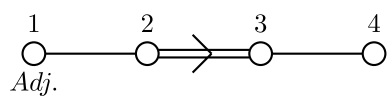
Lecture 22 - F_4

April 19, 2013

1 Review of what we know about F_4

We have two definitions of the Lie algebra $\mathfrak{f}_4$ at this point. The old definition is that it is the exceptional Lie algebra with Dynkin diagram



The Weyl dimension formula gives the following dimensions of the fundamental representations:

$HighestWeight$	$Dimension$
$(1,0,0,0)$	52
$(0,1,0,0)$	1274
$(0,0,1,0)$	273
$(0,0,0,1)$	26

(1)

so there is a representation smaller than the adjoint representation. It is by finding out what this is that F_4 is best understood. The structure of it's subgroups is also important. Obviously B_3 and C_3 are subalgebras. Dynkin's trick shows that $B_4 \approx \mathfrak{o}(9)$ is also a subalgebra. In addition, the long roots of F_4 form a D_4 system, so $\mathfrak{o}(8)$ is another subalgebra, although it is not maximal. It is worth noting that, by dualization in $\mathfrak{h}^*$, that the short roots also form a D_4 system, so the F_4 root system is two copies of the D_4 system, one long and one short.

The second definition of $\mathfrak{f}_4$ is that it is the Lie algebra of the set of Jordan-product preserving automorphisms of $\mathfrak{h}_3(\mathbb{O})$:

$$\begin{aligned} F_4 &\triangleq Aut(\mathfrak{h}_3(\mathbb{O})) \\ \mathfrak{f}_4 &\triangleq \mathfrak{aut}(\mathfrak{h}_3(\mathbb{O})) \end{aligned} \tag{2}$$

We shall see that these two definitions five the same algebra, and that $\mathfrak{aut}(\mathfrak{h}_3(\mathbb{O}))$ is the 26-dimensional representation of $\mathfrak{f}_4$ (note $\mathfrak{h}_3(\mathbb{O})$ is 27-dimensional; the F_4 action splits off multiples of the identity).

2 Spin factors and the $\mathfrak{h}_2(\mathbb{K})$

2.1 Spin factors and projective spaces

Recall the spin factors: $J_n = \mathbb{R}^n \oplus \mathbb{R}$ with product

$$(v,\, \alpha) \circ (w,\, \beta) \;=\; (\alpha w + \beta v,\; \langle v,\, w\rangle \;+\; \alpha\beta)\,. \tag{3}$$

It is natural to regard J_n as Minkowski space with inner product $\|(v, \alpha)\| = \alpha^2 - \|v\|^2$. One easily verifies that $p \in J_n$ is a projector if and only if

$$p \;=\; \frac{1}{2}\,(v,\, 1)\,, \qquad \|v\| = 1. \tag{4}$$

Note that these are just the normalized light-like vectors. Thus the projective spaces associated to the J_n are the *celestial spheres* of the various Minkowski spaces; they are diffeomorphic to $\mathbb{R}^{n-1}$-spheres.

Now let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, and consider $\mathfrak{h}_2(\mathbb{O})$. A matrix $M \in \mathfrak{h}_3(\mathbb{O})$ can be expressed

$$M \;=\; \begin{pmatrix} \alpha+\beta & \psi \\ \bar{\psi} & \alpha-\beta \end{pmatrix} \tag{5}$$

which has a natural Minkowski bilinear form, namely the determinant:

$$det(M) \;=\; \alpha^2 \;-\; \beta^2 \;-\; \|\psi\|^2. \tag{6}$$

This suggests that $\mathfrak{h}_3(\mathbb{O})$ should be $J_n(\mathbb{K} \oplus \mathbb{R}) \approx \mathbb{R}^{n+1} \oplus \mathbb{R}$, and indeed the map

$$\begin{pmatrix} \alpha+\beta & \psi \\ \bar{\psi} & \alpha-\beta \end{pmatrix} \quad\longrightarrow\quad ((\psi,\, \beta),\, \alpha) \tag{7}$$

is a Jordan algebra isomorphism. This shows that, diffeomorphically, we have spheres

$$\mathbb{R}P^1 \;=\; \mathbb{S}^1, \qquad \mathbb{C}P^1 \;=\; \mathbb{S}^2, \qquad \mathbb{H}P^1 \;=\; \mathbb{S}^4, \qquad \mathbb{O}P^1 \;=\; \mathbb{S}^8 \tag{8}$$

2.2 Spin factors and Spinors

Given a point in $(\psi_1,\, \psi_2)^T \in \mathbb{K}^2$ we can obtain a light-like vector in $\mathfrak{h}_2(\mathbb{K}) \approx J_n(\mathbb{K}) \oplus \mathbb{R}$:

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\overline{\psi_1},\, \overline{\psi_2}) \quad = \quad \begin{pmatrix} |\psi_1|^2 & \psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_1} & |\psi_2|^2 \end{pmatrix} \tag{9}$$

Since points in $\mathfrak{h}_2(\mathbb{K})$ are vectors is $\mathbb{R}^{n+1,1}$ Minkowski space and are linear combinations, therefore, of elements in $\mathbb{K}^2$, we have that volume-preserving transformations of $\mathbb{K}^2$ act as orthogonal transformations of $\mathbb{R}^{n+1,1}$. Thus

$$SL(2,\,\mathbb{K}) \;=\; Spin(n+1,\,1). \tag{10}$$

Note that $SL(2,\mathbb{K})$ is itself not easy to define, owing to the fact that

$$\begin{aligned} det(ABA^{-1}) \;&=\; det(B) \\ tr(AB) \;&=\; tr(BA) \end{aligned} \tag{11}$$

only holds when $\mathbb{K}$ is commutative and associative. Still, these can be defined as follows. We take $\mathfrak{sl}(n,\mathbb{K})$ to be the Lie algebra *generated* by the trace-free $n \times n$ matrices. Then $SL(n,\mathbb{K})$ is the exponentiation of the $\mathfrak{sl}(n,\mathbb{K})$. One checks that with these definitions, we indeed obtain that

$$det(AMA^{-1} \;=\; det(M) \tag{12}$$

when $M \in \mathfrak{h}_2(\mathbb{K})$ and $A \in SL(2,\mathbb{K})$. We obtain that

$$\triangle_{2,1} \;\approx\; \mathbb{R}^2, \qquad \triangle_{3,1} \;\approx\; \mathbb{C}^2, \qquad \triangle_{5,1} \;\approx\; \mathbb{H}^2, \qquad \triangle_{9,1} \;\approx\; \mathbb{O}^2 \tag{13}$$

are the spinors for $Spin(2,1) \approx SL(2,\mathbb{R})$, $Spin(3,1) \approx SL(2,\mathbb{C})$, $Spin(5,1) \approx SL(2,\mathbb{H})$, and $Spin(9,1) \approx SL(2,\mathbb{O})$.

We therefore have an orthogonal $Spin(1,9)$ representation on $\mathfrak{h}_2(\mathbb{O})$. Under $Spin(9)$ we obtain the splitting $\mathfrak{h}_2(\mathbb{O}) \approx \mathbb{R}^8 \oplus \mathbb{R}$ into space-like and time-like factors. Now consider the Clifford algebra $Cl_9 \approx \mathbb{R}^{16}$. We then have the spin group $Spin(9) \subset Cl_9^0$ has a representation on $\triangle_9 \approx \mathbb{O}^2$. Note that, when restricted to $Spin(8) \subset Spin(9)$, we split off the two octonionic factors:

$$\mathbb{O}^2 \;\approx\; \triangle_9 \;\approx\; \triangle_8^+ \oplus \triangle_8^- \;=\; \mathbb{O} \oplus \mathbb{O} \tag{14}$$

We have

$$\begin{aligned} \mathbb{R} \oplus \mathbb{O}^2 \oplus \mathfrak{h}_2(\mathbb{O}) \;&\approx\; \mathfrak{h}_3(\mathbb{O}) \\ \mathbb{R} \oplus \mathbb{O}^2 \oplus (\mathbb{R}^9 \oplus \mathbb{R}) \;&\approx\; \mathfrak{h}_3(\mathbb{O}) \\ \big(\alpha,\,(\psi_1,\,\psi_2)^T,\,M\big) \;&\longrightarrow\; \left(\begin{array}{ccc} \alpha & \overline{\psi_1} & \overline{\psi_2} \\ \psi_1 & \multicolumn{2}{c}{\multirow{2}{*}{M}} \\ \psi_2 & & \end{array}\right) \\ \Big(\alpha,\,(\psi_1,\,\psi_2)^T,\,(\vec{V}_9,\,\beta)\Big) \;&\longrightarrow\; \left(\begin{array}{ccc} \alpha & \overline{\psi_1} & \overline{\psi_2} \\ \psi_1 & \multicolumn{2}{c}{\multirow{2}{*}{$\beta \cdot I_{2\times 2} + \vec{V}_9$}} \\ \psi_2 & & \end{array}\right) \end{aligned} \tag{15}$$

But how do we see the Jordan product? It comes precisely from the action of $\vec{V}_9 \in \mathbb{R}^9$ on $\triangle_9 \approx \mathbb{O}^2$ via $\vec{V}_9 \subset Cl_9$, along with the obvious actions $\triangle_9 \otimes \overline{\triangle_9} \to \mathbb{R}$ and the Jordan product on $\mathfrak{h}_2(\mathbb{O})$.

Now consider the orthogonal action of $Spin(9,1)$ on $\mathfrak{h}_2(\mathbb{O})$. This does not preserve the $\mathfrak{h}_3(\mathbb{O})$ product. However, it does preserve the $\mathfrak{h}_3(\mathbb{O})$ determinant; thus we have found a subgroup

$$Spin(9,1) \;\subset\; E_6. \tag{16}$$

Now if we restrict to $Spin(9) \subset Spin(9,1)$, we have an orthogonal action on the $\mathfrak{h}_2(\mathbb{O})$ factor, as well as on the the $\mathbb{O}^2$ factor—this preserves the Jordan product!

Defining $F_4 = Aut(\mathfrak{h}_3(\mathbb{O})$, we have found a subgroup

$$Spin(9) \;\subset\; F_4. \tag{17}$$

A second subgroup of F_4 is given by $O(3,\mathbb{R})$. This is by the conjugation action; there is no issue of non-associativity as anything in $\mathbb{R}$ associates with $\mathbb{O}$. Note that the factors $O(3)$ and $Spin(9)$ do not split inside F_4.

2.3 Projectors in $\mathfrak{h}_3(\mathbb{O})$, and the Octonionic Projective Plane

Now $Spin(9)$, which acts on $\mathfrak{h}_2(\mathbb{O}) = \mathbb{R}^{n+1} \oplus \mathbb{R}$ via orthogonal actions on $\mathbb{R}^n$. We can therefore transform any $\mathfrak{h}_2(\mathbb{O})$ matrix to a completely real matrix. Fixing the resulting real space-like vector, we obtain the subgroup $Spin(8) \subset Spin(9)$.

Lifting now to the $Spin(8)$ action on $\mathfrak{h}_3(\mathbb{O})$, we can transform the $\mathbb{O}^2 \approx \triangle_8^+ \oplus \triangle_8^-$ factor as such. Transforming (ψ_1, ψ_2) via sending ψ_1 to a multiple of the identity, we obtain the $Spin(7)$-action on $\triangle_8^-$, which is transitive on the unit sphere, so we can send the ψ_2-factor to a multiple of the identity. We arrive at the fact that any matrix

$$\begin{pmatrix} \alpha & x & z \\ \bar{x} & \beta & y \\ \bar{z} & \bar{y} & \gamma \end{pmatrix} \tag{18}$$

can be transformed, via $Spin(9) \subset F_4$, into a matrix that is entirely real. Then using the $O(3)$-conjugation we can transform into a diagonal matrix.

Since projectors transform to projectors, we have that any projector is equivalent to

$$\begin{aligned} p_0 \;&=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & p_1 \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ p_2 \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & p_3 \;&=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{19}$$

From our description of the action of $Spin(9)$, we know that the isotropy group of $Spin(9)$ is precisely p_1. Since the $M \in \mathfrak{h}_3(\mathbb{O})$ that are conjugate to p_1 constitute the points of $\mathbb{O}P^2$, we have that

$$\mathbb{O}P^2 \;=\; F_4 \,/\, Spin(9). \tag{20}$$

Therefore we compute $dim\, F_4 \;=\; dim\, Spin(9) + 16 \;=\; 36 + 16 \;=\; 52.$

At this point, it is possible to compute that $Spin(9)$ and $SO(3)$ generate F_4, and that the result is simple. Thus by dimension counting $F_4 = Aut(\mathfrak{h}_3(\mathbb{O}))$ has the Lie algebra the exceptional algebra $\mathfrak{f}_4$.

Finally, we can obtain a description of $\mathbb{O}P^2$. One computes that

$$p \;=\; \left(\begin{array}{c} x \\ y \\ z \end{array}\right)(\bar{x},\, \bar{y},\, \bar{z}) \;\in\; \mathfrak{h}_3(\mathbb{O}) \tag{21}$$

is a projector if and only if x, y, and z associate, and $\|x\|^2 + \|y\|^2 + \|z\|^2 = 1$. The converse holds as well. Thus the points of $\mathbb{O}P^2$ are the equivalence classes

$$[x,\, y,\, z] \tag{22}$$

in which $[x, y, z]$ is equivalent to $[x', y', z']$ if an only if both triples associate, and both determine the same projector (after normalization).

3 Other descriptions of F_4

3.1 Triality description

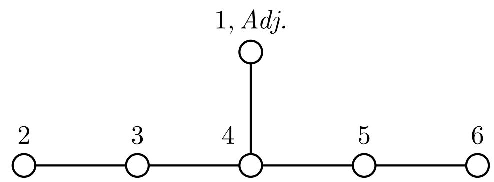
From (20), we see that as vector spaces we have

$$\begin{aligned} \mathfrak{f}_4 \;&=\; \mathfrak{so}(9) \,\oplus\, \mathbb{O}^2 \\ &=\; \mathfrak{so}(8) \,\oplus \mathbb{R}^8 \,\oplus\, \triangle_8^+ \,\oplus\, \triangle_8^-. \end{aligned} \tag{23}$$

The bracket on the $\mathfrak{so}(8)$ factor is the natural bracket, and the brackets between the $\mathfrak{so}(8)$ and the $\mathbb{R}^8$, $\triangle_8^+$, and $\triangle_8^-$ are the three inequivalent actions (one vector, two spinor). The brackets between the $\mathbb{R}^8$, $\triangle_8^+$, and $\triangle_8^-$ and one another are the triality maps. Then there are brackets $\mathbb{R}^8 \otimes \mathbb{R}^8 \to \mathfrak{so}(8)$, $\triangle_8^+ \otimes \triangle_8^+ \to \mathfrak{so}(8)$, and $\triangle_8^- \otimes \triangle_8^- \to \mathfrak{so}(8)$ that are a little more difficult to see, but can be described as follows. Given two vectors $v_1, v_2 \in \mathbb{R}^8$, there is a transformation in $SO(8)$ that takes one line to the other in unit time while leaving the perpendicular 6-space invariant, and with sign given by orientation. The generator is an element of $\mathfrak{so}(8)$, which is then scaled by the lengths of the vectors. If $\psi_1^+, \psi_2^+ \in \triangle^+$ are spinor, there is a geodesic in $Spin(8)$ that takes the line determined by ψ_1^+ to the line determined by ψ_2^+ in unit time, with sign determined by orientation. The generator an the element of $\mathfrak{so}(8)$, which can be scaled according to lengths of ψ_1^+, ψ_2^+.

3.2 Derivation description

4 Descriptions of E_6



$HighestWeight$	$Dimension$
$(1,0,0,0,0,0)$	78
$(0,1,0,0,0,0)$	27
$(0,0,1,0,0,0)$	351
$(0,0,0,1,0,0)$	2925
$(0,0,0,0,1,0)$	351
$(0,0,0,0,0,1)$	27

(24)

We have defined the group E_6 to be the mappings $\mathfrak{h}_3(\mathbb{O}) \to \mathfrak{h}_3(\mathbb{O})$ that preserve the determinant:

$$det(M) \;=\; \frac{1}{3}Tr(M^3) \;-\; \frac{1}{2}Tr(M^2)Tr(M) \;+\; \frac{1}{6}\left(Tr(M)\right)^3 . \tag{25}$$

This gives the 27-dimensional representation. It can be shown that another representation exists, on $\mathfrak{h}_3(\mathbb{C}\otimes\mathbb{O})$, which is indeed a Jordan algebra, although it is not formally real. In this representation, we have $E_6 = Aut(\mathfrak{h}_3(\mathbb{C}\otimes\mathbb{O}))$. Now $\mathfrak{h}_3(\mathbb{C}\otimes\mathbb{O})$ *does* give a certain projective space, called the bi-octonionic projective plane:

$$(\mathbb{C}\otimes\mathbb{O})\,P^2 \tag{26}$$

which carries an homogeneous Riemannian metric with isometry group E_6. It can be shown that $(Spin(10)\times U(1))/\mathbb{Z}_4 \subset E_6$ and that

$$(\mathbb{C}\otimes\mathbb{O})\,P^2 \;=\; E_6\,/\,\left((Spin(10)\times U(1))/\mathbb{Z}_4\right) \tag{27}$$