1 Formal Characters and Characters

We defined a formal character to be any element in the group algebra over the weight lattice. We express such elements as formal linear combinations over $e(\lambda)$, where the $\lambda$ lie in the weight lattice in $g^*$. Given an irreducible representation $V^\Lambda$ (with, of course, $\Lambda$ dominant integral), we define the representation’s formal character to be

$$\chi_\Lambda = \sum_{\lambda} N_\lambda e(\lambda)$$

(1)

where $\lambda$ ranges over the weights in the weight scheme of $V^\Lambda$.

Humphrey’s handles formal characters in an awkward way, which we will not adopt. But to explain, for Humphreys the formal symbols $e(\lambda)$ is interpreted to be a function on $g^*$, namely the characteristic function of $\lambda$. That is, $e(\lambda)(x) = 0$ if $x \neq \lambda$, and $e(\lambda)(\lambda) = 1$. However this does not commute with the usual multiplication of formal characters (namely algebra multiplication in the group algebra), so Humphreys switches to a convolution-type multiplication.

We will not do this. Instead, we treat $e(\lambda)$ as the formal symbol $e^{2\pi i \lambda}$ or $\exp(2\pi i \lambda)$, and interpret it as an operator (nonlinear) on $g^*$ as follows

$$e^{2\pi i \lambda}(x) = e^{2\pi i \lambda(x)}$$

(2)
where the inner product is the Killing inner product. Formal characters are then functions on $\mathfrak{g}^\ast$, given by
\[
ch_{\lambda}(x) = \sum_{\lambda} N_\lambda e^{2\pi i (\lambda, x)}
\] (3)

This interpretation is natural in the following sense. Given a (real) Lie group $G$ with maximal torus $H$, we saw that $H$ preserves the weight-space decomposition. Given a weight $\lambda$, we have a map, $\chi_{\lambda} : H \to \mathbb{S}^1 \subset \mathbb{C}$, given (implicitly) by
\[
X.v_{\lambda} = \xi_{\lambda}(X) v_{\lambda}.
\] (4)
for $X \in H$. This is called the multiplicative character of the representation. Now we have already see that $H = \text{Exp}(2\pi i \mathfrak{h})$, so if $X = \text{Exp}(2\pi i h)$ for some $h \in \mathfrak{h}$, then
\[
\xi_{\lambda}(h) = e^{2\pi i \lambda(h)} = e^{2\pi i (\lambda, h)}.
\] (5)
Then the formal character is
\[
ch_{\lambda}(h) = \text{Tr}_{V^\lambda} (e^{2\pi i h}).
\] (6)

2 Weyl’s Function and Alternating Functions

Given an algebra $\mathfrak{g}$ with base $\Delta$ and positive roots $\Phi^+$, consider the Weyl function $Q$
\[
Q = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})
= e(\delta) \prod_{\alpha \in \Phi^+} (1 - e(-\alpha))
\] (7)
This is an alternating function, in the sense that if $\sigma \in \mathcal{W}$, then
\[
\sigma Q \triangleq Q \circ \sigma = (\text{det } \sigma) \cdot Q.
\] (8)
To see this, note that if $\sigma = \sigma_\beta$, $\beta \in \Delta$ is a basic reflection, then $\sigma_\beta$ sends $\beta$ to $-\beta$ and permutes the remaining roots. Then
\[
Q(\sigma_\beta x) = \prod_{\alpha \in \Phi^+} \left( e^{\pi i (\alpha, \sigma_\beta x)} - e^{-\pi i (\alpha, \sigma_\beta x)} \right)
= \prod_{\alpha \in \Phi^+} \left( e^{\pi i (\sigma_\beta \alpha, x)} - e^{-\pi i (\sigma_\beta \alpha, x)} \right)
\]
\[
= \left( e^{\pi i (\beta, x)} - e^{-\pi i (\beta, x)} \right) \prod_{\alpha \in \Phi^+ \setminus \{\beta\}} \left( e^{\pi i (\beta \alpha, x)} - e^{-\pi i (\beta \alpha, x)} \right)
\] (9)
\[
= - \left( e^{\pi i (\beta, x)} - e^{-\pi i (\beta, x)} \right) \prod_{\alpha \in \Phi^+ \setminus \{\beta\}} \left( e^{\pi i (\alpha, x)} - e^{-\pi i (\alpha, x)} \right)
= -Q(x)
\]
Now given any function $T$ on $\mathfrak{g}^*$ we can alternate it as follows. Define an operator on functions

$$A = \sum_{\sigma \in W} (\det \sigma) \sigma$$

(10)

(where the factor acts on functions via external multiplication) and compose with $T$ to obtain

$$T' = AT.$$ (11)

We check

$$T'(\sigma x) = \sum_{\sigma \in W} (\det \sigma) T(\sigma \sigma, x)$$

$$= \sum_{\sigma \in W} (\det \sigma \sigma) T(\sigma x)$$

$$= (\det \sigma \sigma) \sum_{\sigma \in W} (\det \sigma) T(\sigma x) = (\det \sigma \sigma) T'(x).$$ (12)

We compute a second useful expressions for $Q$.

**Proposition 2.1 (Weyl Denominator Formula)** We have that $Q$ is the alternation of $e(\delta)$. Specifically

$$Q = A e(\delta)$$

$$= \sum_{\sigma \in W} (\det \sigma) e(\sigma \delta)$$

$$Q(x) = \sum_{\sigma \in W} (\det \sigma) e^{2\pi i (\sigma \delta, x)} = \sum_{\sigma \in W} (\det \sigma) e^{2\pi i (\delta, \sigma x)}$$ (13)

Pf. The highest weight exponential in $Q$ is clearly $e^{\frac{1}{2} \sum_{\alpha > 0} \alpha} = e^\delta$, while the lowest weight exponential is $e(-\delta)$. Thus $Q$ is the alternation of a sum of exponentials $e(\rho)$ that can be expressed as follows:

$$Q = A \sum_{\beta} e(\delta - \beta)$$ (14)

where the sum ranges over those $\beta$ that are sums of positive roots. Since every root is $W$-conjugate to a dominant root, we only need consider those $\beta$ so that $\delta - \beta$ has non-negative Dynkin coefficients. However, if $\rho$ has any Dynkin coefficient zero, then $A e(\rho) = 0$. To see this, note that if $(\rho, \alpha_i) = 0$, then $\sigma_i \rho = \rho$ where $\sigma_i$ is the reflection in $\alpha_i$. Then on the one hand $\sigma_i A e(\rho) = -A e(\rho)$, and on the other $\sigma_i A e(\rho) = A e(\sigma_i \rho) = A e(\rho)$. Thus $A e(\rho) = 0$. Therefore

$$Q = A e(\delta)$$ (15)
3 The Kostant Partition Formula

If \( \lambda \) is any weight in the root lattice, meaning \( \lambda = \sum_{\alpha < 0} n_\alpha \alpha \) where \( n_\alpha \in \mathbb{Z} \), define

\[
P(\lambda)
\]

(16) to be the number of ways of expressing \( \lambda \) as a sum of positive roots (not counting multiplicities or order, so for instance \( 2\alpha + \beta, \beta + \alpha + \alpha \), and \( \alpha + \beta + \alpha \) count as just one way). Obviously if \( \gamma \) is a positive sum of negative roots, then \( P(\gamma) = 0 \). One use of the partition function is in expressing the character of a Verma module. We have that \( U(n^-) \) is (vector-space) isomorphic to the polynomial generated by the \( y_\alpha \) for \( \alpha \in \Phi^+ \), and that \( V^\Lambda \) is a vector space over \( U(n^-).v^+ \) where \( v^+ \) is any highest weight vector. Thus

\[
\chi_{V^\Lambda} = \sum_\mu P(\Lambda - \mu) e(\mu) = e(\Lambda) \sum_\nu P(\nu) e(-\nu)
\]

(17)

where \( \mu \) ranges over all points in the weight lattice and \( \nu \) ranges over all points that are finite sums of roots with non-negative integral coefficients. Define \( K \), which we will call the Kostant character, to be

\[
K = \sum_\nu P(\nu)e(-\nu)
\]

(18) so that \( \chi_{V^\Lambda} = e(\Lambda) K \).

**Lemma 3.1** We have

\[
K = \prod_{\alpha > 0} (1 + e(-\alpha) + e(-2\alpha) + \ldots)
\]

(19)

*Pf.* If \( \nu \) is a positive sum of weights, then \( e(-\nu) \) is in the product on the right. Further, its coefficient is precisely the number of ways that \( e(-\nu) \) can be expressed as a product \( e(-\alpha) \ldots e(-\zeta) \) for \( \alpha, \ldots, \zeta \in \Phi^+ \).

The following lemma gives the relation between the Kostant partition function and \( Q \). Namely, \( K \) is just \( Q \) with a phase shift.

**Lemma 3.2** We have

\[
Q \cdot K = e(\delta)
\]

(20)
or

\[
Q(x) \cdot K(x) = e^{2\pi i(\delta,x)}
\]

(21)
\textit{Pf.} Since

\[ Q = \prod_{\alpha > 0} (e(\alpha/2) - e(-\alpha/2)) = e^\delta \prod_{\alpha > 0} (1 - e(-\alpha)) \] (22)

we have

\[ e^{-\delta}Q \cdot K = \prod_{\alpha > 0} (1 - e(-\alpha)) (1 + e(-\alpha) \ldots) = 1. \] (23)

Therefore \( Q \cdot ch_{\mathcal{A}} = e(\Lambda + \delta). \)