Lecture 20 - Duality and Triality

April 8, 2013

References:
Spin Geometry. M. Michelsohn and B. Lawson (1989)
The Octonions. J. Baez (2001)

1 Review

1.1 Representations

Letting $\mathbb{R}^n$ have a positive definite inner product and using the notation $\bigwedge^k = \bigwedge^k \mathbb{R}^n$ and $\bigwedge$ for the exterior algebra, we have the twisted adjoint representation

$$\widetilde{Ad} : Cl_n \longrightarrow O(n) \tag{1}$$

on $\bigwedge^1$. We also have the representation

$$Cl_n \longrightarrow Hom\left(\bigwedge, \bigwedge\right) \tag{2}$$

generated by

$$v.\eta = v \wedge \eta - i_v \eta \tag{3}$$

where $v \in \mathbb{R}^n$ and $\eta \in \bigwedge$. This representation passes to two kind of representations of Cl_n^0 on $\bigwedge$, namely

$$\begin{aligned} Cl_n^0 &\longrightarrow Hom\left(\bigwedge^{even}, \bigwedge^{even}\right) \\ Cl_n^0 &\longrightarrow Hom\left(\bigwedge^{odd}, \bigwedge^{odd}\right) \\ Cl_n^0 &\longrightarrow Hom\left(\bigwedge^{k}, \bigwedge^{k}\right) \end{aligned} \tag{4}$$

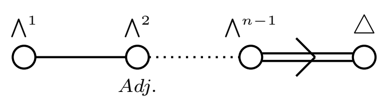
The second, obtained simply by restriction of $\bigwedge^k \to \bigwedge^{k-2} \oplus \bigwedge^k \oplus \bigwedge^{k+2}$ to $\bigwedge^k$, is simply the twisted adjoint representation of $\mathbb{R}^n$ extended in the standard way to $\bigwedge^k \mathbb{R}^n$.

We obtain a $Spin(n) \subset Cl_n^0$ representation on $\bigwedge^k$. This is a vector representation, and not strictly a spin representation. It passes to a derived representation of $\mathfrak{spin}(n) \approx \mathfrak{so}(n)$ on $\bigwedge^k$.

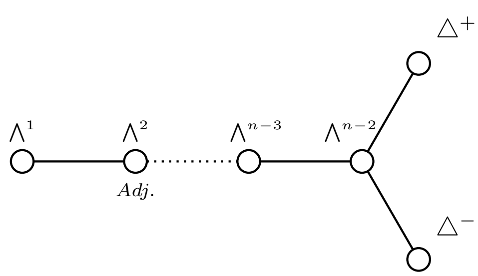
In addition, we have a natural spin representation $\triangle$ for $Spin(n)$ when n is odd, and two representations $\triangle^+$, $\triangle^-$ when n is even. When $n \equiv 2\, mod\, 4$ we have a natural equivalence $\triangle^+ = \overline{\triangle^-}$. When $n \equiv 4\, mod\, 4$ then $\triangle^+$ and $\triangle^-$ are inequivalent.

1.2 Dynkin Diagrams

In the case of B_n we have the following fundamental $\mathfrak{spin}(2n+1)$ representations:



In the case of D_n we have the following fundamental $\mathfrak{spin}(2n)$ representations:



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1.3 Table of Spin Representations

We have

n	Cl_n	Cl_n^0	$Spin(n)$	Vec	$\triangle$	$\triangle^+$	$\triangle^-$
0	$\mathbb{R}$	$-$	$-$	$-$	$-$	$-$	$-$
1	$\mathbb{C}$	$\mathbb{R}$	$\{\pm 1\} \approx O(1)$	$\mathbb{R}$	$\mathbb{R}$		
2	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{S}^1 \xrightarrow{2-1} SO(2)$	$\mathbb{R}^2$	$\mathbb{C} \oplus \overline{\mathbb{C}}$	$\mathbb{C}$	$\overline{\mathbb{C}}$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$Sp(1)$	$\mathbb{R}^3$	$\mathbb{H}$		
4	$\mathbb{H}(2)$	$\mathbb{H} \oplus \overline{\mathbb{H}}^t$	$Sp(1) \oplus Sp(1)$	$\mathbb{R}^4$	$\mathbb{H} \oplus \overline{\mathbb{H}}^t$	$\mathbb{H}$	$\overline{\mathbb{H}}^t$
5	$\mathbb{C}(4)$	$\mathbb{H}(2)$	$Sp(2)$	$\mathbb{R}^5$	$\mathbb{H}^2$		
6	$\mathbb{R}(8)$	$\mathbb{C}(4)$	$SU(4)$	$\mathbb{R}^6$	$\mathbb{C}^4 \oplus \overline{\mathbb{C}}^4$	$\mathbb{C}^4$	$\overline{\mathbb{C}}^4$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(8)$	$Spin(7)$	$\mathbb{R}^7$	$\mathbb{R}^8$		
8	$\mathbb{R}(16)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$Spin(8)$	$\mathbb{R}^8$	$\mathbb{R}^{8+} \oplus \mathbb{R}^{8-}$	$\mathbb{R}^{8+}$	$\mathbb{R}^{8-}$

(6)

The vector representation is simply the orthogonal $\widetilde{Ad}$-representation of Cl_n on $\mathbb{R}^n$, passed to $Spin(n)$, which acts by factoring through $SO(n)$. The $\triangle$ representation is the $Spin(n)$ representation obtained by restricting the basic Cl_n-representation. We have four basic cases:

$n \equiv 1 \, mod \, 4$: Under $Spin(4n+1)$, the Cl_{4n+1} representation splits into two equivalent representations, either one of which defines $\triangle$.

$n \equiv 2 \, mod \, 4$: Under $Spin(4n+2)$, the Cl_{4n+2} representation splits into two equivalent representations, which are naturally complex-conjugates of one another. We have

$$\triangle \;=\; \triangle^+ \oplus \triangle^- \;=\; \triangle^+ \oplus \overline{\triangle^+}.$$

$n \equiv 3 \, mod \, 4$: Under $Spin(4n+3)$, either of the Cl_{4n+3} representations is irreducible, and the two are equivalent; this defines $\triangle$.

$n \equiv 4 \, mod \, 4$: The Clifford representation splits into two inequivalent representations under $Spin(4n)$. We have

$$\triangle \;=\; \triangle^+ \oplus \triangle^-.$$

2 A_n and Duality

A *duality* between real vector spaces V, W is a bilinear map $F : V \otimes W \to \mathbb{R}$ so that given any non-zero $v \in V$ there is a $w \in W$ so that $F(v, w)$ is non-zero, and vice-versa. If V and W are normed vector space, we can define a *normed duality*, which is a duality F so that $|F(v, w)| \leq \|v\|\|w\|$ and so that given any $v \in V$ there is a $w \in W$ for which equality is achieved, and vice-versa.

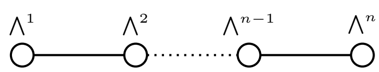
Dualities are common in mathematics; we know that if V and W have a duality, then $W = V^*$. If this duality is normed, we obtain a canonical isometric isomorphism $V \to V^*$.

Now consider the lie algebras $\mathfrak{su}(n+1)$. The complexifications are $\mathfrak{sl}(n, \mathbb{C})$, and the fundamental irreducible representations are the basic representation on $\bigwedge V_{\mathbb{C}}^1 \approx \mathbb{C}$, which lifts to a unitary representation of $SU(n+1)$. Similarly the $\mathfrak{sl}(n+1, \mathbb{R})$ action on $\bigwedge_{\mathbb{R}}^i$ lifts to a $SL(2, \mathbb{R})$ action. By a process similar to that done above for the B_n and D_n algebras, one sees that the fundamental irreducible representations are $\bigwedge^1$, $\bigwedge^2$, $\ldots$ $\bigwedge^{n-1}$.

We know there is a duality relation here: $\left(\bigwedge^i\right)^* \approx \bigwedge^{n+1-i}$, or that we have a natural map

$$F : \bigwedge\nolimits_{\mathbb{F}}^{i} \otimes \bigwedge\nolimits_{\mathbb{F}}^{n+1-i} \;\to\; \bigwedge\nolimits_{\mathbb{F}}^{n+1} \approx \mathbb{F} \tag{7}$$

which we know is a normed duality in the case that $\mathbb{F} = \mathbb{R}$. This gives an explicit realization of the order-2 outer automorphism of the A_n algebras:



The adjoint representation has highest weight $(1, 0, \ldots, 0, 1)$.

3 D_4 and Triality

3.1 Triality

A *triality* among three real vector spaces V_1, V_2, V_3 is a trilinear map

$$F : V_1 \otimes V_2 \otimes V_3 \to \mathbb{R} \tag{8}$$

so that for any non-zero $v_1 \in V_1$, $v_2 \in V_2$, there is a $v_3 \in V_3$ so that $F(v_1, v_2, v_3)$ is non-zero, and likewise for the other two cases. Note that if v_1 is fixed, then $F(v_1, \cdot, \cdot)$ is a duality between V_2 and V_3, and likewise for the other two cases. If V_1, V_2, V_3 are normed, a *normed triality* amongst them is a triality so that for any $v_1 \in V_1$, $v_2 \in V_2$, $v_3 \in V_3$ we have

$$|F(v_1, v_2, v_3)| \;\le\; \|v_1\|\|v_2\|\|v_3\| \tag{9}$$

and so that, given any $v_1 \in V_1$, $v_2 \in V_2$, there is some $v_3 \in V_3$ so that equality is obtained, and likewise for the other two cases.

Trialities are substantially less common than dualities, and normed trialities are very difficult to find. One way to find a normed triality is to build one from a product operation on a division algebra. Assuming we have a division algebra $\mathbb{D}$ and product map $\cdot$ with

$$\cdot : \mathbb{D} \otimes \mathbb{D} \;\to\; \mathbb{D} \tag{10}$$

Dualizing, we obtain

$$\cdot : \mathbb{D} \otimes \mathbb{D} \otimes \mathbb{D}^* \;\to\; \mathbb{R} \tag{11}$$

We know that such algebras have an inner product, so we can dualize to obtain a product

$$\begin{aligned} \cdot : \mathbb{D} \otimes \mathbb{D} \otimes \mathbb{D} \;&\to\; \mathbb{R} \\ \cdot\,(A,\, B,\, C) \;&=\; (A \cdot B,\, C)\,. \end{aligned} \tag{12}$$

To see that this is a normed triality, simply note

$$|(A,\, B,\, C)| \;\le\; \|A \cdot B\|\|C\| \;=\; \|A\|\|B\|\|C\| \tag{13}$$

and that equality is obtained for $C = \pm A \cdot B$.

3.2 Trialities and Division Algebras

Now conversely, given a normed triality F, we obtain a division algebra. Pick any unit vectors $v_1 \in V_1$, and $v_2 \in V_2$. Then there is a unit vector $v_3 \in V_3$ so that $F(v_1, v_2, v_3) = 1$. Since $F(v_1, \cdot, \cdot)$ is a V_2, V_3 duality, we obtain an isometric isomorphism $V_2 \leftrightarrow V_3$ where $v_2 \leftrightarrow v_3$, and likewise $V_1 \leftrightarrow V_3$ and $V_1 \leftrightarrow V_2$, where $v_1 \leftrightarrow v_2 \leftrightarrow v_3$. Thus any triality F is in fact

$$F : V \otimes V \otimes V \;\to\; \mathbb{R} \tag{14}$$

which is canonical only after choosing v_1, v_2. We have

$$F : V \otimes V \;\to\; V^* \tag{15}$$

Now since $F(v_1, \cdot, \cdot)$ is a duality, we have an isometric isomorphism $V \to V^*$ so we obtain a new map which we'll call $\star$:

$$\star : V \otimes V \;\to\; V. \tag{16}$$

Note that $v_1 \star b = b$ and $a \star v_2 = a$, so we can identify v_1, v_2, v_3 with 1, the identity element.

Note that $a \star b$ is an element $c \in V$ so that $|F(a, b, c)| = \|a\|\|b\|\|c\|$ so we have

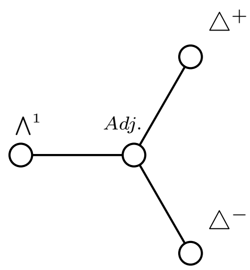
$$\|a \star b\| \;=\; \|a\|\|b\| \tag{17}$$

and we have obtained a division algebra.

3.3 D_4

We have seen that if we can find a triality, we have a division algebra.

The diagram for D_4 is



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This diagram has the largest symmetry group among all Dynkin diagrams; it is the symmetric group on 3 letters, and has a *triality* automorphism: an automorphism A so that $A^3 = 1$. The diagram automorphisms correspond to the outer automorphisms of $\mathfrak{spin}(8)$, which interchange three of the four fundamental representations: $\bigwedge^1$, $\triangle^+$ and $\triangle^-$, which therefore have the same dimension, but are inequivalent as we have already seen.

Now $\triangle^+$ and $\triangle^-$ are the ± 1 eigenspaces of ω, which, recall, commutes with $\mathfrak{spin}(8)$ but anticommutes with anything in Cl_8^1, including $\bigwedge^1 \subset Cl_8^1$. Therefore we obtain a map

$$\bigwedge^1 \otimes \triangle^+ \;\to\; \triangle^- \tag{19}$$

via Clifford multiplication. Because the action of $\bigwedge^1$ is kernel-free, this produces a triality

$$F : \bigwedge^1 \otimes \triangle^+ \otimes \triangle^{-*} \;\longrightarrow\; \mathbb{R} \tag{20}$$

Because the action of $\bigwedge^1$ is isometric (orthogonal), this is a normed triality.

3.4 Automorphisms of the Triality map

The triality map above is $Spin(8)$-invariant. To see this, let ab be any generator of $Spin(8)$, and recall its action on $\bigwedge^1$ is the twisted adjoint action, and on $\triangle^\pm$ is left Clifford multiplicaiton. Then if $v \in \bigwedge^1$ and $s^+ \in \triangle^+$ is any left-handed spinor, we have

$$\left(\widetilde{Ad}_{ab} v\right)\left(ab\, s^+\right) \;=\; abvbaabs^+ \;=\; ab\left(vs^+\right). \tag{21}$$

Notice that the unique irreducible representation of $Spin(7)$ is the spinor representation of dimension 8. Further, one can prove it is transitive on the unit sphere (maybe when there's more time we'll do this). Now there is a smaller $Spin(7)$ representation, the orthogonal representation on $\mathbb{R}^7$. After restricting to $Spin(7) \subset Spin(8)$, obtained by requiring a vector in $\bigwedge^1 \mathbb{R}^8$ be fixed, we obtain representations of $Spin(7)$ on the spaces $\triangle^\pm$. To see that these are the irreducible 8-dimensional $Spin(7)$ representations, note that the Clifford action of Cl_8^0 on $\triangle^{\pm}$ passes to Cl_7 under the standard isomorphism, so that $\triangle^{\pm}$ remain irreducible under Cl_7. If $\triangle^+$, say, were to reduce under Cl_7^0, it must split into $\mathbb{R}^7 \oplus \mathbb{R}$ where the representation on $\mathbb{R}^7$ is the orthogonal representation. However, this is impossible because $\bigwedge^1 \mathbb{R}^7 \subset Cl_7^1$ acts transitively on the unit spheres in both the 7- and 8-dimensional irreducible representations.

Thus G_2 is the subgroup of $Spin(7)$ that fixes a spinor, and

$$Spin(7)/G_2 \;=\; \mathbb{S}^7. \tag{22}$$