *polarization* for $\mathcal{G}$ if $\mathcal{P}$ has an atlas so that the rank of $\mathcal{P}_\alpha$ is positive and constant on U_α (though the rank of $\mathcal{P}$ may vary with α). A polarization $\mathcal{P}$ is called *pure* if $\mathcal{P}$ is a pure $\tilde{g}$-structure. A pure polarization gives the base space the structure of a fibration.

Another way to understand polarizations is as follows. An orbit $\mathcal{O}$ of an F-structure is called a *singular orbit* if the dimension of the orbit is different from the dimension of the stalk of $\mathcal{G}$ at points of $\mathcal{O}$ (note that the dimension of the stalks on $\mathcal{O}$ is constant, due to the third part of *III*). An F-structure is polarized if and only if no singular orbits exist. A polarized F-structure may have stalks of non-constant dimension, however.

2 Basic Examples of F-structures

2.1 Group action of T^k on M^n

Assume there is a group action of T^k on M^n. If $(g, x) \in T^k \times M^n$, denote the action by $(g, x) \mapsto g.x$.

To define an F-structure, we define the sheaf $\mathcal{G}$ is the constant sheaf: if $U \subset M^n$ is any open set besides the empty set then $\mathcal{G}(U) = T^k$, and all structure homomorphisms are the identity. A partial action of $\mathcal{G}(U)$ on U is given as follows: we define the domain $\mathcal{D}_U$ by

$$\mathcal{D}_U \;=\; \left\{\, (g,u) \in T^k \times M^n \mid g.u \in U \,\right\}, \tag{1}$$

and a partial action to be $(g,x) \mapsto g.x$ whenever $x \in U$ and $g.x \in U$. Clearly $\{e\} \times U \subset \mathcal{D}_U$. The local action $[\mathcal{G}(U)]$ on U is the equivalence class of this partial action. Because the restriction homomorphisms are each the identity, it is clear that the local actions commute with the restriction homomorphisms.

Setting $V(x) = T^k$ for every $x \in T^k$ and $\widetilde{V}(x) = V(x)$, the F-structure axioms are satisfied. Since the local covers are trivial, this is an elementary T-structure.

2.2 F-structures on quotients

Assume a manifold M^n admits an action by a torus T^k, and let $\mathcal{G}$ be the associated F-structure. Assume Γ is a discrete group of free actions of M^n, and set $N^n = M^n/\Gamma$.

An F-structure $\mathcal{G}'$ exists on N^n, as follows. Cover N^n with a finite number of open sets U_α that are "small," specifically, that obeys the following two stipulations. If $\widetilde{U}_\alpha$ is any lift of U_α to M^n then $\pi : \widetilde{U}_\alpha \to U_\alpha$ is a homeomorphism, and $U_\alpha \cap U_\beta$, if non-empty, is connected. Choose a basepoint $x \in N^n$, and select a lift $\tilde{x} \in M^n$ of x. Now for each α choose a point $x_\alpha \in U_\alpha$ and a minimizing geodesic γ_α from x to x_α. Lifting the geodesic to a geodesic $\widetilde{\gamma}_\alpha$ that begins at $\tilde{x}$, we see that it terminates at a point $\widetilde{x}_\alpha \in \pi^{-1}(x_\alpha)$. Note that $\tilde{x}_\alpha$ lies in some pre-image $\widetilde{U}_\alpha$ of U_α. Since $\pi : \widetilde{U}_\alpha \to U_\alpha$ is a homeomorphism, we can define $\mathcal{G}'(U_\alpha) = \mathcal{G}(\widetilde{U}_\alpha)$, and also define the local action of $\mathcal{G}'(U_\alpha)$ on U_α as the local action coming from $\mathcal{G}(\widetilde{U}_\alpha)$ on $\widetilde{U}_\alpha$.

The structure homomorphisms will be as follows. If $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ is not empty, then simply define $\rho'_{U_\beta U_\alpha} = \rho_{\widetilde{U}_\beta \widetilde{U}_\alpha}$. If $U_\alpha \cap U_\beta$ is not empty but $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ is empty, then there is a unique element $g \in \Gamma$ so that $g\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ is not empty. Define $\rho'_{U_\beta U_\alpha}$ to be $\rho_{\widetilde{U}_\beta g \widetilde{U}_\alpha} \circ g$. Here $g \in \Gamma$ acts on a section of $\mathcal{G}(\widetilde{U}_\alpha)$ via the holonomy action: if $p \in M^n$ and $a \in \mathcal{G}(\widetilde{U}_\alpha)$ then $g(a)$ acts on $g.p$ via $g \circ a \circ g^{-1}$.

2.3 A pure F-structure of non-positive rank on a compact manifold

Consider the action of $\mathbb{S}^1$ on $\mathbb{R}^3$, given by

$$\theta.\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ y\cos\theta - x\sin\theta \\ z \end{pmatrix}.$$

This gives rise to an F-structure of non-positive rank on $\mathbb{R}^3$. Specifically, the rank is 1, except for points of the form $(0, 0, z)$, where the rank is zero. Restricting this action to $\mathbb{S}^2$, this gives an F-structure on $\mathbb{S}^2$. The stalk at each point is isomorphic to $\mathbb{S}^1$, and this is a constant sheaf (the total space is $\mathbb{S}^1 \times \mathbb{S}^2$). However the action has two singular points, at $(0, 0, 1)^T, (0, 0, -1)^T \in \mathbb{S}^2$, so the F-structure does not have positive rank.

2.4 An F-structure that does not lift to a covering space

Consider the action of $\mathbb{S}^1$ on itself. As in the first example, this produces an F-structure $\mathcal{G}$. Consider the covering map $\pi : \mathbb{R}^1 \to \mathbb{S}^1$. The sheaf pulls back, as do the local actions. However, there is no *complete local action* of the pullback sheaf $\pi^* \mathcal{G}$ on $\mathbb{R}$ (the definition of a complete local action of a sheaf on its base space was given in the previous lecture). Therefore the pullback sheaf with its pullback action is not an F-structure.

2.5 An effective F-structure with non-injective structure homomorphisms

If k is an integer, let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be the map $f(x) = x^k$. This is the standard k-1 cover of the circle on itself. Let X be the mapping torus for f. That is, X is the topological space

$$X \;=\; \mathbb{S}^1 \times [0, 1] \,/ \sim \tag{2}$$

where the equivalence is $(p, 1) \mapsto (f(p), 0)$. Of course X is not a manifold unless $k = 1$; for example if $k = 2$, this space is obtained by a 1-1 gluing of the the boundary circle of the Möbius strip to its center circle.

Define $S \subset X$ to be the image of $\mathbb{S}^1 \times \{0\}$ (or $\mathbb{S}^1 \times \{1\}$) in X. An F-structure $\mathcal{G}$ and its action can best be described using pre-image sets: if $U \subset X$ is open, let $U' \subset \mathbb{S}^1 \times [0, 1]$ be its pre-image. First assume $U \neq \varnothing$ but U does not intersect S. Then U is homeomorphic to U', so we can define $\mathcal{G}(U) = \mathbb{S}^1$ with its local action obtained from the action of $\mathbb{S}^1$ on $\mathbb{S}^1 \times [0, 1]$.

Next assume $U \subset X$ intersects S, but does not intersect, say, the image of $\mathbb{S}^1 \times [\frac{1}{4}, \frac{3}{4}]$ in X; again define $\mathcal{G}(U) = \mathbb{S}^1$. Let $p' \in U'$ be any pre-image of p. If $p' \in U' \cap \mathbb{S}^1[\frac{3}{4}, 1]$ then $p' \mapsto e^{i\theta}.p'$ If $p' \in U' \cap \mathbb{S}^1[0, \frac{1}{4}]$ then $p' \mapsto e^{ki\theta}.p$. If $p \in S$ we can take p' in either $\mathbb{S}^1 \times [\frac{3}{4}, 1]$ or in $\mathbb{S}^1 \times [0, \frac{1}{4}]$, but the actions we defined commute with the identification map, so it does not matter which set we assume p' to be in.

If U is a saturated open set that intersects S but does not intersect the image of $\mathbb{S}^1 \times [\frac{1}{4}, \frac{3}{4}]$, and V is the image of $\mathbb{S}^1 \times (0, 1/4)$, say, then the structure homomorphism $\rho_{U \cap V, U}$ is a k-1 covering map.

2.6 A pure, non-polarized F-structure with a polarization

Consider $\mathbb{S}^3 \subset \mathbb{C}^2$. The Clifford torus, which is the set of points $(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{S}^3 \subset \mathbb{C}^2$, acts on S^3 via multiplication. Let $\mathcal{G}$ be the F-structure obtained from this action. This F-structure is pure but does not have constant rank, so is therefore non-polarized. To see this explicitly by the definition, consider any atlas $\mathcal{A} = \{(U_\alpha, \mathcal{G}_\alpha)\}$. Let $p \in \mathbb{S}^3 \cap \{0\} \times \mathbb{C}$ be a point on one of the primary circles (that is, the intersection of $\mathbb{S}^3$ with either the z- or w-axis). Since $\mathcal{A}$ is an atlas, there is some U_α with $p \in U_\alpha$ and so that $\mathcal{G}_p = \mathcal{G}_{\alpha,p} = T^2$, forcing $\mathcal{G}_\alpha = \mathcal{G}\,|_{U_\alpha}$. The rank of $\mathcal{G}_\alpha$ is therefore not constant on U_α, as the orbit through p is 1-dimensional while orbits through neighboring points will be 2-dimensional.

However we can construct a polarized substructure. Let $S_1 = \mathbb{S}^3 \cap \mathbb{C} \times \{0\}$ and $S^2 = \mathbb{S}^3 \cap \{0\} \times \mathbb{C}$. Recalling that $\mathcal{G}(U) = T^2$ (unless $U = \varnothing$), we define $\mathcal{G}' \subset \mathcal{G}$ be as follows:

$$\mathcal{G}'(U) \;=\; \begin{cases} \{1\} & \text{if } U \;=\; \varnothing \\ \{(e^{i\theta}, 1)\} \subset \mathcal{G}(U) & \text{if } U \text{ intersects } S_1 \text{ but not } S_2 \\ \{(1, e^{i\theta})\} \subset \mathcal{G}(U) & \text{if } U \text{ intersects } S_2 \text{ but not } S_1 \\ \mathcal{G}(U) & \text{if } U \text{ intersects neither } S_2 \text{ nor } S_1 \\ \{1\} \subset \mathcal{G}(U) & \text{if } U \text{ intersects both } S_2 \text{ and } S_1 \end{cases} \tag{3}$$

The restriction homomorphisms for $\mathcal{G}'$ are simply induced by the restriction maps for $\mathcal{G}$. If $p \in \mathbb{S}^3$, the stalks are

$$\mathcal{G}'_p \;=\; \begin{cases} \mathbb{S}^1 = \{(e^{i\theta}, 1)\} & \text{if } p \in S_1 \\ \mathbb{S}^1 = \{(1, e^{i\theta})\} & \text{if } p \in S_2 \\ T^2 & \text{if } p \in \mathbb{S}^3 \setminus \{S_1 \cup S_2\}. \end{cases} \tag{4}$$

For an atlas, let U_1 be a saturated neighborhood of S_1 and U_2 a saturated neighborhood of S_2, with the condition that $U_1 \cap U_2 = \varnothing$, and let $U_3 = \mathbb{S}^3 \setminus (S_1 \cup S_2)$. Let $\mathcal{G}'_1 \subset \mathcal{G}'\,|_{U_1}$ be the substructure given by $\mathcal{G}'_1(U) = \{(e^{i\theta}, 1)\}$ for any nonempty $U \subset U_1$, and similarly for $\mathcal{G}'_2$. Finally define $\mathcal{G}'_3 = \mathcal{G}'\,|_{U_3}$.

To see that $\mathcal{A}' = \{(\mathcal{G}'_1, U_1), (\mathcal{G}'_2, U_2), (\mathcal{G}'_3, U_3)\}$ is an atlas for $\mathcal{G}$, note that $\{U_1, U_2, U_3\}$ form a saturated open cover of $\mathbb{S}^3$, that each $\mathcal{G}'_i$ is a pure structure, and that if $p \in \mathbb{S}^3$ then $\mathcal{G}'_p = \mathcal{G}'_{1,p}$ if $p \in S_1$, $\mathcal{G}'_p = \mathcal{G}'_{2,p}$ if $p \in S_2$, and $\mathcal{G}'_p = \mathcal{G}'_{3,p}$ if $p \notin S_1 \cup S_2$. To see that $\mathcal{G}' \subset \mathcal{G}$ is a polarization, simply note that each structure $\mathcal{G}'_i$ has constant rank.

3 Theorems

Proposition 3.1 (regular atlases) *If the F-structure $\mathcal{G}$ on the manifold X (possibly open) has an atlas $\{(U_\alpha, \mathcal{G}_\alpha)\}$, then $\mathcal{G}$ has an atlas $\{(\underline{U}_\alpha, \mathcal{G}_\alpha)\}$ with the following properties:*

(1) The sets $\underline{U}_\alpha$ have compact closure

(2) If $x \in \underline{U}_{\alpha_1} \cap \cdots \cap \underline{U}_{\alpha_k}$, then (for some ordering) $\mathcal{G}_{\alpha_1,x} \subseteq \cdots \subseteq \mathcal{G}_{\alpha_k,x}$

(3) Given any $x \in \underline{U}_\alpha$, there is at most one $\underline{U}_\beta$ with $\mathcal{G}_{\alpha,x} = \mathcal{G}_{\beta,x}$. If the manifold is compact or if (1) is dropped, we can assume strict inclusion in (2).

<u>*Pf*</u>

(1) is clear.

(2) We argue inductively. Assume $x \in U_\beta \cap U_\gamma$ but $\mathcal{G}_{\beta,x} \nsubseteq \mathcal{G}_{\gamma,x}$ and $\mathcal{G}_{\gamma,x} \nsubseteq \mathcal{G}_{\beta,x}$. Since $\mathcal{G}_y \neq \mathcal{G}_{\beta,y} \neq \mathcal{G}_{\gamma,y}$ for any $y \in U_\beta \cap U_\gamma$, so that $U_\beta \cap U_\gamma$ is covered by other domains in the atlas. Thus we can replace U_β by $U_\beta - \overline{U_\gamma}$ and U_γ by $U_\gamma - \overline{U_\beta}$, and still retain $X = \bigcup U_\alpha$.

(3) First assume (1) an be dropped or that the manifold is compact. Let $U_1, \ldots, U_k$ be a maximal subcollection so that $\bigcup U_i$ is connected and whenever $x \in U_i \cap U_j$, then $\mathcal{G}_{i,x} = \mathcal{G}_{j,x}$. Set $\underline{U}_1 = U_1$ and let $\underline{U}_2, \ldots, \underline{U}_l$ be the connected components of $\bigcup_i U_i$ where the union is over the U_i the have nonzero intersection with U_1. Now consider the U_i that do not intersect U_1, and repeat this process.

Doing this for all such subcollections, the result follows. □

Proposition 3.2 (invariant metrics) *Assume X is a manifold, and let $\mathcal{A} = \{(U_\alpha, \mathcal{G}_\alpha)\}$ be a regular atlas for the F-structure $\mathcal{G}$. Then X has a $\mathcal{G}$-invariant metric.*

<u>*Pf*</u>

Let $\mathcal{A}' \subset \mathcal{A}$. With a partial ordering of the U_α coming from (2) of Proposition 3.1, we can choose U_α to be maximal. Cover U'_α by sets $V(x_1), \ldots, V(x_k)$ with $\overline{V(x_i)} \subset U_\alpha$. Put some metric on $V(x_1)$, lift it to $\tilde{V}(x_1)$, and average it over the action of $\mathcal{G}$ and over the deck action. Project back to $V(x_1)$. Put a metric on $V(x_2)$ that agrees with the invariant metric on $V(x_1)$ on the overlap, and perform the same averaging. Eventually this gives an invariant metric on U'_α. This same procedure can be done on any U'_β, only the starting metrics on the $V(x_i)$ must now agree with the metric on U_α where the intersection is nonempty. □

Proposition 3.3 *If X is a compact manifold that carries an F-structure of positive rank, then $\chi(X) = 0$.*

<u>*Pf*</u>

On each $\tilde{V}(x)$ a torus acts with no common fixed points, so almost all of its elements have a fixed-point free action. Given such an element with no fixed points, one finds a one-parameter subgroup that acts on $\tilde{V}(x)$, and so $\chi(\tilde{V}(x)) = 0$, so $\chi(V(x)) = 0$. Essentially the same argument shows that $\chi(V(x) \cap V(y)) = 0$. Recalling that $\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V)$ and covering X with finitely many $V(x)$, we get the result. □

4 Exercises

1) *Canonical action of a sheaf on its total space.* Let $\mathfrak{g}$ be a locally constant sheaf of topological groups over a manifold X, with projection π. Let $\mathfrak{g}^* = \pi^*(\mathfrak{g})$ denote the pullback sheaf. Show that there is a canonical action of $\mathfrak{g}^*$ on the total space of the sheaf $\mathfrak{g}$. This action is pure, the orbits are just the fibers, and $(\pi^{-1}(X), \mathfrak{g}^*)$ is a pure polarized F-structure.

2) *Nilgeometry and Solvegeometry.* Let A be a matrix $A \in SL(2, \mathbb{Z})$, so A can be considered a map $A : T^2 \to T^2$. Let M^3 be its mapping torus. If A is nilpotent, M^3 is a nilmanifold. Show that it supports a pure F-structure of rank 1, but no F-structure of rank 2. If A has distinct real eigenvalues, it is a solvemanifold. In this case show that there is a pure F-structure of rank 2, with exactly two substructures of rank 1, each corresponding to an eigenvalue of A.