Lecture 6 - F-structures I, Definitions

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This lecture's reference is Cheeger-Gromov's 1986 JDG paper "Collapsing Riemannian manifold while keeping their curvature bounded. I."

1 Definitions

Def A *partial action*, A, of a topological group G on a Hausdorff space X is given by

i. The *domain* of the action: a neighborhood $\mathcal{D} \subset G \times X$ of $\{e\} \times X$.

ii. A continuous map $A : \mathcal{D} \to X$, also written $(g, x) \to gx$, such that $(g_1 g_2)x = g_1(g_2 x)$ whenever $(g_1 g_2, x)$ and $(g_1, g_2 x)$ lie in $\mathcal{D}$.

To emphasize the domain, a partial action A can be written $(A, \mathcal{D})$. An equivalence relation exists on the set of partial actions; an equivalence class is called a *local action*. Two partial actions $(A_1, \mathcal{D}_1)$ and $(A_2, \mathcal{D}_2)$ of a topological group G on a Hausdorff space X are equivalent if for any open subset $\mathcal{D} \subset \mathcal{D}_1 \cap \mathcal{D}_2$ with $\{e\} \times X \subset \mathcal{D}_1 \cap \mathcal{D}_2$, we have $A_1|_{\mathcal{D}} = A_2|_{\mathcal{D}}$; an equivalence class is denote by $[A]$. We list some additional facts and definitions.

- Any global action defines a local action; an equivalence class which has such a member will be called *complete*. Notice that if G is connected, any two global actions in the same equivalence class are identical.
- In the smooth category, the local actions of a Lie group G is in 1-1 correspondence with the homomorphisms from the Lie algebra of G to the Lie algebra of vector fields on X. The completeness of an action is the same as the *global* integrability of the individual vector fields.
- If $[A]$ is a local action, a subset $X_0 \subset X$ is called $[A]$-invariant if whenever $x_0 \in X_0$ and $(g, x_0) \in \mathcal{D}$ then $gx_0 \in X_0$. The intersection of $[A]$-invariant sets is $[A]$-invariant, so any point x lies in a minimal $[A]$-invariant set, called the orbit of x, denoted $\mathcal{O}_x$ or just $\mathcal{O}$.

- The orbits of X form a partition.
- Unlike an action, a local action $[A]$ on X pulls back along any local homeomorphism $f : Y \to X$ to a local action $f^*[A]$ on Y.
- A local action $[A]$ on X can be restricted to any subset $U \subset X$ by restricting the domain $\mathcal{D}$ of any representative of $[A]$ to any open subset $\mathcal{D}'$ that contains $\{e\} \times U$ and which obeys (*ii*) from above.
- If $(A_1, \mathcal{D}_1)$ represents a local action on U_1 and $(A_2, \mathcal{D}_2)$ represents a local action on U_2 and if $A_1|_{U_1 \cap U_2} = A_2|_{U_1 \cap U_2}$, a local action on $U \cup V$ can be taken to be the equivalence class of the partial action $(A_1|_{U_1 \cap U_2}, \mathcal{D}_1 \cap \mathcal{D}_2)$. This equivalence class is uniquely defined.

The last two points suggest compatibility with another topological construction that is perhaps more familiar.

Def A *sheaf* $\mathfrak{F}$ on a topological space X is an association between open sets $U \subset X$ and groups that satisfies the following three axioms.

a) $\mathfrak{F}(U)$ is a group whenever U is an open subset and $\mathfrak{F}(\varnothing) = \{0\}$, of X

b) If $V \subseteq U$ is an inclusion of open sets, there is a homomorphism (the restriction, or structure homomorphism) $\rho_{VU} : \mathfrak{F}(U) \to \mathfrak{F}(V)$ subject to the restrictions (*i*) that $\rho_{UU} = \mathrm{Id}$ and (*ii*) that $W \subseteq V \subseteq U$ implies $\rho_{WU} = \rho_{WV} \circ \rho_{VU}$.

c) If $\{U_\alpha\}$ is an open covering of U and $s_\alpha \in \mathfrak{F}(V_\alpha)$ satisfies $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there exists a unique element $s \in \mathfrak{F}\, U$ so that $s|_{U_\alpha} = s_\alpha$.

If $\mathfrak{F}$ only satisfies (*a*) and (*b*) it is called a *presheaf*. A salient feature of sheafs is the existence of a *stalk* over each point. Let $\mathfrak{F}$ be a sheaf over M, and let $p \in M$. Let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be the family of open sets containing p. Then the $\mathfrak{F}(U_\alpha)$ along with the structure homomorphisms constitute a directed family of groups, the direct limit of which is called the stalk at p. A topology can be put on the space of stalk: a neighborhood base is given by the images of the "sections" $\mathfrak{F}(U)$ in the space of stalks. Stalks can be defined if just a presheaf structure exists, and then sections of the space of stalks constitute a sheaf (the *sheafification* of the presheaf).

Let $\mathfrak{g}$ be a sheaf of connected topological groups (note there is some question about topology here; we just accept that there are two topologies, the sheaf topology, and a topology that makes the stalks into Lie groups— we usually ignore the sheaf topology). An *action* of $\mathfrak{g}$ on X is given by a local action of $\mathfrak{g}(U)$ for each open U such that the local actions agree with the sheaf restriction maps. To be explicit, when $x \in V \subset U$ and $g \in \mathfrak{g}(U)$, we have $gx = \rho_{VU}(g)x$ wherever $\rho_{VU}(g)x$ is defined.

A set $S \subset X$ is called invariant if $S \cap U$ is invariant under $\mathfrak{g}(U)$ for all open subsets $U \subset X$. A minimal invariant set is called an orbit. The orbits partition X, and a set that is the disjoint union of orbits is called *saturated*.

We use $\mathfrak{g}_x$ to denote the stalk of $\mathfrak{g}$ at $x \in X$. If $f : X \to Y$ is a local homeomorphism then we denote by $f^* \mathfrak{g}$ the pullback sheaf.

Def An action of a sheaf $\mathfrak{g}$ is called a *complete local action* if

i. Whenever $x \in X$ there exists a neighborhood $V(x)$ of x and a local homeomorphism $\pi : \widetilde{V}(x) \to V(x)$ so that $\widetilde{V}(x)$ is Hausdorff

ii. If $\tilde{x} \in \pi^{-1}(x)$ and $\tilde{\mathfrak{g}}$ denotes the pullback of $\mathfrak{g}$ along π, then for any open set $\widetilde{W} \subset \widetilde{V}(x)$ with $\tilde{x} \in \widetilde{W}$, the structure homomorphism $\tilde{\mathfrak{g}}(\widetilde{W}) \to \mathfrak{g}_{\tilde{x}}$ is an isomorphism

iii. The local action of $\tilde{\mathfrak{g}}(\widetilde{V}(x))$ on $\widetilde{V}(x)$ is complete.

If $\pi : \widetilde{V}(x) \to V(x)$ is a covering space, the deck transformation group Γ induces a natural action on $\pi^* \mathfrak{g}$, called the *holonomy action*. Specifically, if $g \in \mathfrak{g}(U)$ then $\gamma(g)$ acts on elements $x \in \gamma(U)$ via $\gamma(g) = \gamma \circ g \circ \gamma^{-1}$. One verifies that for $\gamma \in \Gamma$, we have $\gamma(gx) = \gamma(g)\gamma(x)$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ on X is a sheaf, $\mathfrak{g}$, of connected topological groups and a complete local action of $\mathfrak{g}$ on X such that the sets $V(x)$ and $\widetilde{V}(x)$ can be chosen so that

i. $\pi : \widetilde{V}(x) \to V(x)$ is a normal covering map

ii. For all x, $V(x)$ is saturated

iii. For all $\mathcal{O}$, if $x, y \in \overline{\mathcal{O}}$, then $V(x) = V(y)$.

Condition (*iii*) implies that $\mathfrak{g}$ is a locally constant sheaf on $\overline{\mathcal{O}}$, though not necessarily on neighborhoods of $\overline{\mathcal{O}}$.

Def A $\tilde{\mathfrak{g}}$-structure $\mathcal{G}$ is called an F-structure if each stalk $\mathfrak{g}_x$ is isomorphic to a torus, and the local covers $\widetilde{V}(x) \to V(x)$ can be chosen to be finite normal covers. If in addition one can choose $\widetilde{V}(x) = V(x)$, then $\mathcal{G}$ is called a *T-structure*.

2 Exercises

1) Consider the action of the Lie group $G = (\mathbb{R}, +)$ on itself; this determines a complete local action $[A_{\mathbb{R}}]$. Let $X = (a, b) \subset \mathbb{R}$. It was claimed that the local action $[A_{\mathcal{R}}]$ on $\mathbb{R}$ restricts to a local action $[A_X]$ on X. Give an example of a partial action $A \in [A_X]$ by specifying the domain and the action.

2) In the context of the previous example, let $X = (0, 2)$ and $Y = (1, 3)$, and let $(A_X, \mathcal{D}_X)$ and $(A_Y, \mathcal{D}_Y)$ be partial actions on X and Y, respectively. To make this example explicit, write down explicit choices for the domains $\mathcal{D}_X$ and $\mathcal{D}_Y$. Given these choices, describe the consequent partial action on $Z = (0, 3)$ with $\mathcal{D}_Z = \mathcal{D}_X \cap \mathcal{D}_Y$.

3) Let $f : \widetilde{X} \to X$ be a local homeomorphism between the Hausdorff spaces $\widetilde{X}$ and X. Prove that any local action on X lifts to a partial action on X.

4) There is an obvious partial action of $\mathbb{S}^1$ on itself, which is also an action. Since $\mathbb{R}^1 \to \mathbb{S}^1$ is a covering map and therefore a local homeomorphism, there is a local action of $\mathbb{S}^1$ on $\mathbb{R}$. Give a description of some partial action in this local action.

5) The local action of $\mathbb{S}^1$ on $\mathbb{R}$ described in the previous exercise can easily be made into the local action of a sheaf of groups over $\mathbb{R}$, with stalk $\mathbb{S}^1$ at each point (a constant sheaf). Prove that this local action is *not* a complete local action.