Math 600 Homework Assignments

Fall 2014

1 Basics of differentiable manifolds

Due September 29, 2014. Exercises marked with a "*" are optional.

1) Consider the locally Euclidean Hausdorff space $\mathcal{M}^2$ discussed in class. Prove that $\mathcal{M}^2$ is Hausdorff. Find two closed subsets of $\mathcal{M}^2$ that cannot be separated with open sets. Prove that $\mathcal{M}^2$ is non-normal and thus not metrizeable.

1.5)* Build a Hausdorff, locally Euclidean space that has an uncountable discrete subset, is non-normal, but is separable (has a a countable dense subset). *Hint*: Consider the subset of $\mathcal{M}^2$ that includes points in the right-hand half-plane and points of the form $(\gamma^b_m, 0)$, but excludes points of the form (γ^b_m, t) for $t > 0$. This is a "manifold" with boundary. Then do something to create a similar manifold without boundary.

2) Consider the group action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ given as follows. If $a \in \mathbb{Z}_2$ is the generator, then $a.(x, y, z) = (-x, -y, -z)$. Prove that $\mathbb{R}^2/\mathbb{Z}_2$ is homeomorphic to the cone over $\mathbb{R}P^2$.

3) Consider the following two differentiable structures on $\mathbb{R}$: Let $\mathcal{X}$ be the standard differentiable structure, and let $\mathcal{X}'$ be the differential structure where, if $I \subset \mathbb{R}^1$ is any interval and $f(x) = x^{1/3}$, then $f \in \mathcal{X}'(I)$. Find an isomorphism from $(\mathbb{R}, \mathcal{X}')$ to $(\mathbb{R}, C^\infty)$, and conclude that $(\mathbb{R}, \mathcal{X}')$ is indeed a differentiable manifold. Prove that $\mathcal{X}$ and $\mathcal{X}'$ are distinct differentiable structures, although the differentiable manifolds $(\mathbb{R}, \mathcal{X})$ and $(\mathbb{R}, \mathcal{X}')$ are diffeomorphic.

4) Show that $\mathbb{C}P^n$ is a differentiable manifold.

4.5)* Show that $\mathbb{C}P^n$ is a complex manifold.

5) Show that $\mathbb{C}P^1$ is the same manifold as $\mathbb{S}^2$.

6) Assume $F : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, and let $(x_0, y_0) \in \mathbb{R}^2$ be a point so that $F(x_0, y_0) = 0$. Under what conditions can you say that, at least sufficiently near (x_0, y_0), the level-set $\{(x, y) \,|\, F(x, y) = 0\}$ is the graph of some function $y = f(x)$?

7) Show that there is no differentiable space-filling curve. That is, if $f : \mathbb{R} \to \mathbb{R}^n$, $n > 1$, is differentiable, its image contains no open set (this is in marked contrast to the continuous case; look up the Hahn-Mazurkiewicz theorem).

8) (Related to 7) Show that a differentiable map $f : \mathbb{R}^2 \to \mathbb{R}^1$ cannot be 1-1. Conclude, in particular, that $\mathbb{R}^2$ and $\mathbb{R}^1$ are not the same differentiable manifold.

9) Prove that $T\mathbb{S}^1 \approx \mathbb{S}^1 \times \mathbb{R}$.

9.5)* Prove that $T\mathbb{S}^2$ is not $\mathbb{S}^2 \times \mathbb{R}^2$.

10) If M^n is a manifold, prove that the standard projection $TM^n \to M^n$ is a submersion, and the standard inclusion $M^n \hookrightarrow TM^n$ is an embedding.

2 Vector bundles

Due October 15, 2014. Exercises marked with a "*" are optional.

1) To get used to the "twisting" involved in forming vector bundles, we will do a few hands-on exercises with the Dirac monopoles. Vector fields on the charge n-monopole can be visualized by looking at their expressions on each of the two charts $V_i = D_i \times \mathbb{R}^2$ (a vector field on $V_i = D_i \times \mathbb{R}^2$ can, of course, be visualized in the usual way with pointed arrows). On V_1, assume the field is the constant field pointing to the right—specifically, in the chart $V_1 \times \mathbb{R}^2$ it has the form $(p, (1,0)^T)$ where $p \in D^1$. Draw, on V_2, a reasonable completion of the field for the following monopoles:

$$k = 0, \quad k = 1, \quad k = -1, \quad k = 2.$$

Make your drawings as simple as possible, but make sure the field is smooth. You should see sinks and/or sources and/or saddles of various degrees.

2) Show that if $R_k(-\theta)$ is replaced by $R_k(-\theta + c)$, any $c \in \mathbb{R}$, then the differentiable manifold resulting from the monopole construction does not change. Thus using $R_k(-\theta)$ or $-R_k(-\theta)$ makes no difference topologically; however extra factor of -1 creates some computational conveniences, which you'll see in problem (3).

3) Show that $T\mathbb{S}^2$ is the Dirac monopole of charge 2.
Hint: To create $\mathbb{S}^2$ consider the charts $\mathbb{R}^2 \approx U_1 = \big\{(r,\theta) \;\big|\; 0 \le r < \infty, \theta \in [0, 2\pi)\big\}$, and $\mathbb{R}^2 \approx U_2 = \big\{(\rho, \psi) \;\big|\; 0 \le \rho < \infty\big\}$ with transition $\varphi_{21}(r,\theta) = (r^{-1}, -\theta)$. Convert the transition φ_{21} to rectangular, and show that the transitions are $y^1 = \frac{x^1}{(x^1)^2+(x^2)^2}$, $y^2 = \frac{-x^2}{(x^1)^2+(x^2)^2}$ (precisely the stereographic projection transitions!). Then compute the transitions for $\frac{\partial}{\partial x^1}$, $\frac{\partial}{\partial x^2}$. Finally show that, when restricted to the circle $r = 1$, these transitions are exactly the monopole transitions for $k = 2$.

3.5)* If you know, for instance, the "hairy ball' theorem, you can use this with problem 3 to prove that the Dirac monopole of charge 2 is not the trivial bundle.

4) Give $\mathbb{R}P^2$ the homogeneous coordinates $[x^1, x^2, x^3]$. We know that $\mathbb{R}P^2$ is covered by three charts, where the three sets of local coordinates are have the form

$$(x,y) \;=\; (x^1(x^3)^{-1},\, x^2(x^3)^{-1}), \quad (a,b) \;=\; (x^1(x^2)^{-1},\, x^3(x^2)^{-1}), \quad (\alpha,\beta) \;=\; (x^2(x^1)^{-1},\, x^3(x^1)^{-1}).$$

Consider the vector field $\vec{v}$ on $\mathbb{R}P^2$ expressed in the first chart by $x\frac{\partial}{\partial x}$. Express v in the other charts, and show that its natural completion on $\mathbb{R}P^2$ is smooth and differentiable.

5) Suppose $\varphi : M^n \to N^k$ and $\psi : N^k \to P^l$ are differentiable maps, where M^n, N^k, K^l are manifolds. Then if $x \in M^n$ we have the maps

$$(\psi \circ \varphi)_*\big|_x \;:\; T_xM \;\longrightarrow T_{(\psi\circ\varphi)(x)}K, \qquad (\psi\circ\varphi)^*\big|_{(\psi\circ\varphi)(x)} \;:\; T^*_{(\psi\circ\varphi)(x)}K \;\longrightarrow T^*_xM. \tag{1}$$

Prove the two forms of the chain rule:

$$(\psi\circ\varphi)_*\big|_x = \psi_*\big|_{\varphi(x)} \circ \varphi_*\big|_x, \qquad (\psi\circ\varphi)^*\big|_{(\psi\circ\varphi)(x)} = \varphi^*\big|_{\varphi(x)} \circ \psi^*\big|_{(\psi\circ\varphi)(x)}. \tag{2}$$

(you may take for granted the standard chain rule from multivariable calculus). Hint: show these are equivalent, so you only have to prove one.

3 Linear Algebra and the Exterior Derivative

Due November 3, 2014. Exercises marked with a "*" are optional.

1) Show that the three classical vector operators on $\mathbb{R}^3$ (the gradient, divergence, and curl) can be expressed using the exterior derivative:

$$\nabla f \;=\; (df)^\sharp\,, \qquad \nabla \cdot X \;=\; d^* X_\flat\,, \qquad \nabla \times X \;=\; (*dX_\flat)^\sharp\,. \tag{3}$$

2) Verify the classical vector calculus identities *curl grad* $=$ 0 and *div curl* $= 0$ using properties of the exterior derivative.

2.5*) Further vector calculus identities can be proven. For instance the cross product on $\mathbb{R}^3$ can be expressed in terms of the exterior product and Hodge star: specifically $X \times Y = (*\,(X_\flat \wedge Y_\flat))^\sharp$. Prove the vector calculus identity $\nabla \cdot (X \times Y) \;=\; -\langle X,\, \nabla \times Y\rangle \;+\; \langle Y,\, \nabla \times X\rangle\,.$ Can you think of any other vector identities you can easily prove using the exterior calculus? (Maybe look at the vector calculus identity page on Wikipedia.)

3) We showed that for the operator $* : \bigwedge^k \to \bigwedge^{n-k}$ we have $** = (-1)^{k(n-k)}$. Thus on $\mathbb{R}^4$, we have $* : \bigwedge^2 \to \bigwedge^2$, and $** : \bigwedge^2 \to \bigwedge^2$ is the identity operator. Show that this $*$ operator has eigenvalues ± 1. The eigenspaces of eigenvalue $+1$ and -1 are denoted $\bigwedge^+$ and $\bigwedge^-$, respectively. Show that $\frac{1}{2}\,(1+*) : \bigwedge^2 \to \bigwedge^+$ and $\frac{1}{2}\,(1-*) : \bigwedge^2 \to \bigwedge^*$ form a complete set of orthogonal projection operators. Find bases for $\bigwedge^+$ and $\bigwedge^-$.

4) We have three norms for k-forms over an n-dimensional oriented inner product space V: the tensor norm, the determinant norm, and the duality norm. These norms are equivalent, though obviously this is tedious to show. These norms are equivalent on just $\bigwedge^n V$, and find the constants of proportionality amongst them.

4.5*) For arbitrary k, consider $\bigwedge^k V$, and show that the tensor norm, the determinant norm, and the duality norm are all equivalent, and find the constants of proportionality amongst them.

5) Show that any oriented Riemannian 2-manifold is a symplectic manifold. Using the Hodge star, prove that any oriented Riemannian 2-manifold is also an almost complex manifold.

5.5*) Prove that any oriented Riemannian 2-manifold is a complex manifold. To do so, verifying the vanishing of the Nijenhuis tensor, and cite the Newlander-Nirenburg theorem.

4 Diffeomorphism invariance and the Lie Derivative

Due November 12, 2014.

1) In class I gave the formula $\mathcal{L}_X = [d, i_X]$ as operators on k-forms. Prove that this is indeed the case. (There are several ways to do this, some easy and some hard!)

2) Prove that $[\mathcal{L}_X,\, i_Y] \;=\; i_{[X,Y]}$ as operators on forms. (Hint: It might be easier to prove first that $[d, [\mathcal{L}_X, i_Y]] = \mathcal{L}_{[X,Y]}$, and then maybe prove that $[\mathcal{L}_X,\, i_Y]$ is linear and work axiomatically, or maybe on a basis.)

3) Consider the following diffeomorphism flow on $\mathbb{R}^2$:

$$\varphi_t(x^1,\, x^2) \;=\; \left(x^1 \,+\, tx^1,\, x^2 \,+\, tx^2\right)^T. \tag{4}$$

Show that the direction field is time-dependent, and in fact

$$X(x^1, x^2, t) \;=\; \frac{x^1}{1+t}\frac{\partial}{\partial x^1} \;+\; \frac{x^2}{1+t}\frac{\partial}{\partial x^2}. \tag{5}$$

4) Consider again $\mathbb{R}^2$, with the radial field $X = x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ and the translational field $Y = \frac{\partial}{\partial x^1}$.

   a) Compute $[X, Y]$ the traditional (and easy) way.

   b) Find the diffeomorphism flow φ_t for X.

   c) Compute the pushforwards $\varphi_{t*}Y$.

   d) Compute $\mathcal{L}_X Y$ using the definition, and verify that it equals $[X, Y]$ from (*a*).

5) Do #15, Ch 5, in Spivak (actually, this gives another way to do problem (1)).

5 Lie Groups

Due Monday Dec 8, 2014

1) A Lie algebra is any vector space $\mathfrak{g}$ over some field $\mathbb{F}$ along with a bilinear product operation

$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{F},$$

called the bracket, that satisfies bilinearity and the Jacobi identity.

a) Show that x, y, h, where

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

is a basis for $\mathfrak{sl}(2,\mathbb{R})$, the Lie algebra of $SL(2,\mathbb{R})$.

b) The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ above, characterized by $[x,y] = h$, $[h,x] = 2x$, $[h,y] = -2y$, can be transformed into the cross product algebra $\mathfrak{su}(2,\mathbb{R})$ by means of a *Wick rotation*—that is, a partial rotation into a complex direction. Specifically, if $\{X, Y, Z\}$ is the standard $\mathbb{R}$-basis of $\mathfrak{su}(2,\mathbb{R})$, show there is an $\mathbb{R}$-isomorphism given by $x = X + iY$, $y = -X + iY$, $h = 2iZ$.

Aside: this transformation occurs frequently in quantum mechanics, eg. in the quantization of angular momentum. The operators x, y are known as the magnetic creation and annihilation operators, respectively, and h is the azimuthal quantum operator.

*2) In class, we discussed the Lie algebra $\mathfrak{g}$ associated to a Lie group $\mathcal{G}$, and we asserted, but didn't prove, that if the Lie group $\mathcal{G}$ is a matrix group, then the matrix commutator is the topological commutator (a formal proof would divert us into the analytic Cambpell-Baker-Hausdorff formula). Any vector $\vec{v} \in T_p\mathcal{G}$ can be extended to a left-invariant vector field that we shall denote $\vec{V}$. The so-called canonical 1-form (or Cartan 1-form) is defined by $\theta : TM \to \mathfrak{g}$ where $\theta(\vec{v}) = \vec{V}$. More precisely, we can say that θ is a $\mathfrak{g}$-valued 1-form:

$$\theta \in \left(\bigwedge^1 \mathcal{G}\right) \otimes \mathfrak{g}. \tag{7}$$

If $\omega \in \bigwedge^1 \mathcal{G} \otimes \mathfrak{g}$ is any such $\mathfrak{g}$-valued 1-form, expressed as $\omega = \omega_i^j \eta^i \otimes \mathfrak{g}_i$ where $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_n\} \in \mathfrak{g}$ is any vector-space basis of $\mathfrak{g}$, then we have an exterior derivative $d : \bigwedge^1 \mathcal{G} \otimes \mathfrak{g} \to \bigwedge^2 \otimes \mathfrak{g}$ defined simply by taking the $\mathfrak{g}$-factor (and therefore the chosen $\mathfrak{g}$-basis) to be d-constant. Again with θ the canonical 1-form, prove the Cartan formula

$$d\theta + \frac{1}{2}[\theta, \theta] = 0. \tag{8}$$

(Hint: This is straightforward if you work in a left-invariant basis and use the formula, incidentally also due to Cartan, that for 1-forms ω we have $d\omega(\vec{v}, \vec{w}) = \vec{v}\,(\omega(\vec{w})) - \vec{w}\,(\omega(\vec{v})) - \omega([\vec{v}, \vec{w}])$.)

3) The cross product on $\mathbb{R}^3$ can be understood in algebraic terms as the quaternionic commutator: if $\mathbb{R}^3$ is identified with the purely imaginary quaternions, then $\vec{v} \times \vec{w} = \frac{1}{2}[\vec{v}, \vec{w}]$

a) Explain why extending the quaternionic cross product to $\mathbb{R}^4$ leads to nothing new.

b) Show that quaternionic conjugation is not a homomorphism. Explain why there are two cross products, one associated to the quaternions (where $ijk=-1$) and one associated to the complex conjugate of the quaternions (where $kji=-1$).

c) As a vector space, the purely imaginary octonions are just $\mathbb{R}^7$. Show that there is an $\mathbb{R}^7$ cross product, but that it does not obey the Jacobi identity (just give a specific example).

**d*) The automorphism group of $\mathbb{O}$ is denoted G_2; this is the group of maps $A:\mathbb{R}^8\to\mathbb{R}^8$ so that $A(a)A(b)=A(ab)$. Show that G_2 preserves the norm, and show that if $A\in G_2$ then $A(1)=1$. Therefore that $G_2\subset O(7)$.

**e*) A *basic triple* $e_1,e_2,e_3\in\mathbb{O}$ is any three elements so that the four elements e_1,e_2,e_1e_2,e_3 are mutually perpendicular. Show that any map in $SO(7)$ that takes a basic triple to a basic triple is actually a map in G_2. Using this, show that $SU(2)\subset G_2$, and therefore uncountably many $\mathbb{R}^7$ cross products exist.

4) In the following hands-on exercises, we shall uncover a few additional spin groups. Recall that if $SO(k,l)$ is an orthogonal group, then the group $Spin(k,l)$ is defined to be its double cover. In class, we discussed the following basis for the trace-free anti-Hermitian matrices

$$\begin{aligned}\hat{\imath} &= \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix} = -\sqrt{-1}\sigma_x \\ \hat{\jmath} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\sqrt{-1}\sigma_y \\ \hat{k} &= \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} = -\sqrt{-1}\sigma_z\end{aligned} \tag{9}$$

and noted that under the linear isomorphism $(x,y,z)\mapsto x\hat{\imath}+y\hat{\jmath}+z\hat{k}$ the standard vector-space norm is simply the determinant (and the cross product is half the commutator). Now extend this to the not-necessarily-trace-free anti-Hermitian matrices by including a "time" direction, whose basis vector is

$$\hat{t} \;=\; \sqrt{-1}\,Id_{2\times 2}. \tag{10}$$

a) Show that *det* now corresponds to a Lorenzian norm, so we should regard the vector space of anti-Hermitian matrices as $\mathbb{R}^{3,1}$.

b) Show that conjugation by $SL(2,\mathbb{C})$ preserves the anti-Hermitian matrices, and also preserves the determinant. The "conjugation" referred to here is the map

$$(A,g)\;\longmapsto\; Ag\overline{A}^T. \tag{11}$$

c) Part (*b*) shows the existence of a homomorphism $SL(2,\mathbb{C})\to O(3,1)$. Show that this is a 2-to-1 map onto the special orthochronous group $SO^+(3,1)$, so therefore $Spin(3,1)\approx SL(2,\mathbb{C})$.

**d*) By considering the purely imaginary vector subspace of the anti-Hermitian matrices, similarly show that $Spin(2,1)\approx SL(2,\mathbb{R})$.

6 Homology and Cohomology

Not Collected.

1) Prove that singular homology is a diffeomorphism invariant. That is, if $f : M^n \to N^n$ is a diffeomorphism, then $f_* : H_k(M^n, \mathbb{F}) \to H_k(N^n, \mathbb{F})$ is a linear isomorphism. Obviously this follows from homotopy invariance, but I want you to verify this theorem directly, without using the machinery we've developed. This is to test you on your knowledge of the basic definitions.

2) We shall compute the deRham cohomology groups of the circle.

   a) Prove that $H^0(\mathbb{S}^1, \mathbb{R}) \approx \mathbb{R}$. More precisely, show it is isomorphic to the vector space of constant functions.

   b) Prove that $H^i(\mathbb{S}^1, \mathbb{R}) \approx \{0\}$ when $i \geq 1$.

   c) Of course *every* $\eta \in A^1(\mathbb{S}^1, \mathbb{R})$ is closed. Show that $\int_{\mathbb{S}^1} \eta = 0$ if and only if $\eta = df$ for some function f.
   Hint: Of course one direction is easy. To prove the " $\Longrightarrow$ " direction, first consider $\mathbb{S}^1 \setminus \{pt\}$. Since this is contractible, use homotopy invariance to conclude that some function f exists that satisfies $\eta = df$ on $\mathbb{S}^1 \setminus \{pt\}$. Then show that $\int_{\mathbb{S}^1} \eta = 0$ implies that f can be defined on $\{pt\}$ in such a way that f is actually continuous. Then show that f is continuously differentiable, then infinitely continuously differentiable and satisfies $df = \eta$ everywhere.

   d) Prove that $\int_{\mathbb{S}^1} : H^1(\mathbb{S}^1, \mathbb{R}) \to \mathbb{R}$, given by $[\eta] \mapsto \int_{\mathbb{S}^1} \eta$ is well defined, and an isomorphism.

3) A general principle of homology theories is that "short exact sequences of chain groups leads to long exact sequences in homology." We shall explore this in the context of the "Meyer-Veitoris sequence." Assume $M^n = N \cup K$ where N, K are open subsets of M. Then we have three singular chain complexes

$$\begin{aligned} 0 &\longrightarrow C_0(M, \mathbb{F}) \xrightarrow{\partial} C_1(M, \mathbb{F}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_i(M, \mathbb{F}) \xrightarrow{\partial} \dots \\ 0 &\longrightarrow C_0(N, \mathbb{F}) \xrightarrow{\partial} C_1(N, \mathbb{F}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_i(N, \mathbb{F}) \xrightarrow{\partial} \dots \\ 0 &\longrightarrow C_0(K, \mathbb{F}) \xrightarrow{\partial} C_1(K, \mathbb{F}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_i(K, \mathbb{F}) \xrightarrow{\partial} \dots \end{aligned} \tag{12}$$

and three resulting sets of singular homology groups $H_*(M, \mathbb{F})$, $H_*(N, \mathbb{F})$, and $H_*(K, \mathbb{F})$. There are obvious inclusion maps

$$\begin{aligned} I_N &: C_i(N, \mathbb{F}) \to C_i(M, \mathbb{F}) \\ I_K &: C_i(K, \mathbb{F}) \to C_i(M, \mathbb{F}) \end{aligned} \tag{13}$$

and inclusions

$$\begin{aligned} J_N &: C_i(N \cap K, \mathbb{F}) \to C_i(N, \mathbb{F}) \\ J_K &: C_i(N \cap K, \mathbb{F}) \to C_i(K, \mathbb{F}) \end{aligned} \tag{14}$$

   a) Show that

$$0 \longrightarrow C(N \cap K, \mathbb{F}) \xrightarrow{J_N \oplus J_K} C(N, \mathbb{F}) \oplus C(K, \mathbb{F}) \xrightarrow{I_N - I_K} C(M, \mathbb{F}) \longrightarrow 0 \tag{15}$$

   is an exact sequence.

b) Show that this sequence induces maps in homology

$$H_k(N \cap K,\, \mathbb{F}) \;\longrightarrow\; H_k(N,\, \mathbb{F}) \oplus H_k(K, \mathbb{F}) \;\longrightarrow\; H_k(M,\, \mathbb{F}) \tag{16}$$

which is exact in the center.

c) As explained in class, give a detailed argument showing that the boundary map is well defined. This gives a "long sequence" in homology

$$\dots \xrightarrow{\partial_*} H_k(N \cap K, \mathbb{F}) \xrightarrow{J_{N*} - J_{K*}} H_k(N, \mathbb{F}) \oplus H_k(K, \mathbb{F}) \xrightarrow{I_{N*} \oplus I_{K*}} H_k(N \cap K, \mathbb{F}) \xrightarrow{\partial_*} H_{k-1}(N \cap K, \mathbb{F}) \xrightarrow{J_{N*} - J_{K*}} \dots \tag{17}$$

d) Show that this sequence is exact at

$$\dots \xrightarrow{I_{N*} \oplus I_{K*}} H_k(N \cap K, \mathbb{F}) \xrightarrow{\partial_*} \dots \tag{18}$$

e) Show that the sequence is exact at

$$\dots \xrightarrow{\partial_*} H_k(N \cap K, \mathbb{F}) \xrightarrow{J_{N*} - J_{K*}} \dots \tag{19}$$

and that therefore the "long sequence" in homology is actually a "long exact sequence."

4) Using your knowledge of the homology groups of a point along with a Mayer-Vietoris sequence, show that the homology groups of $\mathbb{S}^1$ are

$$\begin{aligned} H^0(\mathbb{S}^1,\, \mathbb{Z}) \;&\approx\; \mathbb{Z} \\ H^1(\mathbb{S}^1,\, \mathbb{Z}) \;&\approx\; \mathbb{Z} \end{aligned} \tag{20}$$

Using this, along with another Mayer-Vietoris sequence and possibly homotopy equivalence, show that the homology groups of $\mathbb{S}^2$ are

$$\begin{aligned} H^0(\mathbb{S}^1,\, \mathbb{Z}) \;&\approx\; \mathbb{Z} \\ H^1(\mathbb{S}^1,\, \mathbb{Z}) \;&\approx\; \{0\} \\ H^2(\mathbb{S}^1,\, \mathbb{Z}) \;&\approx\; \mathbb{Z}. \end{aligned} \tag{21}$$