Lecture 5 - Hausdorff and Gromov-Hausdorff Distance

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1 Definition and Basic Properties

Given a metric space X, the set of closed sets of X supports a metric, the Hausdorff metric. If A is a set in X and $r > 0$, we define the r-thickening, or r-neighborhood, of A to be the set $A^{(r)}$ defined by

$$A^{(r)} \;=\; \bigcup_{x\in A} B_x(r) \tag{1}$$

where $B_x(r)$ is the (open) ball of radius r about x. If $A, B \subset X$ are closed sets, define their *Hausdorff distance* $d_H(A,B)$ to be the number

$$d_H(A,B) \;=\; \inf\left\{\, r>0 \mid B \,\subset A^{(r)} \text{ and } A \,\subset B^{(r)}\,\right\}. \tag{2}$$

Recall that the infimum of an empty set is regarded to be $+\infty$. A equivalent definition is as follows. Given a point $p \in X$ and a closed set $A \subset X$, define

$$d(p,A) \;=\; \inf_{y\in A}\, \mathrm{dist}(p,y). \tag{3}$$

Then the Hausdorff distance is

$$d_H(A,B) \;=\; \max\left\{ \sup_{x\in A} d(x,B),\; \sup_{y\in B} d(y,A) \right\} \tag{4}$$

That is, $d_H(A,B)$ is the farthest distance any point of B is from the set A, or the farthest any point of A is from B, whichever is greater. Again, this could be infinite.

Theorem 1.1 *If (X,d) is a bounded metric space, the set of closed sets of X is itself a metric space with the Hausdorff metric.*

Pf We verify the metric space axioms. First, the symmetry of d_H is clear by definition. Second, d_H satisfies the triangle inequality because if C is in the r-neighborhood of B and

B is in the s-neighborhood of A, then C is in the $(r+s)$-neighborhood of A. Likewise A is in the $(r+s)$-neighborhood of C. Thus $d(A,C) \leq d(A,B) + d(B,C)$. Finally $d_H(A,B) = 0$ implies $A \subseteq \overline{B} = B$, because if B is in every r-neighborhood of A then every point of A is a limit point of B. Likewise $B \subseteq \overline{A} = A$. □

If X is not bounded, the metric space axioms continue to hold except that possibly closed sets A and B exist with $d_H(A,B) = \infty$. This could be rectified by restricting to compact subsets of X, although this is not natural in some cases.

2 Compactness Properties

Let (X,d) be a metric space and denote the set of closed subsets of X by $\mathfrak{C}(X)$ (or just $\mathfrak{C}$ for short). Given a closed set A and a number r, let $\mathfrak{B}_A(r)$ be the set of all $D \in \mathfrak{C}$ with $d_H(B,A) < r$ (that is, the r-ball around A in $\mathfrak{C}$). Since d_H is a metric on $\mathfrak{C}$, we know that the balls $\mathfrak{B}_A(r)$ are open, and form a neighborhood base.

Obviously the balls with rational radius also form a base, so the topology on $\mathfrak{C}$ induced by d_H is first countable. All metric spaces are Hausdorff, so $(\mathfrak{C}, d_H)$ is Hausdorff. One can state this directly: since distinct closed sets are separated by a finite distance, say ϵ, so the balls of radius, say, $\epsilon/4$ around each is disjoint.

Theorem 2.1 *If (X,d) is a compact metric space, then $(\mathfrak{C}(X), d_H)$ is compact.*

<u>*Pf*</u>

The proof can go as in the proof of Gromov's precompactness theorem; we leave it as an exercise. □

It should be clear that if (X,d) is non-compact, then $(\mathfrak{C}(X), d_H)$ is non-compact. This can be seen by the existence of an obvious isometric embedding $X \hookrightarrow \mathfrak{C}(X)$, and by noting that if a sequence in X converges in $\mathfrak{C}(X)$, its limit must be a point, and therefore again an element of X.

If (X,d) is bounded, then it is locally compact if and only if $(\mathfrak{C}(X), d_H)$ is locally compact. It can be proven that $(\mathfrak{C}(X), d_H)$ is paracompact whenever (X,d) is bounded. The proofs are left as exercises.

In sharp contrast, if (X,d) is unbounded, then $(\mathfrak{C}(X), d_H)$ need not be locally compact nor even locally paracompact. For instance if the base space X is unbounded and nondiscrete (it has the property that, given any point $x \in X$ and any number $\epsilon > 0$, there is a point $y \in X$ with $d(x,y) < \epsilon$), then it is not locally compact. As an example, we will will show that $\mathbb{R}$ is not locally compact. Let $A = [0,\infty)$ be the half-line, and consider its r-neighborhood $B_A(r)$ (wlog assume $r < \frac{1}{2}$). Define the A_i inductively by setting $A_0 = A$ and $A_i = A_{i-1} \setminus (i, i + r/2)$. We have $d_H(A_i, A_j) = r/2$ for any $i \neq j$, so there are no Cauchy subsequences and therefore no convergent subsequences.

A topology does exist on $\mathfrak{C}(X)$ that is both locally compact and compact, whether X is bounded or not. Let a base for this topology be set of the form $N_{K,\epsilon}(A)$, where $K \subset X$ is compact, $A \subset X$ is closed, and $\epsilon > 0$, where we define

$$N_{K,\epsilon}(A) \;=\; \{\, B \in \mathfrak{C}(X) \mid d_H(A \cap K, B \cap K) \;<\; \epsilon \,\}.$$

This topology on $\mathfrak{C}(X)$ is called the *pointed Hausdorff topology.* If X is compact, it is the metric topology. If X is noncompact, this topology is not induced by any metric.

3 The Gromov-Hausdorff distance

The Gromov-Hausdorff distance significantly extends the idea of the Hausdorff distance. Given two closed metric spaces A and B, we define

$$d_{GH}(A, B) \;=\; \inf_{f,g} d_H(f_{A\to X}(A),\, g_{B\to X}(B)) \tag{5}$$

where the notation $f_{A\to X}$ (resp. $g_{B\to X}$) denotes an isometric embedding of A into some metric space X (resp. an isometric embedding of B into X) and the infimum is taken over all possible such embeddings. Note that d_{GH} could well be infinty; however it is clearly symmetric. To show that $d_{GH}(A, B) = 0$ iff A and B are isometric, we first give an equivalent definition of d_{GH}.

Proposition 3.1 *The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is the infimum of the Hausdorff distances between X and Y taken among all metrics on $X \coprod Y$ that restrict to the given metrics on X and on Y.*

Pf Exercise. □

Proposition 3.2 *If (X, d_X) and (Y, d_Y) are metric spaces that admit compact exhaustions, and that $d_{GH}(X, Y) = 0$. Then (X, d_X) and (Y, d_Y) are isometric.*

Pf

If X and Y are isometric then clearly $d_{GH}(X, Y) = 0$.

Conversely assume $d_{GH}(X, Y) = 0$, and for the moment assume X and Y are compact. Then there is a sequence of distance functions d_i on $X \coprod Y$ with $d_i|_X = d_X$ and $d_i|_Y = d_Y$ so that $d_{i,H}(X, Y) \to 0$. Let $\epsilon_j > 0$ be a sequence that converges to 0. For each j construct finite sets of points $\mathcal{X}_j = \{x_k\}$ and $\mathcal{Y}_j = \{y_k\}$ with the following properties: $\mathcal{X}_j$ is ϵ_j-dense in X, $\mathcal{Y}_j$ is ϵ_j-dense in Y, and for large enough i the sets $\mathcal{X}_j$ and $\mathcal{Y}_j$ are ϵ_j-close in the Hausdorff metric. We also require that $\mathcal{X}_j \subset \mathcal{X}_{j+1}$ and $\mathcal{Y}_j \subset \mathcal{Y}_{j+1}$, so that $\mathcal{X} = \bigcup_j \mathcal{X}_j$ is dense in X and $\mathcal{Y} = \bigcup_j \mathcal{Y}_j$ is dense in Y.

Now consider the distance functions $\{d_i\}$ restricted to $\mathcal{X}_j \cup \mathcal{X}_j$. Because $\mathcal{X}_j \cup \mathcal{X}_j$ is finite, a subsequence d_{i_j} converges to a limiting pseudometric $\overline{d}_j$. Passing to refined subsequences as j increases and taking a diagonal subsequence, we get convergence to a pseudometric $\overline{d}$ on $\mathcal{X} \cup \mathcal{Y}$, a dense subset of $X \coprod Y$, and therefore convergence on $X \coprod Y$.

Given any ϵ_j, a given point $x \in X$ is ϵ_j-close to a point $x_j \in \mathcal{X}$, which is ϵ_j-close to a point of $y_j \in \mathfrak{Y}$. Taking a limit $y = \lim_j y_j$ we have that $\overline{d}(x, y) = 0$; because X is a metric space, this point x is unique (similarly given a point $y \in Y$ we can find a unique point $x \in X$ with $\overline{d}(x, y) = 0$). Finally send x to the unique point $y \in Y$ with $\overline{d}(x, y) = 0$.

The fact that this is an isometry follows from the triangle inequality: if $x_1, x_2 \in X$ are sent to $y_1, y_2 \in Y$, respectively, then

$$\begin{aligned} d_X(x_1, x_2) &= \overline{d}(x_1, x_2) \leq \overline{d}(x_1, y_1) + \overline{d}(y_1, y_2) + \overline{d}(y_2, x_2) = \overline{d}_Y(y_1, y_2) \\ d_Y(y_1, y_2) &= \overline{d}(y_1, y_2) \leq \overline{d}(y_1, x_1) + \overline{d}(x_1, x_2) + \overline{d}(x_2, y_2) = \overline{d}_X(x_1, x_2), \end{aligned}$$

so that $d_X(x_1, x_2) = d_Y(y_1, y_2)$. □

4 Gromov-Hausdorff Approximations

We mention what is often a more useful formulation of the Gromov-Hausdorff distance. A map $f : X \to Y$ (not necessarily continuous) between metric spaces is called an ϵ-GHA (for "Gromov-Hausdorff approximation") if $|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \epsilon$ for all $x_1, x_2 \in X$, and Y is in the ϵ-neighborhood of $f(X)$. We can define a new distance function between metric spaces, called $\widehat{d_{GH}}$, by setting

$$\widehat{d_{GH}}(X, Y) = \inf\{\, \epsilon > 0 \mid \text{there are}\, \epsilon\text{−GHA's}\, f : X \to Y \,\text{and}\, g : Y \to X \,\}.$$

It is a simple exercise to prove that this is a metric: if there is an ϵ_1-GHA $f : X \to Y$ and an ϵ_2-GHA $g : Y \to Z$, then the composition satisfies

$$\begin{aligned} &|d_Z(gf(x_1), gf(x_2)) - d_X(x_1, x_2)| \\ &\quad \leq |d_Z(gf(x_1), gf(x_2)) - d_Y(f(x_1), f(x_2))| + |d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| \\ &\quad \leq \epsilon_1 + \epsilon_2 \end{aligned}$$

and it is also easy to show that the $(\epsilon_1 + \epsilon_2)$-neighborhood of $fg(X)$ is Z. Taking infima, we have that $\widehat{d_{GH}}(X, Z) \leq \widehat{d_{GH}}(X, Y) + \widehat{d_{GH}}(Y, Z)$.

Proposition 4.1 *The metrics $\widehat{d_{GH}}$ and d_{GH} are equivalent (though they are not the same).*

Proof. Exercise. □

5 Compactness Properties

Proposition 5.1 *The Gromov-Hausdorff topology on the set of compact metric spaces is second countable.*

<u>*Pf*</u> Exercise. (Hint: If a topology is Hausdorff and separable it is second countable.) □

Lemma 5.2 (Gromov's Precompactness Lemma) *Let $N : \mathbb{N} \to \mathbb{N}$ be monotonic. Assume $\mathfrak{M}$ is a collection of metric spaces so that each $M \in \mathfrak{M}$ has a $\frac{1}{j}$-dense discrete subset of cardinality $\leq N(j)$. Then $\mathfrak{M}$ is precompact.*

<u>*Proof*</u>. Let $\{M_i\} \subset \mathfrak{M}$, and let $\tilde{M}_{i,j} \subset M_i$ be a $\frac{1}{j}$-dense subset of cardinality $\leq N(j)$. By replacing $N(j)$ with $\sum_{i=1}^{j} N(i)$ we can assume that $\tilde{M}_{i,j} \subset \tilde{M}_{i,j+l}$. Fixing j and letting $i \to \infty$ we get convergence of $\tilde{M}_{i,j}$ along a subsequence to a space $\tilde{M}_j$. Passing to further refinements of the subsequence and taking a diagonal sequence, we get a sequence of distance functions d_k that converge on each $\tilde{M}_j$, and therefore on $\tilde{M} = \overline{\bigcup_j \tilde{M}_j}$. Now given $\epsilon > 0$ there is an i so that $\tilde{M}_i$ is ϵ-close to $\tilde{M}$, and there is a j so that $M_{i,j}$ is ϵ-close to both $\tilde{M}_i$ and to M_i. Thus M_i converges to $\tilde{M}$. □

In general the topology associated to the Gromov-Hausdorff distance is neither locally compact nor locally paracompact. To redress this we define the *pointed Gromov-Hausdorff topology*, which is locally compact and compact. On the space of compact metric spaces, this will be the same as the original Gromov-Hausdorff topology. However the pointed topology is not induced by any norm.

The pointed Gromov-Hausdorff topology is defined on the set of pointed metric spaces (defined to be pairs (A, p, d) where (A, d) is a closed metric space and $p \in A$). A local base for this topology are the sets of the form $N_{K,\epsilon}(A)$ (where A is closed, $K \subset A$ is compact and $p \in K$, and $\epsilon > 0$); we define $N_{K,\epsilon}(A)$ to be the set of pointed closed sets (B, q) so that there exists a compact subset $J \subset B$, $q \in J$, and so that there are isometric embeddings $f : A \cap K \to X$ and $g : B \cap J \to X$ into some space X so that $f(p) = g(q)$ and the Hausdorff distance satisfies $d_H(f(A \cap K),\, g(B \cap J)) < \epsilon$.

6 Exercises

1) Give an example of a bounded metric space that is not locally compact.

2) Prove that if (X, d) is bounded, then it is locally compact iff $(\mathfrak{C}(X), d_H)$ is locally compact.

3) Prove that if (X, d) is bounded, then $(\mathfrak{C}(X), d_H)$ is paracompact.

4) Let $(\mathbb{R}, d)$ be the real line with the standard metric. Construct an uncountable discrete subset of $(\mathfrak{C}(\mathbb{R}), d_H)$.

5) Prove that if $(\mathbb{R}, d)$ is the real line with the standard metric, the topology induced by $(\mathfrak{C}(\mathbb{R}), d_H)$ is not locally paracompact (note that d_H is not a metric on $\mathfrak{C}(\mathbb{R})$).

6) Prove Theorem 2.1, namely that a compact metric space induces a compact Hausdorff metric.

7) Prove Theorem 4.1, namely that $\widehat{d_{GH}}$ is equivalent to d_{GH}.

8) Prove that the pointed Gromov-Hausdorff topology is not second countable.

9) In a previous lecture, we constructed a sequence of metrics g_δ on a nilmanifold $\Gamma\backslash N$, where N was the Heisenberg group and Γ was the integer lattice. Prove that given any ϵ, there is a δ to that the map $\Gamma\backslash N \mapsto pt$ is an ϵ-GHA.

10) In a previous lecture, we constructed a family of metrics g_t, the Berger metrics, on the manifold $\mathbb{S}^3$. Prove that the Hopf map $\mathbb{S}^3 \mapsto \mathbb{S}^2$ induces a πt-GHA from $(\mathbb{S}^3, g_t)$ to $(\mathbb{S}^2, 4g_{\mathbb{S}^2})$ where $g_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$.