Lecture 1 - Basic Concepts I - Riemannian Geometry

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These lectures are entirely expository and no originality is claimed. Where necessary, references are indicated in the text.

1 Collapsing

Collapse in Riemannian geometry is the phenomenon of injectivity radii limiting to zero, while sectional curvatures remain bounded.

Any compact Riemannian manifold converges to a point by multiplying distances by a constant δ and letting $\delta \to 0$. However only in the flat case does this lead to collapse with bounded curvature, as sectional curvatures scale by δ^{-2}. What is meant by "converge to a point" will be made precise later, but note that such a manifold's volume and diameter both converge to zero.

However many Riemannian manifolds do admit some collapsing process that leaves curvature bounded, and most of these manifolds admit no flat metric whatsoever. The first example, both historically, and in these notes, is the family of Berger 3-spheres. This is a sequence of metrics on $\mathbb{S}^3$ that converges, in an appropriate sense, to the round metric of half radius on $\mathbb{S}^2$.

For reasons that are not apparent at the outset, Lie groups are fundamental objects in the study of collapse. However in a way this is fortunate. Lie groups, and their finite quotients, have so much structure that a precise understanding of their geometry is often possible. In the first two lectures, we review the aspects of Lie group theory, differential topology, and Riemannian geometry most needed for our study.

The definitions involved in the study of collapse are, unfortunately, rather complex in places. To put a statement of Isaacs' into different words, the heart of mathematics is examples and theorems. Definitions, as they are just putting a name to something, are essentially empty of content. But they are a necessary evil, as one does, after all, have to know what one is talking about. In this series of lectures I will put an emphasis on examples, where one obtains a real feel for collapsing. The fundamental theorems in the subject are the Bieberbach theorem, Gromov's almost flat manifold theorem, the Cheeger-Gromov F-structure theorem, Fukaya's theorems, and the Cheeger-Fukaya-Gromov N-structure theorem.

2 Why Study Collapsing?

In mathematics, we seek information on large classes of objects (eg. all compact n-manifolds with bounded diameter, curvature, and injectivity radius, or all 4-dimensional compact Einstein manifolds with two-sided volume bounds), rather than on single objects (as, for instance, in the work of engineers). Still, it is much easier to study individual objects. If we want to know if a property inheres in a certain class of objects, it is useful to take a sequence of objects, then ask questions about properties of the limit.

But in geometry, in what sense can limits be taken? Under what circumstances do they exist?

Recall the statement of the Cheeger diffeofiniteness theorem. The set of Riemannian manifolds with bounded sectional curvature, bounded diameter, and bounded injectivity radius has finitely many diffeomorphism types. Thus, given a sequence of such manifolds, a subsequence can be chosen with the same underlying differentiable structure. Limits of tensors, such as the metric, can then be taken in an ordinary sense, modulo diffeomorphism. On any compact manifold, one of the three quantities can be controlled simply by scaling; it is normal to assume $\max|sec| = 1$. If the diameter is unbounded, limits can still be taken, once a fixed point is specified. However, if the injectivity radius collapses, there is no immediate way to know what can happen.

Another fundamental difficulty comes from geometric analysis. It is very natural to study elliptic differential equations on Riemannian manifolds. Elementary (and indispensable) tools in the analysis on manifolds include integration by parts, the maximum principle, the Bochner technique, and the Sobolev inequality (which, for instance, makes Moser iteration possible). While the other tools enjoy unrestricted applicability on compact manifolds, the Sobolev inequality is equivalent to the isoperimetric inequality, so if injectivity radii degenerate, the isopoerimetric inequality degenerates as well. The loss of this fundamental tool has broad consequences in, for instance, the study of special metrics such as Einstein metrics or extremal Kähler metrics.

Unfortunately it is difficult, if not impossible, to control the injectivity radius in terms of either local (curvature) or global (topological) data. It can be thought of as a, semi-local, or semi-global variable, that is nearly (though admittedly not totally) independent of other geometric and topological data. Since injectivity radii cannot themselves be controlled, then understanding geometric and topological phenomena in the presence of collapsing injectivity radii is a fundamental question of Riemannian geometry.

3 Riemannian Geometry

3.1 The Metric

The metric on a manifold is the basic object of Riemannian geometry. Given a differentiable manifold M, the metric g is simply an assignment of a bilinear map $T_pM \otimes T_pM \to \mathbb{R}$ at each point $p \in M$ with the following properties:

i) $g(\mathbf{v}, \mathbf{w}) = g(\mathbf{w}, \mathbf{v})$ (symmetry)

ii) $g(\mathbf{v}, \mathbf{v}) > 0$ when $\mathbf{v} \neq 0$ (positive definiteness), and

iii) g varies differentiably.

Example: $\mathbb{S}^3$ and stereographic projection

We will look at a very concrete case, that of the ordinary sphere. As a subset of $\mathbb{R}^4$, it is defined to be

$$\mathbb{S}^4 \quad \triangleq \quad \{\, (x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1 \,\}. \tag{1}$$

A vector $\mathbf{v} = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z} + \delta\frac{\partial}{\partial w}$ at the point (x, y, z, w) lies tangent to $\mathbb{S}^3$ if $\alpha x + \beta y + \gamma z + \delta w = 0$. We take the inner product at $T_{(x,y,z,w)}\,\mathbb{S}^3$ to be simply the restriction of the $\mathbb{R}^4$ inner product to tangent vectors. That is, if $\mathbf{v}_1, \mathbf{v}_2 \in T_{(x,y,z,w)}\,\mathbb{S}^3$ are

$$\begin{aligned} \mathbf{v}_1 &= \alpha_1\frac{\partial}{\partial x} + \beta_1\frac{\partial}{\partial y} + \gamma_1\frac{\partial}{\partial z} + \delta_1\frac{\partial}{\partial w} \\ \mathbf{v}_2 &= \alpha_2\frac{\partial}{\partial x} + \beta_2\frac{\partial}{\partial y} + \gamma_2\frac{\partial}{\partial z} + \delta_2\frac{\partial}{\partial w} \end{aligned} \tag{2}$$

then we define

$$g(\mathbf{v}_1,\, \mathbf{v}_2) \quad = \quad \alpha_1\alpha_2 \,+\, \beta_1\beta_2 \,+\, \gamma_1\gamma_2 \,+\, \delta_1\delta_2 \tag{3}$$

It is easily seen that (*i*)-(*iii*) are satisfied by this definition.

This definition of the metric is elegant, but is awkward for carrying out computations (although one has the Gauss and Codazzi equations). For explicit computations, it is useful to choose a coordinate system. One particularly natural coordinate system is obtained by *stereographic projection*, which is a mapping of $\mathbb{S}^2 \setminus \{(0,0,0,1)\}$ diffeomorphically onto $\mathbb{R}^3$. Letting (a, b, c) be the coordinates on $\mathbb{R}^3$, then the diffeomorphism $\varphi : \mathbb{S}^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ and its inverse φ^{-1} are

$$\begin{aligned} \varphi(x,\, y,\, z,\, w) &= \left(\frac{x}{1-w},\, \frac{y}{1-w},\, \frac{z}{1-w}\right) \\ \varphi^{-1}(a,\, b) &= \left(\frac{2a}{r^2+1},\, \frac{2b}{r^2+1},\, \frac{2c}{r^2+1},\, \frac{r^2-1}{r^2+1}\right) \end{aligned} \tag{4}$$

where we have abbreviated $r^2 = a^2 + b^2 + c^2$. Viewing the coordinate components a, b, x, y, and z strictly as functions on $\mathbb{S}^3$, we have the identifications

$$\begin{aligned} a &= \frac{x}{1-w}, \quad b = \frac{y}{1-w}, \quad c = \frac{z}{1-w} \\ x &= \frac{2a}{r^2+1}, \; y = \frac{2b}{r^2+1}, \; z = \frac{2c}{r^2+1}, \; w = \frac{r^2-1}{r^2+1} \end{aligned} \tag{5}$$

To compute the metric as a function of the coordinates (a, b), we compute

$$\begin{aligned} \frac{\partial}{\partial a} &= \frac{\partial x}{\partial a}\frac{\partial}{\partial x} + \frac{\partial y}{\partial a}\frac{\partial}{\partial y} + \frac{\partial z}{\partial a}\frac{\partial}{\partial z} + \frac{\partial w}{\partial a}\frac{\partial}{\partial w} \\ &= \frac{2r^2+2-4a^2}{(r^2+1)^2}\frac{\partial}{\partial x} + \frac{-4ab}{(r^2+1)^2}\frac{\partial}{\partial y} + \frac{-4ac}{(r^2+1)^2}\frac{\partial}{\partial z} + \frac{4a}{(r^2+1)^2}\frac{\partial}{\partial z} \\ &= (1-x^2-w)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} - xz\frac{\partial}{\partial z} + x(1-w)\frac{\partial}{\partial w} \\ \frac{\partial}{\partial b} &= -yx\frac{\partial}{\partial x} + (1-y^2-w)\frac{\partial}{\partial y} - yz\frac{\partial}{\partial z} + y(1-w)\frac{\partial}{\partial w} \\ \frac{\partial}{\partial c} &= -zx\frac{\partial}{\partial x} - zy\frac{\partial}{\partial y} + (1-z^2-w)\frac{\partial}{\partial z} + z(1-w)\frac{\partial}{\partial w} \end{aligned} \tag{6}$$

Therefore

$$\begin{aligned} g\left(\frac{\partial}{\partial a}, \frac{\partial}{\partial a}\right) &= (1-x^2-w)^2 + (xy)^2 + (xz)^2 + (x-wx)^2 = (1-w)^2 \\ &= \frac{4}{(r^2+1)^2} \\ g\left(\frac{\partial}{\partial b}, \frac{\partial}{\partial b}\right) &= \frac{4}{(r^2+1)^2}, \quad g\left(\frac{\partial}{\partial c}, \frac{\partial}{\partial c}\right) = \frac{4}{(r^2+1)^2} \end{aligned} \tag{7}$$

and all other inner products are zero. Therefore the metric in the (a, b, c)-coordinate system is written

$$g = \frac{4}{(a^2+b^2+c^2+1)^2}\left(da \otimes da + db \otimes db + dc \otimes dc\right) \tag{8}$$

Note that g is conformal to the Euclidean metric on $\mathbb{R}^3$.

3.2 The Connection

The second most basic object in Riemannian geometry is the connection ∇. Given a vector field X, the connection is an operator $\nabla_X : \bigotimes^{r,s} TM \to \bigotimes^{r,s} TM$, and the tensor field $\nabla_X T$ is called the *covariant derivative* of the tensor T in the X direction. The operator ∇_X is defined by requiring that it have the following characteristics:

I) ∇ is tensorial in the first variable (the "X" variable)

II) ∇ is linear in the second variable

III) ∇ is the ordinary derivative on functions: $\nabla_X f = X(f)$ (equivalently, $\nabla f = df$)

IV) ∇ obeys the Leibnitz rule for contractions: $\nabla_X\left(i_Y\eta\right) \;=\; i_{\nabla_X Y}\eta \,+\, i_Y\nabla_X\eta$ where Y is a vector field and η is a covector field, as well as for tensor products: $\nabla_X(T\otimes S) = (\nabla_X T)\otimes S \,+\, T\otimes(\nabla_X S)$ where T and S are arbitrary tensor fields.

V) the metric is covariant-constant: $\nabla_X g \;\equiv\; 0$

VI) ∇ is "torsion-free": $\nabla_X Y \,-\, \nabla_Y X \;=\; [X,\,Y]$.

An operator ∇ that satisfies (I)-(IV) is called an affine connection (or just a connection), if it also satisfies (V) it is called a metric connection, and if it satisfies (I)-(VI) it is called the Riemannian or Levi-Civita connection. From (I)-(VI), the covariant derivative of any vector field Y is determined uniquely by the *Koszul formula*

$$\begin{aligned} 2\left\langle \nabla_X Y,\, Z\right\rangle = \;& X\left\langle Y,\,Z\right\rangle \,+\, Y\left\langle X,\,Z\right\rangle \,-\, Z\left\langle X,\,Y\right\rangle \\ & +\left\langle [X,Y],\,Z\right\rangle \,-\, \left\langle [Y,Z],\,X\right\rangle \,+\, \left\langle [Z,X],\,Y\right\rangle . \end{aligned} \tag{9}$$

Example: $\mathbb{S}^3$

Using the coordinates (a,b,c) on $\mathbb{S}^3$, the Koszul formula allows us to compute

$$\begin{aligned} \nabla_{\partial/\partial a}\partial/\partial a &= -\frac{2a}{r^2+1}\frac{\partial}{\partial a} + \frac{2b}{r^2+1}\frac{\partial}{\partial b} + \frac{2c}{r^2+1}\frac{\partial}{\partial c} \\ \nabla_{\partial/\partial b}\partial/\partial b &= \frac{2a}{r^2+1}\frac{\partial}{\partial a} - \frac{2b}{r^2+1}\frac{\partial}{\partial b} + \frac{2c}{r^2+1}\frac{\partial}{\partial c} \\ \nabla_{\partial/\partial a}\partial/\partial b \;=\; \nabla_{\partial/\partial b}\partial/\partial a &= -\frac{2b}{r^2+1}\frac{\partial}{\partial a} - \frac{2a}{r^2+1}\frac{\partial}{\partial b} \\ \nabla_{\partial/\partial a}\partial/\partial c \;=\; \nabla_{\partial/\partial c}\partial/\partial a &= -\frac{2c}{r^2+1}\frac{\partial}{\partial a} - \frac{2a}{r^2+1}\frac{\partial}{\partial c} \\ \nabla_{\partial/\partial b}\partial/\partial c \;=\; \nabla_{\partial/\partial c}\partial/\partial b &= -\frac{2b}{r^2+1}\frac{\partial}{\partial c} - \frac{2c}{r^2+1}\frac{\partial}{\partial b} \end{aligned} \tag{10}$$

3.3 Curvature

The third and final basic object in Riemannian geometry is the Riemann tensor. At a point p the Riemann tensor $\mathrm{Rm} : T_pM\otimes T_pM\otimes T_pM \to T_pM$ is defined by

$$\mathrm{Rm}\left(X,\,Y\right)\,Z \quad=\quad \nabla_X\nabla_Y Z \,-\, \nabla_Y\nabla_X Z \,-\, \nabla_{[X,Y]}Z. \tag{11}$$

This tensor measures the failure of mixed second partial derivatives of vector fields to commute. Thus the Riemann tensor is fundamentally an analytic object.

Another viewpoint is that the Riemann tensor at a point measures *infinitesimal holonomy*, or, intuitively, the failure of parallel translation to be Euclidean-like. Let X and Y be vector fields. Imagine a path starting from a point p, following the integral line for X for a length of time t, then an integral line for Y for time t, then X again for time $-t$, then Y for time $-t$. One terminates at a point p', which is not necessarily the starting point. However, $\mathrm{dist}(p', p)/t \to 0$ as $t \searrow 0$; sometimes this is stated "boxes always close to first order". Now imagine parallel-transporting the vector Z around the four segments to p', and then back to p via a short geodesic. At p we have 2 vectors, the original vector Z, and the translated vector Z' which is a function of t. It can be shown that $\mathrm{Rm}(X, Y)Z = -\frac{1}{2}\frac{d^2}{dt^2}Z'$.

A third interpretation of the Riemann tensor, equally fundamental, is that it measures the "bending" of the manifold at a point. Thus the Riemann tensor controls geometric phenomena such as volume growth of balls and the lensing of geodesics. The sectional, or Gaussian, curvature of the infinitesimal plane spanned by the vectors X and Y at a point is

$$sec(X,\,Y) \;=\; \frac{\langle \mathrm{Rm}(X,\,Y)\,Y,\,X\rangle}{|X|^2|Y|^2-\langle X,\,Y\rangle^2}. \tag{12}$$

The Riemann tensor can be recovered if sec is known for all X and Y.

The Riemann tensor contains all the local geometric information on a Riemannian manifold. In coordinates, Rm is an $n\times n\times n\times n$ array at each point, and so carries a huge amount of information (of course there are some redundancies, due to the Bianchi identity, etc). A curvature tensor that is more manageable, but has less information in general that Rm is the Ricci tensor

$$\mathrm{Ric}_{ij} \;=\; \mathrm{Rm}_{ijkl}\,g^{jk}. \tag{13}$$

Even simpler yet is the scalar curvature function

$$R \;=\; \mathrm{Ric}_{ij}\,g^{ij} \tag{14}$$

which unfortunately carries very little information in general.

Example: $\mathbb{S}^3$

One computes (preferably with the help of a computer) that

$$\begin{aligned}
\mathrm{Rm}\left(\frac{\partial}{\partial a},\frac{\partial}{\partial b}\right)\frac{\partial}{\partial b} &= \frac{4}{(r^2+1)^2}\,\frac{\partial}{\partial a}\\
\mathrm{Rm}\left(\frac{\partial}{\partial b},\frac{\partial}{\partial c}\right)\frac{\partial}{\partial c} &= \frac{4}{(r^2+1)^2}\,\frac{\partial}{\partial b}\\
\mathrm{Rm}\left(\frac{\partial}{\partial c},\frac{\partial}{\partial a}\right)\frac{\partial}{\partial a} &= \frac{4}{(r^2+1)^2}\,\frac{\partial}{\partial c}\\
sec\left(\frac{\partial}{\partial a},\frac{\partial}{\partial b}\right) = 1,\quad sec\left(\frac{\partial}{\partial b},\frac{\partial}{\partial c}\right) &= 1,\quad sec\left(\frac{\partial}{\partial c},\frac{\partial}{\partial a}\right) = 1.
\end{aligned} \tag{15}$$

Warped Products

A very fundamental example is the 2-dimensional warped product. Consider the topological manifold $I \times \mathbb{S}^1$, where $I = (a, b)$ is an open interval. We can use coordinates (r, θ), where r is the projection on to I, and $\theta \in (-\pi, \pi)$ parametrizes the circle outside of a single point. Note that the fields $\frac{\partial}{\partial \theta}$ and $d\theta$ can be taken to be global.

We put a metric on $I \times \mathbb{S}^1$ that leaves distances along on I unchanged but scales distances along $\mathbb{S}^1$ by a function of r alone. In coordinates we have

$$g \;=\; dr \otimes dr \,+\, f(r)^2 \, d\theta \otimes d\theta \tag{16}$$

for some positive smooth $f : I \to \mathbb{R}$. We easily compute the connections

$$\begin{aligned} \nabla_{\partial/\partial r} \partial/\partial r \;&=\; 0 \\ \nabla_{\partial/\partial \theta} \partial/\partial \theta \;&=\; -f(r)\, f'(r)\, \frac{\partial}{\partial r} \\ \nabla_{\partial/\partial \theta} \partial/\partial r \;&=\; \nabla_{\partial/\partial r} \partial/\partial \theta \;=\; \frac{f'(r)}{f(r)} \frac{\partial}{\partial \theta}, \end{aligned} \tag{17}$$

the non-zero components of the Riemann tensor

$$\begin{aligned} \mathrm{Rm}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial \theta} \;&=\; -f(r) f''(r) \frac{\partial}{\partial r} \\ \mathrm{Rm}\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} \;&=\; \frac{f''(r)}{f(r)} \frac{\partial}{\partial \theta}, \end{aligned} \tag{18}$$

and the sectional curvature

$$sec\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right) \;=\; -\frac{f''(r)}{f(r)}. \tag{19}$$

Notice that this provides an enormous number of examples of collapse with bounded curvature, for if f is multiplied by any constant c, then sec is unchanged. Letting $c \to 0$ provides examples of 2-dimensional Riemannian manifolds that "converge" to line segments.

4 Exercises

1) Verify (10), (15), (17), (18), and (19).

2) Write down a warped product metrics for $\mathbb{R}^2$, $\mathbb{S}^2$, and $\mathcal{H}^2$ (hyperbolic 2-space). (Hint: set $-f''/f$ equal to a constant.)

3) There is an obvious $\mathbb{R}$-linear vector space isomorphism $\mathbb{R}^4 \to \mathbb{C} \oplus \mathbb{C}$, so we can view $\mathbb{S}^3$ as sitting inside $\mathbb{C}^2$. Give $\mathbb{C}^2$ the coordinates (z, w). The z-axis is defined to be the complex line $\{w = 0\}$, and similarly for the w-axis. Note that the intersection of $\mathbb{S}^3$ with either axis is a circle. Under stereographic projection, what are the images of the z and w axes?

4) The 3-sphere is enormously symmetric. The embedding $\mathbb{S}^2 \hookrightarrow \mathbb{C}^2$ gives us one view of this. There is a free, isometric $\mathbb{S}^1$-action on $\mathbb{S}^3$ given by the restriction of the action $e^{i\theta} : \mathbb{C}^2 \to \mathbb{C}^2$, $(z, w) \mapsto (e^{i\theta} z, e^{i\theta} w)$ to $\mathbb{S}^3$ (note that no free action of $\mathbb{S}^1$ exists on $\mathbb{S}^2$). This is called the *Hopf action*. Under stereographic projection, what are the orbits?

5) There is a mapping $\mathbb{S}^3 \to \mathbb{C}^*$ (where $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ and ∞ is the point at infinty), called the *Hopf map*, given by $(z, w) \mapsto zw^{-1}$. Since $\mathbb{C}^*$ is the same as $\mathbb{S}^2$, we can regard this as a map $\mathbb{S}^3 \to \mathbb{S}^2$. Prove that this is a submersion, and that the fibers of this submersion are just the orbits of the Hopf action.