Addendum - Further Modes of Collapse, and References

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1 Additional Theorems

The Bieberbach theorem characterizes compact flat manifolds. A *crystallographic group* is a subgroup of $\mathcal{O}(n) \ltimes \mathbb{R}^n$ that acts freely on $\mathbb{R}^n$ and has compact fundamental domain.

Theorem 1.1 (Bieberbach, 1911) *Assume $G \subset \mathcal{O}(n) \ltimes \mathbb{R}^n$ acts freely on $\mathbb{R}^n$. If $\alpha \in G$ then all principle rotational angles of $A(\alpha)$ are rational. If the translational parts of G span some subspace $S \subset \mathbb{R}^n$, then the pure translations of G span E.*

Since the translational parts of elements of a crystallographic group spans $\mathbb{R}^n$, it follows that a flat manifold is covered by a torus. Bieberbach was also able to use this with a theorem on group extensions to prove that there are only finitely many quotients. As a special case, Gromov's almost-flat manifold theorem implies stronger form of Bieberbach's theorem.

Theorem 1.2 *Let G be a crystallographic group. Then*

i) There is a translational, normal subgroup $\Gamma \triangleleft G$ of finite index (indeed $ind(\Gamma : G) < 2(4\pi)^{\frac{1}{2}n(n-1)}$).

ii) If $\alpha \in G$ then either α is a translation or the smallest nonzero principle angle $A(\alpha)$ is greater than $\frac{1}{2}$.

iii) Further, if $\alpha \in G$ and $0 < \theta_1 < \cdots < \theta_k$ are the nonzero principle rotational angles of $A = A(\alpha)$, then

$$\theta_l \geq \frac{1}{2}(4\pi)^{l-k}$$

Via (i), this formulation directly shows that there are finitely many flat manifolds of a given dimension.

Emplying a series of strikingly original techniques, Gromov proved a radical extension of the Bieberbach theorem. Let $d = d(M)$ is the diameter of M and $K = \max_{p \in M} |sec_p|$ is the largest sectional curvature that appears on M.

Theorem 1.3 (Gromov's almost flat manifold theorem) *There is an $\epsilon > 0$ so that if M is a compact Riemannian manifold and $d^2K < \epsilon$, then $\pi_1(M)$ has a nilpotent subgroup Γ of finite index, and M is a finite quotient of a nilmanifold.*

Fukaya proved an extension of Gromov's theorem. He assumes that a Riemannian manifold converges with bounded curvature, not to a point, but to another manifold, possibly of lower dimension.

Theorem 1.4 (Fukaya's Theorem) *Given n, $\mu > 0$, there is a number ϵ so that whenever N^n, M are Riemannian manifolds with $|sec| \le 1$, $\mathrm{inj}(N) > \mu$, and $d_{GH}(N, M) < \epsilon$, then there is a submersion $f : M \to N$ so that (M, N, f) is a fiber bundle, the fibers are quotients of nilmanifolds, and $e^{-\tau(\epsilon)} < |df(\xi)|/|\xi| < e^{\tau(\epsilon)}$.*

2 Further References

The original references for F-structures are the works of Cheeger-Gromov:

J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded I*, Journal of Differential Geometry, Vol. 23, No. 3 (1986) 309–346

J. Cheeger and M. Gromov, *Collapsing Riemannian manifolds while keeping their curvature bounded II*, Journal of Differential Geometry, Vol. 32, No. 1 (1990) 269–298

The first systematic study of collapse was Gromov's work on almost flat manifolds. This paper is difficult to read, but follow-on studies by Buser and Karcher clarified the proof:

M. Gromov, *Almost flat manifolds* Journal of Differential Geometry, Vol. 13 (1978) 231–241

P. Buser and H. Karcher, *The Bieberbach case in Gromov's almost flat manifold theorem*, Global Differential Geometry and Global Analysis (1981) Springer

H. Karcher, *Report on Gromov's almost flat manifolds* Séminaire Bourbaki Vol. 1978/79 Exposés 525–542 (1980) Springer

P. Buser and H. Karcher, *Gromov's almost flat manifold's*, *Astérique*, Soc. Math. France, Vol. 81 (1981) 1–148

In a series of papers, Fukaya studied the phenomena of the collapse of one manifold to another manifold of lower dimension:

K. Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions*, Journal of Differential Geometry, Vol. 25, No. 1 (1988) 139–156

K. Fukaya, *A boundary of the set of Riemannian manifolds with bounded curvatures and diameters*, Journal of Differential Geometry, Vol. 28, No 1 (1988) 1–21

K. Fukaya, *Collapsing Riemannian manifolds to ones of lower dimensions II*, Journal of the Mathematical Society of Japan, Vol. 41, No. 2 (1989) 333–356

N-structures were introduced by Cheeger-Fukaya-Gromov in 1992. It combined ideas of Gromov, Cheeger-Gromov, and Fukaya. It gives a complete picture of sufficiently collapsed Riemannian manifolds.

J. Cheeger, K. Fukaya, and M. Gromov, *Nilpotent structures and invariant metrics on collapsed manifolds*, Journal of the American Mathematics Society, Vol. 5, No. 2 (1992) 327–372

The existence of polarized F-structures in dimension 4 was studied by Rong:

X. Rong, *The existence of polarized F-structures on volume collapsed 4-manifolds*, Geometrics and Functional Analysis, Vol. 3, No. 5 (1993) 474–501

In higher dimensions, polarized F-structures were studied by Cheeger-Rong:

J. Cheeger and X. Rong, *Existence of polarized F-structures on collapsed manifolds with bounded curvature and diameter*, Geometric and Functional Analysis, Vol. 6, No. 3 (1996) 411–429

A recent paper of Naber-Tian explores the kinds of length spaces that can arise out of collapse with bounded curvature:

A. Naber and G. Tian, *Geometric structures of collapsing Riemannian manifolds, I*, arXiv:0804.2275