Lecture 21 - Jordan Algebras and Projective Spaces

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References:
Jordan Operator Algebras. H. Hanche-Olsen and E. Stormer
The Octonions. J. Baez

1 Jordan Algebras

1.1 Definition and examples

In the 1930's physicists, looking for a larger context in which to place quantum mechanics, settled on the following axioms for an algebra of observables:

i) $\mathcal{A}$ is a vector space over $\mathbb{R}$

ii) The product $\circ$ is bilinear

iii) The product is commutative: $A \circ B = B \circ A$

iv) The product obeys the *Jordan identity*: $A \circ (B \circ (A \circ A)) \;=\; (A \circ B) \circ (A \circ A)$

An algebra obeying these axioms is called a *Jordan algebra*. A Jordan algebra is called *formally real* if it satisfies

v) $A_1 \circ A_1 \,+\, \ldots \,+\, A_k \circ A_k = 0$ if and only if $A_1 = \cdots = A_k = 0$.

The motivating example was the set of physical observables; in the finite dimensional case, these are the hermitian $n \times n$ matrices, $\mathfrak{h}_n(\mathbb{C})$ with the product

$$A \circ B \;=\; \frac{1}{2}\left(AB \,+\, BA\right). \tag{1}$$

Two things can be noted here: this is formally real (unlike the full matrix algebra $\mathbb{C}(n)$), and the Jordan product comes from the product in a larger associative algebra. A Jordan algebra whose product comes from an associative algebra via (1) is called a *special Jordan algebra*. The goal of the early explorers of this field was to come up with a broader theory of general Jordan algebras, into which $\mathfrak{h}_n(\mathbb{C})$ might fit as some sort of special case.

Three series of finite dimensional formally real special Jordan algebras are

- The $n \times n$ real symmetric matrices $\mathfrak{h}_n(\mathbb{R})$
- The $n \times n$ complex Hermitian matrices $\mathfrak{h}_n(\mathbb{C})$
- The $n \times n$ quaternionic Hermitian matrices $\mathfrak{h}_n(\mathbb{H})$

In addition there is are the algebras of *spin factors*:

- $\mathbb{R}^n \oplus \mathbb{R}$ with product $(X, a) \circ (Y, b) \ = \ (bX \ + \ aY, \ \langle X, Y \rangle \ + \ ab\,)$

1.2 Additional Facts

Normally we have left- and right- multiplication operators. Due to commutativity, we simple define the Jordan multiplication operator

$$T_a(b) \ = \ a \circ b. \tag{2}$$

The algebra of multiplication operators is associative (trivially); however due to non-associativity

$$\begin{aligned} T_a T_b \ &\neq \ T_{ab} \\ T_a T_b \ &\neq \ T_b T_a. \end{aligned} \tag{3}$$

Two elements a, b are said to *operator commute* if $T_a T_b = T_b T_a$. The *center* of a Jordan algebra is the set of all elements that operator commute. The Jordan identity can be expressed

$$T_a T_{a^2} \ = \ T_{a^2} T_a \tag{4}$$

or $[T_a, T_{a^2}] = 0$. The Jordan identity implies the following identity, called the *linearized Jordan identity*:

$$[T_a, T_{b \circ c}] \ + \ [T_b, T_{c \circ a}] \ + \ [T_c, T_{a \circ b}] \ = \ 0 \tag{5}$$

The first substantial theorem is that Jordan algebras are power-associative.

Theorem 1.1 *Given a Jordan algebra $\mathcal{J}$, we have the following identities:*

$$\begin{aligned} a^m a^n \ &= \ a^{m+n} \\ T_{a^m} T_{a^n} \ &= \ T_{a^n} T_{a^m} \end{aligned} \tag{6}$$

$\square$

We define a triple product

$$\{ABC\} \;=\; (A \circ B) \circ C \,+\, (B \circ C) \circ A \,-\, (A \circ C) \circ B. \tag{7}$$

In a special Jordan algebra one checks this is

$$\{ABC\} \;=\; \frac{1}{2}(ABC + CBA) \tag{8}$$

We define the operator $U_{a,c}$ by

$$\begin{aligned} U_{a,c}(b) \;&\triangleq\; \{abc\} \\ U_{a,c} \;&=\; T_a T_c + T_c T_a - T_{a \circ c}. \end{aligned} \tag{9}$$

and the operator $U_a = U_{a,a}$, which is

$$U_{a,a} \;=\; 2(T_a)^2 - T_{a^2}. \tag{10}$$

Theorem 1.2 *The Following identities hold in any Jordan algebra:*

$$\begin{aligned} (U_a)^n \;&=\; U_{a^n} \\ 2T_{a^l} U_{a^m,a^n} \;&=\; U_{a^{m+l},a^n} + U_{a^m,a^{n+l}} \\ 2U_{a^m,a^n} T_{a^l} \;&=\; U_{a^{m+l},a^n} + U_{a^m,a^{n+l}} \end{aligned} \tag{11}$$

A surprising fact is MacDonald's theorem

Theorem 1.3 (MacDonald) *Assume $P(x, y, z)$ is any polynomial in three variables that has degree at most 1 in z. If $P(x, y, z) = 0$ for all x, y, z in all special Jordan algebras, $P(x, y, z) = 0$ in all Jordan algebras.*

Theorem 1.4 (Shirshov-Cohn) *Any Jordan algebra generated by two elements is special. In the unital case, any Jordan algebra generated by two elements and 1 is special.*

Theorem 1.5 (Central Theorem) *The center of any Jordan algebra is an associative algebra.*

Theorem 1.6 (Complete Reducibility) *If $\mathcal{I} \subset \mathcal{J}$ is an ideal in a finite dimensional Jordan algebra, then $\mathcal{J} = \mathcal{I} \oplus \mathcal{I}'$.*

1.3 Projectors and the Classification

A projection in a Jordan algebra is an element p with $p^2 = p$. Lie algebras can be considered algebras of derivations; in an analogous sense, Jordan algebras can be considered algebras over projectors. As an example of this idea, consider $\mathfrak{h}_2(\mathbb{C})$. If $(x,\, y) \in \mathbb{C}^2$ has $|x|^2+|y|^2 = 1$, then

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} (x,\, y) \;=\; \begin{pmatrix} |x|^2 & \bar{x}y \\ \bar{y}x & |y|^2 \end{pmatrix} \tag{12}$$

is a projector in $\mathfrak{h}_2(\mathbb{C})$. Linear combinations of these constitute the entire algebra.

Theorem 1.7 *The finite dimensional formally real Jordan algebras are the following:*

- $\mathfrak{h}_n(\mathbb{R})$, $n \geq 3$
- $\mathfrak{h}_n(\mathbb{C})$, $n \geq 3$
- $\mathfrak{h}_n(\mathbb{H})$, $n \geq 3$
- *the spin factors* J_n
- *the Albert algebra, also known as the exceptional Jordan algebra.*

The Albert algebra is $\mathfrak{h}_3(\mathbb{O})$ *with Jordan product* $a \circ b = \frac{1}{2}(ab + ba)$.

Pf. See the reference. The idea is that Jordan algebras have so many projectors, that projections onto the entries of an $n \times n$ matrix can be found. □

This theorem was a disappointment to the physicists of the 1950's. It meant there was no "general type" formally real Jordan algebra, only the special algebras and a single exceptional algebra.

1.4 Projectors and Projective Spaces

Given two projectors $p : \mathbb{K} \to V_p$, $p : \mathbb{K} \to V_q$ we say $q \leq p$ if $V_q \subseteq V_p$ (with strict inequality corresponding to strict inequality. Given p, we can generally construct chains of inequalities of projectors:

$$0 \;=\; p_0 \;<\; p_1 \;<\; \ldots \;<\; p_n \;=\; p \tag{13}$$

Given $p \in \mathcal{J}$, the largest possible such n is called the *rank* of p in $\mathcal{J}$.

The projectors in $\mathcal{J}$ can be used to construct projective spaces. The points are the rank-1 projectors, the lines are the rank-2 projectors, and so forth. Inclusion is the inequality, so given a rank 1 projector p_1 and a rank 2 projector p_2, we can say the point p_1 is in the line p_2 provided $p_1 < p_2$.

As one expects, this process carried out in $\mathfrak{h}_n(\mathbb{R})$ gives $\mathbb{R}P^{n-1}$, in $\mathfrak{h}_n(\mathbb{C})$ gives $\mathbb{C}P^{n-1}$, and in $\mathfrak{h}_n(\mathbb{H})$ gives $\mathbb{H}P^{n-1}$. Carried out in the spin factors, one obtains a series of 1-dimensional projective planes. The algebra $\mathfrak{h}_2(\mathbb{O})$ is a spin factor, and has associated to it the octonionic projective line, a manifold of 8 real dimensions diffeomorphic to $\mathbb{S}^8$. The algebra $\mathfrak{h}_3(\mathbb{O})$ produces octonionic projective plane, a manifold of 16 real dimensions.

If $\mathbb{P}$ is a projective space generated by a one of the $\mathfrak{h}_n(\mathbb{K})$, we have that any $\vec{x} = (x_1, \ldots, x_n)$ determines a projection by first normalizing to unit length, and then defining

$$p_{\vec{x}} \;=\; \overline{\vec{x}^T}\,\vec{x} \;\in\; \mathfrak{h}_n(\mathbb{K}) \tag{14}$$

gives a point in in $\mathbb{K}P^{n-1}$.

2 Octonionic Projective Spaces

2.1 Geometric construction

Due to non-associativity, it is impossible to produce any kind of $\mathbb{O}P^n$ for $n \geq 3$. However 1- and 2-dimensional octonionic projective planes have reasonable constructions.

By analogy with other octonionic lines, we can define $\mathbb{O}P^1$ as the 1-point compactification of $\mathbb{O} \approx \mathbb{R}^8$, which is $\mathbb{S}^8$. The building of $\mathbb{O}P^2$ can take place via the octonionic Hopf map. We have the unit sphere

$$\mathbb{S}^{15} \subset \mathbb{O}^2 \tag{15}$$

and therefore a map $\mathbb{S}^{15} \to \mathbb{S}^8$ given by $(x, y) \mapsto xy^{-1}$. This gives $\mathbb{S}^{15}$ as an $\mathbb{S}^7$-bundle over $\mathbb{S}^8$. We define

$$\mathbb{O}P^2 \;=\; \left(\mathbb{D}_1 \cup \mathbb{O}P^1\right) / \, equiv \tag{16}$$

where $\mathbb{D}_1$ is the unit disk in $\mathbb{O}^2$, and the equivalence is the the Hopf map on $\partial\mathbb{D}_1 = \mathbb{S}^{15} \to \mathbb{S}^8 \approx \mathbb{O}P^1$.

2.2 The $\mathbb{O}P^n$ as projective spaces

In terms of the projective spaces $\mathbb{K}P^n$ defined above, two vectors

$$\vec{x} \;=\; (x_1,\, \ldots,\, x_{n+1}), \qquad\qquad \vec{y} \;=\; (y_1,\, \ldots,\, y_{n+1}) \tag{17}$$

determine the same point iff $p_{\vec{x}} = p_{\vec{y}}$. This determines an equivalence relation on these vectors:

$$\vec{x} \; equiv \; \vec{y} \quad \text{iff} \quad p_{\vec{x}} = p_{\vec{y}}. \tag{18}$$

An equivalence class will be denoted

$$[\vec{x}] \; = \; [x_1, \, \ldots, \, x_n] \tag{19}$$

If the base field is $\mathbb{O}$, then

$$\begin{aligned} &[x_1, \, x_2] \; \neq \; \left\{ (\alpha x_1, \, \alpha x_2) \;\middle|\; \alpha \in \mathbb{O} \right\} \\ &[x_1, \, x_2, \, x_3] \; \neq \; \left\{ (\alpha x_1, \, \alpha x_2, \, \alpha x_3) \;\middle|\; \alpha \in \mathbb{O} \right\} \end{aligned} \tag{20}$$

3 The Punchline

In the non-commutative case (let alone the non-associative case) it is difficult to define a determinant. However in the $\mathfrak{h}_2(\mathbb{K})$ case it is always possible:

$$\det \begin{pmatrix} \alpha + \beta & x \\ \overline{x} & \alpha - \beta \end{pmatrix} \; = \; \alpha^2 \, - \, \beta^2 \, - \, |x|^2 \tag{21}$$

which gives a Lorenzian metric on $\mathbb{R}^{n+1} \oplus \mathbb{R}$, and an automorphism

$$SL(2, \mathbb{K}) \to SO_0(1, 3), \tag{22}$$

assuming, of course, that $SL(2, \mathbb{H})$ and $SL(2, \mathbb{O})$ can be independently defined! Indeed they can, and we have the expected isomorphisms

$$\begin{aligned} SL(2, \mathbb{R}) \; &\approx \; Spin(1, 2) \\ SL(2, \mathbb{C}) \; &\approx \; Spin(1, 3) \\ SL(2, \mathbb{H}) \; &\approx \; Spin(1, 5) \\ SL(2, \mathbb{O}) \; &\approx \; Spin(1, 9) \end{aligned} \tag{23}$$

The first two of these we saw in a previous lecture.

There will be more about this in the homeworks—in fact $SL(n, \mathbb{H})$ and $\mathfrak{sl}(n, \mathbb{H})$ always have a reasonable definition, and also $SL(2, \mathbb{O})$, $\mathfrak{sl}(2, \mathbb{O})$ can be defined.

In the case of $A \in \mathfrak{h}_3(\mathbb{O})$ we have the following definition of the determinant:

$$\begin{aligned} det\, A \; &= \; \frac{1}{3} Tr(A^3) \, - \, \frac{1}{2} Tr(A^2) Tr(A) \, + \, \frac{1}{6} (Tr(A))^3 \\ det \begin{pmatrix} \alpha & \overline{z} & \overline{y} \\ z & \beta & x \\ y & \overline{x} & \gamma \end{pmatrix} \; &= \; \alpha\beta\gamma \, - \, \big(\alpha|x|^2 \, + \, \beta|y|^2 \, + \, \gamma|z|^2\big) \, + \, 2Re(xyz). \end{aligned} \tag{24}$$

The automorphisms that preserve the Hermitian inner product on $\mathbb{O}^2$ and $\mathbb{O}^3$ are the groups $SU(2,\mathbb{O})$ and $SU(3,\mathbb{O})$ corresponding to the groups that preserve the Jordan product in these dimensions. We have

$$SU(2,\, \mathbb{O}) \;\approx\; SO(9), \tag{25}$$

the isometry group of $\mathbb{O}P^1 \approx \mathbb{S}^8$.

The group of automorphisms of $\mathfrak{h}_3(\mathbb{O})$ that preserve the Jordan product is a real form of F_4. In an appropriate sense, we may therefore regard

$$F_4 \;\approx\; SU(3,\, \mathbb{O}). \tag{26}$$

The group of maps $\mathfrak{h}_3(\mathbb{O}) \to \mathfrak{h}_3(\mathbb{O})$ that preserve the determinant is a real form of E_6. In an appropriate sense, we may therefore regard

$$E_6 \;\approx\; SL(3,\, \mathbb{O}). \tag{27}$$

We will give justifications next time, and begin the constructions that culminate in the "magic square."