Special Lecture - The Octionions

March 15, 2013

1 $\mathbb{R}$

1.1 Definition

Not much needs to be said here. From the "God given" natural numbers, we algebraically build $\mathbb{Z}$ and $\mathbb{Q}$. Then create a topology from the distance function (subtraction) and complete $\mathbb{Q}$ to obtain $\mathbb{R}$, the unique complete well-ordered field.

Symmetry comes into play when we look at maps that preserve structures.

1.2 Automorphisms

Of course there is the ring structure, but due to uniqueness there is nothing there—there are no non-trivial automorphisms.

1.3 Vector spaces and transformations

Forming real vector spaces, we have the linear maps, which preserve the vector space structure

$$L(n;\mathbb{R}) \;=\; \left\{\, M:\mathbb{R}^n\to\mathbb{R}^n \;\middle|\; M(a\vec{x}+b\vec{y}) = aM(\vec{x})+bM(\vec{y}) \,\right\} \tag{1}$$

This is a monoid, not a group, as inverses may not exist. The first group is the general linear group, which preserve dimensionality

$$GL(n;\mathbb{R}) \;=\; \left\{\, M\in L(n) \;\middle|\; M \text{ is invertible} \,\right\}. \tag{2}$$

Further, we have the area preserving maps

$$SL(n;\mathbb{R}) \;=\; \left\{\, M\in L(n) \;\middle|\; det(M)=1 \,\right\}. \tag{3}$$

the maps that preserve a symmetric bilinear form

$$O(p, q; \mathbb{R}) \;=\; \left\{\, M \in L(n) \;\middle|\; M^T I_{p,q} M = I_{p,q} \,\right\}. \tag{4}$$

and the maps that preserve an antisymmetric bilinear form

$$Sp(2n; \mathbb{R}) \;=\; \left\{\, M \in L(n) \;\middle|\; M^T J M = J \,\right\}. \tag{5}$$

2 $\mathbb{C}$

2.1 Construction

This is defined as the unique complete, algebraically complete field that contains $\mathbb{R}$. We explain two ways to obtain it. The first is to adjoin the number i and enforce $i^2 = -1$. This produces a twisted product structure on the vector space $\mathbb{R} \oplus \mathbb{R}$, which can be proved to be algebraically complete.

The second method is to consider the ring of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{6}$$

for real a, b, and to observe that the product rule for complex numbers is retained. To split hairs, the mapping $\mathbb{C} \to \mathbb{R}(2)$ is called *realification*, whereas the construction of the complex numbers as the subalgebra of $\mathbb{R}(2)$ spanned by matrices of the form (6) is called the *Cayley-Dickson construction*.

Notice that complex conjugation corresponds to transposition in the Cayley-Dickson construction, the norm is given by the determinant, and the real part is given by half the trace.

2.2 Automorphisms

A map $\varphi : \mathbb{C} \to \mathbb{C}$ is an automorphism provided $\varphi(a)\varphi(b) = \varphi(ab)$. Such a map is linear, preserves 1 since $\varphi(1)\varphi(1) = \varphi(1)$ implies $\varphi(1) = 1$, and preserves the imaginary line, as $\varphi(i)^2 = -1$. The only possibilities are $\varphi(x) = x$ and $\varphi(x) = \overline{x}$. Therefore

$$Aut(\mathbb{C}) \;=\; \mathbb{Z}_2. \tag{7}$$

2.3 Vector spaces and transformations

We have the usual transformations of $\mathbb{C}^n$

$$\begin{aligned} L(n;\,\mathbb{C}) &= \left\{ M : \mathbb{R}^n \to \mathbb{R}^n \,\middle|\, M(a\vec{x}+b\vec{y}) = aM(\vec{x}) + bM(\vec{y}) \right\} \\ GL(n;\,\mathbb{C}) &= \left\{ M \in L(n) \,\middle|\, M \text{ is invertible} \right\} \\ SL(n;\,\mathbb{C}) &= \left\{ M \in L(n) \,\middle|\, det(M) = 1 \right\}. \end{aligned} \tag{8}$$

We also have the maps that preserve a symmetric bilinear form

$$O(n;\,\mathbb{C}) = \left\{ M \in L(n) \,\middle|\, M^T I_n M = I_n \right\}. \tag{9}$$

and the maps that preserve an antisymmetric bilinear form

$$Sp(2n;\mathbb{C}) = \left\{ M \in L(n) \,\middle|\, M^T J M = J \right\}. \tag{10}$$

In addition, we have the maps that preserve an hermitian form of signature (p, q)

$$\begin{aligned} U(n;\,\mathbb{C}) &= \left\{ M \in L(n) \,\middle|\, M^T I_n \overline{M} = I_n \right\} \\ SU(n;\,\mathbb{C}) &= U(n,\,\mathbb{C}) \cap SL(n,\,\mathbb{C}). \end{aligned} \tag{11}$$

3 $\mathbb{H}$

3.1 Construction

First we can set

$$\mathbb{H} = span_{\mathbb{R}} \{ 1,\, i,\, j,\, k,\, \} \tag{12}$$

and stipulate Hamilton's equations:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{13}$$

making $\mathbb{H}$ into a non-commutative algebra. Second, in our study of Clifford algebras we saw it was natural to consider

$$\mathbb{C} = \{ a + b\, ij \} \approx Cl_2^0 \approx Cl_1 \tag{14}$$

so that

$$\mathbb{H} = \mathbb{C} \oplus \overline{\mathbb{C}}\, j \approx Cl_2 \tag{15}$$

with a twisted product. Third, we can use the Cayley-Dickson construction. Again writing $q = a + bij + (c - dij)j = \alpha + \overline{\beta} j$, set

$$q_M = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}. \tag{16}$$

One checks that quaternionic multiplication holds. As in the complex case, we attempt to interpret the matrix operations. We define the *norm* of a quaternion $q = \alpha + \overline{\beta} j$ to be the determinant of its matrix

$$N(q) \ = \ det \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \ = \ |\alpha|^2 \ + \ |\beta|^2. \tag{17}$$

and the trace to be one-half the trace of its matrix

$$\tau(q) \ = \ Tr \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \ = \ \frac{1}{2}(\alpha + \overline{\alpha}). \tag{18}$$

Lastly we define the complex conjugate as

$$\overline{q} \ = \ 2\tau(q) \ - \ q \tag{19}$$

which corresponds to the complex-conjugate transpose of the matrix. Notice that

$$\overline{q_1 q_2} \ = \ \overline{q_2}\,\overline{q_1} \tag{20}$$

and $\overline{\overline{q_1}} = q_1$.

By the properties of the determinant, N is *multiplicative*

$$N(q_1 q_2) \ = \ N(q_1)N(q_2). \tag{21}$$

Since N is a quadratic form, we can polarize to obtain an inner product

$$(q_1,\, q_2) \ = \ \frac{1}{2}\left(N(q_1 + q_2) \ - \ N(q_1) \ - \ N(q_2)\right). \tag{22}$$

This is of course a purely real, positive definite inner product, and in fact

$$(q_1,\, q_2) \ = \ \frac{1}{2} Tr(q_1 \overline{q_2} + \overline{q_1} q_2) \ = \ Re(q_1 \overline{q_2}) \tag{23}$$

which is simply the dot product in $\mathbb{R}^4$. Note that Ad_q, given by $Ad_q(x) = qxq^{-1}$, is orthogonal, and $L_q = q\cdot$ is skew-Hermitian:

$$\begin{aligned} \left(qxq^{-1},\, qyq^{-1}\right) \ &= \ (x,\, y) \\ (qx,\, y) \ &= \ (x,\, \overline{q}y)\,. \end{aligned} \tag{24}$$

Finally we can recover both the dot and cross product in $\mathbb{R}^3$. If $\vec{A}_1 = (x_1, y_1, z_1)$ and $\vec{A}_2 = (x_2, y_2, z_2)$ are vectors in $\mathbb{R}^3$ and A_1, A_2 the corresponding purely imaginary quaternions, then

$$A_1 A_2 \in \mathbb{H} \quad \textit{corresponds to} \quad \vec{A}_1 \cdot \vec{A}_2 \ + \ A_1 \times \vec{A}_2 \in \mathbb{R}^4 \tag{25}$$

3.2 Euler's formula and rotations

Any quaternion can be written in the form

$$q \;=\; a \,+\, \theta\, u \tag{26}$$

where $a, \theta \in \mathbb{R}$ and u is a unit-length imaginary quaternion. We have $u^2 = -u\overline{u} = -1$, so

$$\begin{aligned} Exp(\theta\, u) \;&=\; \sum_{n=0}^{\infty} \frac{1}{n!}(\theta\, u)^n \\ &=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\theta^{2n} \,+\, u \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\theta^{2n+1} \\ &=\; \cos\theta \,+\, u \sin\theta. \end{aligned} \tag{27}$$

Therefore

$$exp\, q \;=\; e^a \left(\cos\theta \,+\, u\,\sin\theta\right). \tag{28}$$

Note that if $q = \theta\, u$ is purely imaginary, then $N(Exp(q)) = 1$, just as in the complex case.

3.3 Automorphisms

If $\varphi : \mathbb{H} \to \mathbb{H}$ is an automorphism, then φ is the identity on $\mathbb{R} \subset \mathbb{H}$ and so in particular φ preserves the trace. Thus

$$(\varphi(q_1),\, \varphi(q_2)) \;=\; \tau(\varphi(q_1)\varphi(q_2)) \;=\; \tau(\varphi(q_1 q_2)) \;=\; \tau(q_1 q_2) \;=\; (q_1,\, q_2) \tag{29}$$

so any automorphism is orthogonal, and so is a subgroup of $O(3)$. Now if u is any unit-length quaternion, se can set $\varphi_u(q) = uqu^{-1}$. Then $\varphi_u(1) = 1$ and

$$\varphi_u(q_1)\varphi_u(q_2) \;=\; uq_1u^{-1}uq_2u^{-1} = uq_1q_2u^{-1} = \varphi(q_1q_2). \tag{30}$$

But $\varphi_u = Id$ precisely when $u = \pm 1$, so we have a 2-1 cover from $\mathbb{S}^3$ to $SO(3)$, the connected component of the identity in $O(3)$. The two sheets of $O(3)$ are interchanged by quaternionic conjugation, which is NOT an algebra isomorphism $((-i)(-j)(-k) = +1)$. So we obtain

$$Aut(\mathbb{H}) \;=\; SO(3). \tag{31}$$

4 $\mathbb{O}$

The octonions are not associative, so we cannot hope to obtain them in the form of any matrix group. Still, we consider an octonion to be a pair of quaternions (a, b), but we set the product

$$(a, b) \cdot (r, s) \;=\; (ra - \overline{b}s,\; sa \,+\, b\overline{r}). \tag{32}$$

Another way to create the octonions is to append the symbol l to the quaternionic i, j, k, and enforce $l^2 = -1$ and l anti-commutes with i, j, k, to obtain a twisted product on

$$\mathbb{O} \;=\; \mathbb{H} \oplus \mathbb{H}l \tag{33}$$

For instance, with the product from (32), we obtain $j(il) = kl$ but $(ji)l = -kl$. We define a trace function on $\mathbb{O}$ in the usual way: if $x = (h_1, h_2)$ where $h_1, h_2 \in \mathbb{H}$, then

$$\tau(x) \;=\; \tau(h_1). \tag{34}$$

This gives us a complex conjugate

$$\overline{x} \;=\; \tau(x) \,-\, x \tag{35}$$

and a norm

$$N(x) \;=\; x\overline{x}. \tag{36}$$

Given a quaternion x we can write $x = a + bv$ where $a, b \in \mathbb{R}$ and $v^2 = -1$. We have

$$N(x) \;=\; \tau(x)x - x^2 \;=\; 2a(a+bv) \,-\, \big(a^2 - b^2 - 2abv\big) \;=\; a^2 + b^2 \;\geq 0 \tag{37}$$

with equality iff $x = 0$. One verifies the following additional facts:

$$\overline{xy} \;=\; \overline{y}\overline{x} \tag{38}$$

$$N(xy) \;=\; N(x)N(y). \tag{39}$$

We have an inner product

$$\langle x,\, y\rangle \;=\; \frac{1}{2}\left(N(x+y) \,-\, N(x) \,-\, N(y)\right) \tag{40}$$

from which one computes

$$\langle x,\, y\rangle \;=\; \frac{1}{2}\left(\tau(y)x + \tau(y)x \,-\, (xy+yx)\right) \;=\; \frac{1}{2}\left(y\bar{x} + \overline{x}y\right) \tag{41}$$

The trace is then

$$\tau(x) \;=\; \langle 2,\, x\rangle \tag{42}$$

One checks that the inner product is the positive definite inner product on $\mathbb{R}^8$ with an orthonormal basis being $\{1, i, j, k, l, il, jl, kl\}$. We therefore have inverses:

$$x^{-1} \;=\; x\frac{1}{N(x)}. \tag{43}$$

We can also prove that L_q for $q \in \mathbb{O}$ is skew-Hermitian:

$$\begin{aligned} 2\,\langle qx, y\rangle \;&=\; N(qx+y) \,-\, N(qx) \,-\, N(y) \\ &=\; N(q(x+q^{-1}y)) \,-\, N(q)N(x) \,-\, N(q)N(q^{-1}y) \\ &=\; N(q)\left(N(x+q^{-1}y)) \,-\, N(x) \,-\, N(q^{-1}y)\right) \\ &=\; 2N(q)\left\langle x,\, q^{-1}y\right\rangle \\ &=\; 2\,\langle x,\, \overline{q}y\rangle \end{aligned} \tag{44}$$

and, for each q, to two orthogonal maps $\varphi^1_q(x) = q(xq^{-1})$ and $\varphi^2_q(x) = (qx)q^{-1}$.

Theorem 4.1 (Power Law, Alternative Laws, and the Moufang Identities) *Given any octonion x and any $m, n \in \mathbb{Z}$ we have the power law*

$$x^n x^m = x^{m+n}. \tag{45}$$

Given any two octonions x, y we have the alternative laws:

$$x(xy) \;=\; x^2 y \qquad (yx)x = yx^2 \qquad (xy)x = x(yx) \tag{46}$$

and for any three octonions $x, y, a \in \mathbb{O}$ we have the Moufang identities

$$\begin{aligned} (xy)(ax) \;&=\; x(ya)x \\ (xax)y \;&=\; x(a(xy)) \end{aligned} \tag{47}$$

In any of the identities, a single x of the pair can be replaced with $\overline{x} = 2\tau(x) - x$ to obtain identities such as

$$\overline{x}(xy) \;=\; |x|^2 y, \qquad x^{-1}(xy) = y, \qquad (xy)^{-1} = y^{-1}x^{-1} \tag{48}$$

4.1 Special Structures

Just as the purely imaginary quaternions create a cross product on $\mathbb{R}^3$, the purely imaginary octonions imbue $\mathbb{R}^7$ with a cross product:

$$v \times w \;=\; \frac{1}{2}\left(vw \,-\, wv\right). \tag{49}$$

Due to non-associativity, this does not obey the Jacobi identity. However clearly we have

$$vw \;=\; v \times w \,-\, \langle v,\, w\rangle \tag{50}$$

from which we obtain the familiar

$$|v \times w| \;=\; |v|^2|w|^2 - \langle v,\, w\rangle^2 \;=\; |v|^2|w|^2\cos^2\theta. \tag{51}$$

We can check that

$$\langle v \times w,\, z\rangle \;=\; -\langle v \times z,\, w\rangle \tag{52}$$

which produces a 3-form, which is non-degenerate in the sense that, given any distinct v, w, there is a z so that the triple product is non-zero. Also, the inner product can be expressed in terms of the cross product:

$$\langle a,\, b\rangle \;=\; -\frac{1}{6}tr(a \times (b \times \cdot)). \tag{53}$$

Second we have the associator:

$$[x, y, z] \;=\; (xy)z \,-\, x(yz) \tag{54}$$

Although it is slightly more difficult to see, both the associator its dualized 4-form

$$\langle [x, y, z],\, w\rangle \tag{55}$$

are purely antisymmetric. If e_1, e_2 are any prurely imaginary, orthogonal unit vectors, then $1, e_1, e_2, e_1e_2$ form a copy of the quaternions. We see that the triple product is unity, and the associator vanishes. On the other hand, if e_1, e_2, e_3 are unit, purely imaginary vectors so that e_3 is also orthogonal to e_1e_2, then the triple product vanishes but the associator does not. Further, the associator 4-form is non-vanishing only on $e_1, e_2, e_3, (e_1e_2)e_3$. Thus we see that that the triple product and the associator 4-forms are dual to each other, in the Hodge-star sense.

4.2 Automorphisms

Are there any automorphisms of the octonions? Due to non-associativity, we cannot expect φ_q^1 or φ_q^2 to be automorphisms, except in the trivial case where $q \in \mathbb{R}$. Should any automorphisms exist, they form a group called G_2.

Let φ be any automorphism. As before, it fixes the real line, and therefore the trace, and $\varphi(\overline{x}) = \overline{\varphi(x)}$. Thus it fixes the norm, and so $G_2 \subseteq O(7)$. Given points $x, y \in \mathbb{O}$ we can define a function $f_{x,y} : O(7) \to \mathbb{R}^8$ by $f_{x,y}(A) = A(xy) - A(x)A(y)$. Since G_2 is the set of common zeros of the continuous functions $f_{x,y}$, we know that G_2 is a compact subgroup of $O(7)$.

Basic Triples: Any ordered triple of purely imaginary octonions e_1, e_2, e_3 that are mutually perpendicular and so e_3 is perpendicular to e_1e_2. Then $1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, (e_1e_2)e_3$ is an orthonormal basis for the octonions. Thus any map taking a basic triple to a basic triple is in G_2, and vice-versa. This is clearly connected. Dimension is 14.

We can find a copy of $SU(3)$, which is a maximal subgroup. Fix a purely imaginary quaternion, say l, and consider its isotropy group in G_2. This group permutes the six-dimensional space V orthogonal to 1 and l. Now if $v \in V$, then

$$\langle lv,\, 1\rangle \;=\; \left\langle v, \overline{l}\right\rangle \;=\; 0 \tag{56}$$

$$\langle lv,\, l\rangle \;=\; -\left\langle v, \overline{l}l\right\rangle \;=\; 0 \tag{57}$$

$$\tag{58}$$

so l leaves V invariant. Choosing $i, j \in V$ so that i, j, l is a basic triple, we have a basis

$$i, j, ij, li, lj, l(ij). \tag{59}$$

Since $l^2 = -1$ we can consider V to be the vector space $\mathbb{C}^3$ with scalars of the form $a + bl$. An hermitian metric $(\cdot, \cdot)$ is usually had by defining $(v, w) = \langle v, \bar{w}\rangle$. We can effect the same thing by setting

$$(v,\, w) \;=\; \langle v,\, w\rangle \,+\, l\,\langle lv,\, w\rangle\,. \tag{60}$$

Since l is fixed by definition, and $\langle , \rangle$ is fixed by anything in G_2, we have that the isotropy group is a subgroup of $U(3)$. On the other hand, the element of $U(3)$ that generates the commuting circle-action is multiplication by $e^{\theta l}$, which is not a part of G_2 unless $\theta = \pi n$. Thus the isotropy group is a subset of $SU(3)$. It is a compact subgroup, and therefore reductive, so that possibilities are $\mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1$, $SU(2)$, $U(2)$, and $SU(3)$. There is no commuting subgroup, as it would pass to a commuting subgroup of the automorphisms of the quaternions. Thus the possibilities are $SU(2)$ and $SU(3)$. However $SU(2)$ has dimension 3, and there the automorphism group must strictly contain the autorphism group of the quaternions. Therefore the automorphism group is $SU(3)$.

The group G_2 has no center, so it is semisimple. It strictly contains $SU(3)$, and since $dim(G_2) = dim(SU(3)) + 6$, we have two possibilities: G_2 is simple, $G_2 = SU(3) \oplus SO(4)$. The second possibility is ruled out by realizing that the action of $SU(3)$ on its compliment is non-trivial.

Therefore G_2 is simple.