# Lecture 18 - Clifford Algebras and Spin groups

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*Reference: Lawson and Michelsohn, Spin Geometry.*

## 1 Universal Property

If $V$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$, let $q$ be any quadratic form, meaning a map $q : V \to W$ with $q(\alpha v) = |\alpha|^2 q(v)$ for scalars $\alpha$. Given a pair $(V, q)$ we can construct the Clifford algebra $Cl(V, q)$ as the quotient of the tensor algebra $\bigotimes V$ by the ideal generated by $v \otimes v + q(v)$, as $v$ ranges over $V$.

Note that if $q = 0$, then $Cl(V, q) = \bigwedge V$.

Ordinarily we consider the case that $q$ is nondegenerate. Over the complex numbers, nondegenerate quadratic forms are all equivalent up to change of basis, so the only invariant in the dimension of the space $V$. Setting $dim_{\mathbb{C}}(V) = n$ we obtain the algebras $\mathbb{C}l_n$. In the real setting, besides the dimension of the vector space $dim_{\mathbb{R}}(V) = n$, the only other invariant of a quadratic form is its signature, which is characterized by a pair of non-negative integers $(r, s)$ where $r + s = n$. We obtain the Clifford algebras $Cl_{r,s}$.

Clifford algebras have the following universal property. If $(V, q)$ is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with quadratic form $q$ and $\varphi : V \to GL(W, W)$ is a linear map so that $\varphi(v)\varphi(v) = -q(v)^2$, then $\varphi$ extends uniquely to a Clifford representation map $\varphi : Cl(V, q) \to GL(W, W)$.

## 2 Reason for Studying Clifford Algebras

Let $\mathbb{D}$ be a normed division algebra over reals, with norm $\|\cdot\|$. We obtain an inner product

$$\langle x,\, y\rangle \;=\; \frac{1}{2}\left(\|x\,+\,y\|^2\,-\,\|x\|^2\,-\,\|y\|^2\right). \tag{1}$$

Since $\|ax\| \;=\; \|a\|\|x\|$, we have that $L_a$ is an orthogonal transformation whenever $\|a\| = 1$.

Consider the subspace of vectors in $\mathbb{D}$ orthogonal to 1; we shall call this $Im(\mathbb{D})$. The unit sphere in $\mathbb{D}$ is a group of orthogonal transformation of $\mathbb{D}$, and the tangent space to the unit sphere at 1, its Lie algebra, can be identified with $Im(\mathbb{D})$. Therefore $L_a$ for $a \in Im(\mathbb{D})$ is skew-symmetric. If $a$ is also a unit vector, then

$$\left\langle 1,\, a^2 \right\rangle \;=\; -\left\langle a,\, a\right\rangle \;=\; -1 \tag{2}$$

so that $a^2 = -1$. If $dim(\mathbb{D}) = n$, this gives a faithful representation of $Cl_{n-1}$ on $\mathbb{D}$.

# 3 Periodicity

**Proposition 3.1** *We have the following algebra isomorphisms*

$$\begin{aligned} Cl_{n,0} \otimes Cl_{0,2} \;&\approx\; Cl_{0,n+2} \\ Cl_{0,n} \otimes Cl_{2,0} \;&\approx\; Cl_{n+2,0} \\ Cl_{r,s} \otimes Cl_{1,1} \;&\approx\; Cl_{r+1,s+1} \end{aligned} \tag{3}$$

*Pf.*

We do only the first isomorphism; the rest are entirely analogous. Consider the vector space $\mathbb{R}^{n+2}$ with orthonormal basis $e_1, \ldots, e_{n+2}$, with negative definite inner product. Also let $f_1, \ldots, f_n$ and $g_{n+1}, g_{n+2}$ be orthonormal bases of $\mathbb{R}^n$ with positive definite inner product, and $\mathbb{R}^2$ with negative definite inner product. Let $F : \mathbb{R}^{n+2} \to Cl_{n,0} \otimes Cl_{0,2}$ be the map

$$F(e_i) \;=\; \begin{cases} f_i \,\otimes\, g_{n+1} g_{n+2} & , \quad \text{if } i \in \{1, \ldots, n\} \\ 0 \,\otimes\, g_i & , \quad \text{if } i \in \{n+1, n+2\} \end{cases} \tag{4}$$

Noting that $F(e_i)F(e_j) = -F(e_j)F(e_i)$ when $i \neq j$ and $F(e_i)^2 = 1$, this extends to the desired algebra isomorphism. □

**Theorem 3.2 ("Bott periodicity")** *We have the following isomorphisms*

$$\begin{aligned} Cl_{n+8,0} \;&\approx\; Cl_{n,0} \,\otimes_{\mathbb{R}}\, Cl_{8,0} \;\approx\; Cl_{n,0} \,\otimes_{\mathbb{R}}\, \mathbb{R}(16) \\ Cl_{0,n+8} \;&\approx\; Cl_{0,n} \,\otimes_{\mathbb{R}}\, Cl_{0,8} \;\approx\; Cl_{0,n} \,\otimes_{\mathbb{R}}\, \mathbb{R}(16) \\ \mathbb{C}l_{n+2} \;&\approx\; \mathbb{C}l_n \,\otimes_{\mathbb{C}}\, \mathbb{C}l_2 \;\approx\; \mathbb{C}l_n \,\otimes_{\mathbb{C}}\, \mathbb{C}(2) \end{aligned} \tag{5}$$

*Pf.*

Noting that $\mathbb{H}^{\otimes 2} \approx \mathbb{R}(4)$, we can use the previous lemma to obtain

$$\begin{aligned} Cl_{n+8,0} \;&\approx\; Cl_{n,0} \otimes (Cl_{0,2} \otimes Cl_{2,0})^{\otimes 2} \\ &\approx\; Cl_{n,0} \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \\ &\approx\; Cl_{n,0} \otimes \mathbb{R}(4) \otimes \mathbb{R}(4) \\ &\approx\; Cl_{n,0} \otimes \mathbb{R}(16) \end{aligned} \tag{6}$$

The other cases are similar. □

## 4 Pin, Spin, and Orthogonal Groups

The "group of units" in $Cl_{p,q}$, denoted $Cl_{p,q}^{\times}$, is the group of all invertible elements. The associated Lie algebra is denoted $\mathfrak{cl}_{p,q}^{\times}$. In fact we have $\mathfrak{cl}_n^{\times} \approx Cl_{p,q}$ with the Lie bracket $[x, y] = xy - yx$.

We have an adjoint map $Ad : Cl_{p,q}^{\times} \to Aut(Cl_{p,q})$

$$Ad_{\varphi}(x) \;=\; \varphi\, x\, \varphi^{-1} \tag{7}$$

whose derivative (in the appropriate sense) is the adjoint map $ad : \mathfrak{cl}_{p,q}^{\times} \to Der(Cl_{p,q})$

$$ad_y(x) \;=\; [y,\, x]. \tag{8}$$

Let $(V, q)$ be an $n$-dimensional vector space with non-degenerate quadratic form $q$. Polarizing the quadratic form $q$ of signature $(p, q)$ we obtain an inner product $(\cdot,\, \cdot)$. Given a vector $v \in V$, let $\sigma_v$ be the reflection about the hyperplane orthogonal to $v$, so that

$$\sigma_v(x) \;=\; x \;-\; 2\frac{(x,\, v)}{(v,\, v)}\, v. \tag{9}$$

An important subgroup of $Cl^{\times}(V, q)$ is the group $P(V, q)$ generated by elements $v \in V$ with $q(v) \neq 0$. Quotienting out by constants, we obtain the Pin group. Specifically, $Pin(V, q)$ (or $Pin_{p,q}$) is the group generated by elements $v \in V$ with $q(v) = \pm 1$. Further define the spin groups to be

$$Spin(V,\, q) \;=\; Pin(V,\, q) \,\cap\, Cl^0(V,\, q). \tag{10}$$

**Proposition 4.1** *The adjoint representation of $Pin(V, q)$ on $Cl(V, q)$ restricts to a representation of $Pin(V, q)$ on $V$. For any $v \in V \cap Cl^{\times}(V, q)$ we have*

$$-Ad_v\big|_V \;=\; \sigma_v. \tag{11}$$

Thus we have found a map

$$Pin_{p,q} \;\longrightarrow\; O(p, q). \tag{12}$$

As subsets of the Clifford algebras, the spin groups contain only monomials with even numbers of generators. Thus we have found a map

$$Spin_{p,q} \;\longrightarrow\; SO(p, q). \tag{13}$$

The adjoint representation is somewhat inadequate, due to the negative. For instance, if $V$ is odd dimensional, then $Ad_v$ always preserves orientation. We introduce the twisted adjoint map

$$\widetilde{Ad}_{\varphi}(x) \;=\; \varphi\, x\, \alpha(\varphi^{-1}) \tag{14}$$

where $\alpha$ is the parity involution. We have that $\widetilde{Ad}_v$ is a representation (as $\widetilde{Ad}_\varphi \widetilde{Ad}_\eta = \widetilde{Ad}_{\varphi\eta}$) and that

$$\widetilde{Ad}_v \;=\; \sigma_v. \tag{15}$$

when $v \in V$.

**Theorem 4.2** *The $\widetilde{Ad}$ maps*

$$\begin{aligned} Pin_{r,s} &\longrightarrow O(r,s) \\ Spin_{r,s} &\longrightarrow SO(r,s) \end{aligned} \tag{16}$$

*are surjective. Their kernels are $\mathbb{Z}_2 \approx \{\pm 1\}$. Except in the cases $(r,s) = (1,1), (1,0), (0,1)$, these two-sheeted coverings are connected over any connected component of $O(r,s)$ (resp. $SO(r,s)$).*

*Pf.* The only thing to prove is that 1 and $-1$ can be joined by a path in $Spin_{r,s} \subset Pin_{r,s}$. Chosing orthogonal unit vectors $e_1, e_2$ with $q(e_1) = q(e_2)$ (either both $+1$ or both $-1$), we have that $e_1 \cos(t) + e_2 \sin(t)$ and $-e_1 \cos(t) + e_2 \sin(t)$ are unit vectors and

$$\begin{aligned} \gamma(t) &= (e_1 \cos(t) + e_2 \sin(t))\,(-e_1 \cos(t) + e_2 \sin(t)) \\ &= \pm\left(\cos^2(t) \,-\, \sin^2(t)\right) \,+\, 2e_1 e_2 \cos(t)\sin(t) \\ &= \pm\cos(2t) \,+\, e_1 e_2 \sin(2t) \end{aligned} \tag{17}$$

is a path from 1 to $-1$ in $Spin_{r,s}$. □

# 5 Spin Representations

## 5.1 Complete Reducibility

A basic theme in the study of Clifford representation is the reduction of the representation to a Pin or Spin representation. For instance, any representations of a Clifford algebra is completely reducible. To see this, note that any Clifford representation $V$ is a Pin representation, so $V$ is completely reducible as a Pin representation. But $Pin_{r,s}$ contains a basis of $Cl_{r,s}$, so the action of the Clifford algebra preserves the factors.

### 5.2 Fundamental Irreducible Clifford Representations

We have the following table

| $n$ | $Cl_n$ | $Cl_n^0$ |
|---|---|---|
| 0 | $\mathbb{R}$ | $-$ |
| 1 | $\mathbb{C}$ | $\mathbb{R}$ |
| 2 | $\mathbb{H}$ | $\mathbb{C}$ |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{H}$ |
| 4 | $\mathbb{H}(2)$ | $\mathbb{H} \oplus \mathbb{H}$ |
| 5 | $\mathbb{C}(4)$ | $\mathbb{H}(2)$ |
| 6 | $\mathbb{R}(8)$ | $\mathbb{C}(4)$ |
| 7 | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ | $\mathbb{R}(8)$ |
| 8 | $\mathbb{R}(16)$ | $\mathbb{R}(8) \oplus \mathbb{R}(8)$ |

(18)

This table can be used to give the irreducible representations of $Cl_n$. In particular, with $n \neq 3\,(mod\,4)$ we have a single irreducible representation, and with $n = 3\,mod\,4$, we have two representations. For $n = 3\,mod\,8$, these representations are inequivalent.

However, we proceed more theoretically. Consider the Clifford group, the finite group $E_n$ generated by an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. We have that the group algebra $\mathbb{R}E_n$ is nearly the Clifford algebra: in particular

$$Cl_n \;\approx\; \mathbb{R}E_n \,/\, \{(-1) \,+\, 1 = 0\}. \tag{19}$$

Note that

$$E_n \;=\; \left\{ \pm \prod_{i=1}^{m} e_i^{k_i} \;|\; k_i \;=\; \pm 1 \right\} \tag{20}$$

is a subgroup of $Cl_n$ with $2^{n+1}$ elements. Set $E_n^0 \;=\; E_n \cap Cl_n^0$, so that $E_n^0$ has $2^n$ elements. Finally, if $n$ is even, then the center of $E_n$ is $\{\pm 1\}$ and if $n$ is odd, its center is $\{\pm 1,\, \pm\omega\}$ where $\omega = e_1 \ldots e_n$.

Now consider that each element of $E_n$ acts on a 1-dimensional subspace, so we have a $2^{n+1}$-dimensional representation of $\mathbb{R}E_n$, which passes to a $2^n$ dimensional representation of $Cl_n$, which we shall call $W_n$.

**Proposition 5.1 (Clifford Representations)** *In the cases $n \equiv 0, 1, 2\,mod\,4$, we the Clifford algebra $Cl_n$ has a single irreducible representation $W_n$ of dimension $2^n$. In the case that $n \equiv 3 \mod 4$, the Clifford algebra $Cl_n$ has two inequivalent irreducible representations $W_n^+$ and $W_n^-$ of dimensions $2^{n-1}$.*

*Pf.* We consider the case of $n \equiv 3\,mod\,4$. Let $W_n$ be the algebra constructed above. We know that $\omega$ is central and $\omega^2 \,=\, 1$. Therefore $\pi^{\pm} \,=\, \frac{1}{2}\,(1 \pm \omega)$ splits the representation:

$W_n^{\pm} = \pi^{\pm} W_n$. To see that theses representations are inequivalent, notice that $\omega$ acts as 1 on $W_n^+$ and $-1$ on $W_n^-$. □

Notice that if $n = 4m$, then $\omega^2 = 1$ and $W_n$ also splits as a vector space: $W_n = W_n^+ \oplus W_n^-$ where $W_n^{\pm} = \pi^{\pm} W$. However $\pi^{\pm}$ is not central, so the factors are exchanged by elements of $Cl_n^1$. But $Cl_n^0$ preserves the factors.

We can restrict these representations $Cl_n$ on $W_n$ to $Spin_n \subset Cl_n^0 \subset Cl_n$; we denote the resulting representation $S_n$.

**Theorem 5.2 (Fundamental Spin Representations)** *If $n \equiv 1, 2, 3 \, mod \, 4$ then $S_n$ is the direct sum of two identical representations, which we call $\triangle_n$. If $n \equiv 0 \, mod \, 4$ then $S_n$ is the direct sum of two different, non-isomorphic representations; we denote $S_n = \triangle_n = \triangle_n^+ \oplus \triangle_n^-$ where $\triangle_n^{\pm} = \pi_{n-1}^{\pm} S_n$.*

*Pf.* Note that the parity involution $\alpha$ preserves $Spin_n \subset Cl_n^0 \subset Cl_n$, and that the projectors $\pi^0 = \frac{1}{2}(1 + \alpha)$, $\pi^1 = \frac{1}{2}(1 - \alpha)$ commute and sum to unity. Thus the representation of $Spin_n$ on $Cl_n$ splits into representations on $Cl_n^0$ and $Cl_n^1$.

Case $n = 2m + 1$ is odd. We have that $\omega \notin spin(n)$. If $m$ is even ($n \equiv 1 \, mod \, 4$) then the representation on $Cl_n^0$ is irreducible because $Cl_n^0 \approx Cl_{n-1}$ is irreducible. Further, the factors $Cl_n^0$ and $Cl_n^1$ are exchanged by $\omega$, which is central; hence the representations on $Cl_n^0$ and $Cl_n^1$ are equivalent. If $m$ is odd ($n \equiv 3 \, mod \, 4$) then we have a secondary decomposition $Cl_n = Cl_n^+ \oplus Cl_n^-$, where $Cl_n^0$ sits in the sum diagonally. However $Cl_n^0 \approx Cl_{n-1}$ is irreducible, and projects onto either factor isomorphically. Thus again we obtain two equivalent representations.

Case $n = 2m$ is even. We have that $\omega \in spin(n)$, and that $\omega$ is central in $Spin(n)$ (although it is not central in $Cl_n$). If $m$ is odd so $n \equiv 2 \, mod \, 4$ then we get the usual even-odd splitting.

If $m$ is even so $n \equiv 0 \, mod \, 4$ then $\omega^2 = 1$ and we can decompose $Cl_n$ into two factors, corresponding to the $\pm 1$ eigenvalues on $\omega$. These factors are clearly inequivalent representations, and $\omega \in Spin(n)$ acts differently. Also,

□

An element of a fundamental spin representation $v \in \triangle_n$ is called a spinor.