Lecture 4 - The Basic Examples of Collapse

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1 Berger Spheres

Let X, Y, and Z be the left-invariant vector fields on $\mathbb{S}^3$ that restrict to i, j, and k at the identity. This is a global frame for $\mathbb{S}^3$. Let η, μ, and λ be the dual frame. Let g_t be the metric

$$g_t \;=\; t^2\eta\otimes\eta \,+\, \mu\otimes\mu \,+\, \lambda\otimes\lambda. \tag{1}$$

This is clearly left-invariant for each t, but bi-invariant only when $t = 1$. We compute (maybe using software)

$$\begin{array}{lll} \nabla_X X \;=\; 0 & \nabla_X Y \;=\; (2-t^2)Z & \nabla_X Z \;=\; (-2+t^2)Y \\ \nabla_Y X \;=\; -t^2 Z & \nabla_Y Y \;=\; 0 & \nabla_Y Z \;=\; X \\ \nabla_Z X \;=\; t^2 Y & \nabla_Z Y \;=\; -X & \nabla_Z Z \;=\; 0 \end{array} \tag{2}$$

and

$$\begin{array}{lll} \mathrm{Rm}(X,Y)Y \;=\; t^2 X & \mathrm{Rm}(X,Z)Z \;=\; t^2 X & \mathrm{Rm}(X,Y)Z \;=\; 0 \\ \mathrm{Rm}(Y,X)X \;=\; t^4 Y & \mathrm{Rm}(Y,Z)Z \;=\; (4-3t^2)Y & \mathrm{Rm}(Y,Z)X \;=\; 0 \\ \mathrm{Rm}(Z,X)X \;=\; t^4 Z & \mathrm{Rm}(Z,Y)Y \;=\; (4-3t^2)Z & \mathrm{Rm}(Z,X)Y \;=\; 0. \end{array} \tag{3}$$

Therefore

$$sec(X,Y) \;=\; t^2, \qquad sec(X,Z) = t^2, \qquad sec(Y,Z) = 4-3t^2. \tag{4}$$

Note that, in any g_t, X integrates out to a geodesic path. The integral curves obtained are simply a loops of length $2t\pi$. Therefore inj $\le t\pi$ (actually equality holds), so we indeed have injectivity radius collapse, and at the same time bounded curvature.

It appears that $(\mathbb{S}^3, g_t)$ is converging to a 2-dimensional space of constant curvature 4. In a way that will be made precise later (using the Gromov-Hausdorff distance), we shall prove that indeed $\mathbb{S}^3$ is converging to the 2-sphere of half-radius.

2 Effective Torus Actions

The examples of the warped product and of the 3-sphere have a common generalized. Berger's collapse is just the scaling of the metric along the orbits of the circle-action while leaving the metric unchanged on the perpendicular distribution. Now suppose a torus T^k acts effectively and isometrically on a Riemannian manifold M^{n+k}, and that isotropy groups are all discrete. Now M supports an integrable tangential distribution (of dimension k) and a perpendicular distribution (of dimension n). The metric can be likewise decomposed: letting $T_p \subset T_pM$ and $P_p \subset T_pM$ indicate the tangential and perpendicular distributions, respectively, we can write $g = g_T + g_P$.

Set $g_\delta = \delta^2 g_T + g_P$, and pick a point $p \in M$; we will estimate the sectional curvatures at p. Let $N^n \subset M^{n+k}$ be a transverse submanifold, defined in a neighborhood of p, that contains p. Let $y^1, \ldots, y^n$ be coordinates on N with $p = (0, \ldots, 0)$. Let $\tilde{x}^1, \ldots, \tilde{x}^k$ be coordinates on the torus T^k with the identity e having coordinates $x^i = 0$. Since isotropy groups are discrete, we can push these coordinate functions forward to functions $x^1, \ldots, x^k$ locally near p, where $x^1 = \cdots = x^k = 0$ on the transversal N. Finally extend the functions $y^1, \ldots, y^n$ to a neighborhood of p by projection along the fibers onto N. This gives a coordinate system $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ in a neighborhood of p.

The coordinate fields $\frac{\partial}{\partial x^i}$ are tangent to the fibers, although the fields $\frac{\partial}{\partial y^i}$ are not necessarily tangent to the transversal. Write $\frac{\partial}{\partial y^i} = X_i + V_i$ where the fields X_i are parallel to the fibers and the fields V_i are perpendicular to the fibers.

The original metric has the form

$$g = \begin{pmatrix} A & B \\ B & C + D \end{pmatrix},$$

where $A_{ij} = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle$, $B_{ij} = \left\langle \frac{\partial}{\partial x^i}, X_j \right\rangle$, $C_{ij} = \langle X_i, X_j \rangle$, and $D_{ij} = \langle V_i, V_j \rangle$. Note that these matrices are functions of the coordinates y^i only, since the torus action is isometric. The new metrics have the form

$$g_\delta = \begin{pmatrix} \delta^2 A & \delta^2 B \\ \delta^2 B & \delta^2 C + D \end{pmatrix}.$$

This metric is singular in the limit and it is not clear that sectional curvature remains bounded. Now make a change of coordinates: $u^i = \delta x^i$, and let $\tilde{A}_{ij} = \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle$ and $\tilde{B}_{ij} = \left\langle \frac{\partial}{\partial u^i}, X_i \right\rangle$. Then in the new coordinates we have

$$g_\delta = \begin{pmatrix} \delta^2 \tilde{A} & \delta^2 \tilde{B} \\ \delta^2 \tilde{B} & \delta^2 C + D \end{pmatrix} = \begin{pmatrix} A & \delta B \\ \delta B & \delta^2 C + D \end{pmatrix}$$

$$\lim_{\delta \to 0} g_\delta = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

a generalized warped product metric. It is clear now that the g_δ have bounded curvature. It is also possible to prove that injectivity radii converge to 0, so we indeed have a prototype for collapse with bounded curvature.

Theorem of Cheeger-Gromov: This example is in essence the only kind of collapse with bounded curvature, at least on the scale of the injectivity radius.

3 $\mathbb{S}^3$ again

$\mathbb{S}^3$ has a variety of circle actions, not all of then induced by left- or right-invariant fields. Consider the circle action

$$t \mapsto \left(e^{2\pi i p t} z,\, e^{2\pi i q t} w\right)$$

where p/q is rational. Shrinking directions along these flow lines and leaving perpendicular directions unchanged, we get collapse with bounded curvature from the procedure in Section 2. The limiting object is an orbifold with zero, one, or two singular points.

4 Nilmanifolds

A fundamentally different form of collapse is seen in the collapse of a nilmanifold to a point, which we now describe. Nilmanifolds provide the prototype for collapse with bounded curvature.

4.1 Nilpotent Groups

Let N be a connected, simply-connected nilpotent Lie group, with Lie algebra $\mathfrak{n}$. It can be proven that any such group is a subgroup of some group of upper triangular matrices with 1's along the diagonal, and is therefore diffeomorphic to some $\mathbb{R}^n$.

Any (finite-dimensional) nilpotent Lie group N is isomorphic to a group of $n \times n$ upper-triangular matrices with 1's along the diagonal; it's Lie algebra $\mathfrak{n}$ is a Lie algebra of strictly upper triangular matrices (ie. with 0's along the diagonal). Given $q > 0$ let g_q be the inner product formed by polarizing the norm

$$|A|_q^2 \;=\; \sum_{ij} (a_{ij})^2\, q^{2(j-i)} \tag{5}$$

on $\mathfrak{n}$, and extend to a left-invariant metric on N. The Cauchy-Schwartz inequality easily produces the estimate

$$|[X,Y]|_q^2 \;=\; |XY - YX|_q^2 \;\le\; 2|X|_q^2 |Y|_q^2, \tag{6}$$

which we use to derive the estimates

$$|\nabla_X Y|_q \;\le\; 3|X|_q^2 |Y|_q^2 \tag{7}$$

$$|\,\mathrm{Rm}(X,Y)Z|_q \;\le\; 24|X|_q |Y|_q |Z|_q \tag{8}$$

This implies that sectional curvatures are bounded independently of q.

4.2 The Heisenberg group

To see this is action, consider again $\mathcal{H}$. The metric is $|P|_q^2 = |Q|_q^2 = q^2$ and $|h|_q^2 = q^4$. We compute

$$\begin{array}{lll} \nabla_P P \;=\; 0 & \nabla_Q P \;=\; -\frac{1}{2}h & \nabla_h P \;=\; -\frac{1}{2}Qq^2 \\ \nabla_P Q \;=\; \frac{1}{2}h & \nabla_Q Q \;=\; 0 & \nabla_h Q \;=\; \frac{1}{2}Pq^2 \\ \nabla_P h \;=\; -\frac{1}{2}Qq^2 & \nabla_Q h \;=\; \frac{1}{2}Pq^2 & \nabla_h h \;=\; 0 \end{array} \tag{9}$$

and

$$\begin{array}{ll} \mathrm{Rm}(P,Q)Q \;=\; -\frac{3}{4}Pq^2 & \mathrm{Rm}(P,h)h \;=\; \frac{1}{4}Pq^4 \\ \mathrm{Rm}(Q,h)h \;=\; \frac{1}{4}Qq^2 & \mathrm{Rm}(Q,P)P \;=\; -\frac{3}{4}Qq^2 \\ \mathrm{Rm}(h,P)P \;=\; \frac{1}{4}hq^2 & \mathrm{Rm}(h,Q)Q \;=\; \frac{1}{4}hq^2. \end{array} \tag{10}$$

Finally

$$sec(P,Q) \;=\; -\frac{3}{4}, \quad sec(P,h) \;=\; \frac{1}{4}, \quad \text{and} \quad sec(Q,h) \;=\; \frac{1}{4}, \tag{11}$$

so sectional curvatures actually remain constant.

4.3 Compact Nilmanifolds

A nilmanifold is a quotient of a nilpotent Lie group by a co-compact lattice.

Let $\Gamma \subset N$ be the integer lattice, or any other co-compact discrete subgroup. This is a discrete subgroup, but unless $N \subset Z(H)$, it is not normal. Still, we can form the left-coset space $\Gamma\backslash N$.

Any left-invariant metric descends to the quotient. Note also that left-invariant vector fields descend to the quotient. Because they integrate out to right translations, and because right translations commute with the action of Γ which is on the left, of course the flow passes to the quotient as well.

To make use of the example from Subsection 4.2, let $N = \mathcal{H}$ and let $\Gamma \subset N$ the group of upper triangular matrices with integral entries and 1's along the diagonal. We see that the quotient is a twisted circle-bundle over a torus; this is the prototype for Thurston's nilgeometry.

The metric on $\mathbb{H}$ is invariant under left translation by Γ so the metric descends to the quotient $M = \Gamma\backslash\mathcal{H}$. As $q \to 0$ the diameter of M goes to zero while its sectional curvature remains bounded. Thus M is an example of an "almost-flat manifold", one which supports a sequence of metrics with $\text{diam}(M)^2 \cdot \max|\sec(M)| \searrow 0$. Later we shall see that there is no flat metric on M.

Gromov's almost-flat manifold theorem states that the only manifolds that collapse to a point with bounded curvature are quotients of nilmanifolds.

5 Solvemanifolds

This previous example can be extended to the solvegeometry; this is an example of Fukaya. Let G be a solvable Lie group, given by the upper triangular $n \times n$ matrices, with Lie algebra $\mathfrak{g}$. Define a sequence of normal subgroups by $G^0 = G$, $G^1 = G' = [G, G]$ and $G^k = [G', G^{k-1}]$, and put on a metric g_q so that $v \in G^i$ and v is perpendicular to G^{i-1} gives $|v|^2 = q^{2i}$.

Let $\Gamma \subset G$ be a cocompact discrete subgroup, for instance the integer subgroup. Now sending $q \to 0$ we get collapse to a manifold $\Gamma\backslash(G/G^1)$, which is isometric to the torus T^n. The "collapsed" directions constitute a fibration by nilmanifolds, each isomorphic to $(G^1 \cap \Gamma)\backslash G$. Indeed this produces a fiber bundle

$$(G^1 \cap \Gamma)\backslash G^1 \to \Gamma\backslash G \to \Gamma\backslash G/G^1.$$

This is an example of Fukaya's theorem, than any collapse to a lower dimensional Riemannian manifold produces a fiber bundle, where the fibers are nilmanifolds.

6 Exercises

1) On the Berger spheres, we cannot expect left-invariant fields to be Killing. Which left-invariant fields, if any, remain Killing?

2) Above we considered the $\mathbb{R}$-linear map $P : \mathbb{H} \to \mathbb{C} \oplus \mathbb{C}$, which of course in not algebraic, given by

$$P \;:\; q \;=\; z + wj \quad \mapsto \quad (z, w). \tag{12}$$

Note the existence of a second map $P' : \mathbb{H} \to \mathbb{C} \oplus \mathbb{C}$

$$P' \;:\; q \;=\; a + jb \quad \mapsto \quad (a, b). \tag{13}$$

Prove that $P'(q)$ and $P(q)$ are related by complex conjugation in the second variable (so properly we should write $P' : \mathbb{H} \to \mathbb{C} \oplus \overline{\mathbb{C}}$). Prove that any $\mathbb{R}$-linear map from $\mathbb{H}$ to $\mathbb{C} \oplus \mathbb{C}$ with $1 \mapsto (1, 0)$ is, up to quaternionic conjugation, one of these two.

3) With quaternions written as $q = z + wj$, let $H : \mathbb{S}^3 \to \mathbb{C}^* \approx \mathbb{S}^2$ be given by $q \mapsto P(q) = (z, w) \mapsto z\overline{w}^{-1}$; this is called the *Hopf map*. A second map, $\overline{H}$, is given by

$q \mapsto P'(q) = (z, \overline{w}) \mapsto zw^{-1}$, the *anti-Hopf map*. Show that any flow line of the left-invariant field X_L on $\mathbb{S}^3$ is mapped to a point under H, and any flow line of the right-invariant field X_R is mapped to a point by $\overline{H}$.

4) Prove (6), (7), and (8).