Lecture 16 - Weyl's Character Formula I: The Weyl Function and the Kostant Partition Function

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References:
A. Knapp, Lie Groups Beyond an Introduction. Ch V
Fulton-Harris, Representation Theory. Ch 24, 25
R. Cahn, Semi-Simple Lie Algebras and Their Representations. Ch XIII

1 Formal Characters and Characters

We defined a formal character to be any element in the group algebra over the weight lattice. We express such elements as formal linear combinations over $e(\lambda)$, where the λ lie in the weight lattice in $\mathfrak{g}^*$. Given an irreducible representation V^Λ (with, of course, Λ dominant integral), we define the representation's formal character to be

$$\chi_\Lambda \;=\; \sum_\lambda N_\lambda \, e(\lambda) \tag{1}$$

where λ ranges over the weights in the weight scheme of V^Λ.

Humphrey's handles formal characters in an awkward way, which we will not adopt. But to explain, for Humphreys the formal symbols $e(\lambda)$ is interpreted to be a function on $\mathfrak{g}^*$, namely the characteristic function of λ. That is, $e(\lambda)(x) = 0$ if $x \neq \lambda$, and $e(\lambda)(\lambda) = 1$. However this does not commute with the usual multiplication of formal characters (namely algebra multiplication in the group algebra), so Humphreys switches to a convolution-type multiplication.

We will not do this. Instead, we treat $e(\lambda)$ as the formal symbol $e^{2\pi i\lambda}$ or $Exp(2\pi i\lambda)$, and interpret it as an operator (nonlinear) on $\mathfrak{g}^*$ as follows

$$e^{2\pi i\lambda}(x) \;=\; e^{2\pi i(\lambda,\, x)} \tag{2}$$

where the inner product is the Killing inner product. Formal characters are then functions on $\mathfrak{g}^*$, given by

$$ch_\Lambda(x) \;=\; \sum_\lambda N_\lambda \, e^{2\pi i(\lambda,\, x)} \tag{3}$$

This interpretation is natural in the following sense. Given a (real) Lie group G with maximal torus H, we saw that H preserves the weight-space decomposition. Given a weight λ, we have a map, $\chi_\lambda : H \to \mathbb{S}^1 \subset \mathbb{C}$, given (implicitly) by

$$X.v_\lambda \;=\; \xi_\lambda(X)\, v_\lambda. \tag{4}$$

for $X \in H$. This is called the multiplicative character of the representation. Now we have already see that $H = Exp(2\pi i\, \mathfrak{h})$, so if $X = Exp(2\pi i h)$ for some $h \in \mathfrak{h}$, then

$$\xi_\lambda(h) \;=\; e^{2\pi i \lambda(h)} \;=\; e^{2\pi i(\lambda,\, h_\flat)}. \tag{5}$$

Then the formal character is

$$ch_\Lambda(h_\flat) \;=\; Tr_{V^\Lambda}\left(e^{2\pi i h}\right). \tag{6}$$

2 Weyl's Function and Alternating Functions

Given an algebra $\mathfrak{g}$ with base $\triangle$ and positive roots Φ^+, consider the *Weyl function* Q

$$\begin{aligned} Q \;&=\; \prod_{\alpha\in\Phi^+} \left(e(\alpha/2) \,-\, e(-\alpha/2)\right) \\ &=\; e(\delta) \prod_{\alpha\in\Phi^+} \left(1 \,-\, e(-\alpha)\right) \end{aligned} \tag{7}$$

This is an alternating function, in the sense that if $\sigma \in \mathcal{W}$, then

$$\sigma\, Q \;\triangleq\; Q \circ \sigma \;=\; (det\, \sigma) \cdot Q. \tag{8}$$

To see this, note that if $\sigma = \sigma_\beta$, $\beta \in \triangle$ is a basic reflection, then σ_β sends β to $-\beta$ and permutes the remaining roots. Then

$$\begin{aligned} Q(\sigma_\beta\, x) \;&=\; \prod_{\alpha\in\Phi^+} \left(e^{\pi i(\alpha,\, \sigma_\beta x)} \,-\, e^{-\pi i(\alpha,\, \sigma_\beta x)}\right) \\ &=\; \prod_{\alpha\in\Phi^+} \left(e^{\pi i(\sigma_\beta \alpha,\, x)} \,-\, e^{-\pi i(\sigma_\beta \alpha,\, x)}\right) \\ &=\; \left(e^{\pi i(\sigma_\beta \beta,\, x)} \,-\, e^{-\pi i(\sigma_\beta \beta,\, x)}\right) \prod_{\alpha\in\Phi^+\setminus\{\beta\}} \left(e^{\pi i(\sigma_\beta \alpha,\, x)} \,-\, e^{-\pi i(\sigma_\beta \alpha,\, x)}\right) \\ &=\; -\left(e^{\pi i(\beta,\, x)} \,-\, e^{-\pi i(\beta,\, x)}\right) \prod_{\alpha\in\Phi^+\setminus\{\beta\}} \left(e^{\pi i(\alpha,\, x)} \,-\, e^{-\pi i(\alpha,\, x)}\right) \\ &=\; -Q(x) \end{aligned} \tag{9}$$

Now given any function T on $\mathfrak{g}^*$ we can alternate it as follows. Define an operator on functions

$$\mathcal{A} \;=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma)\;\sigma \tag{10}$$

(where the factor acts on functions via external multiplication) and compose with T to obtain

$$T' \;=\; \mathcal{A}T. \tag{11}$$

We check

$$\begin{aligned} T'(\sigma_\alpha x) \;&=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma)T(\sigma\sigma_\alpha x) \\ &=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma_\alpha\sigma)T(\sigma x) \\ &=\; (det\,\sigma_\alpha)\sum_{\sigma\in\mathcal{W}} (det\,\sigma)T(\sigma x) \;=\; (det\,\sigma_\alpha)\,T'(x). \end{aligned} \tag{12}$$

We compute a second useful expressions for Q.

Proposition 2.1 (Weyl Denominator Formula) *We have that Q is the alternation of $e(\delta)$. Specifically*

$$\begin{aligned} Q \;&=\; \mathcal{A}\,e(\delta) \\ &=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma)e(\sigma\delta) \\ Q(x) \;&=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma)\,e^{2\pi i(\sigma\delta,\,x)} \;=\; \sum_{\sigma\in\mathcal{W}} (det\,\sigma)\,e^{2\pi i(\delta,\,\sigma x)} \end{aligned} \tag{13}$$

Pf. The highest weight exponential in Q is clearly $e^{\frac{1}{2}\sum_{\alpha>0}\alpha} = e^\delta$, while the lowest weight exponential is $e(-\delta)$. Thus Q is the alternation of a sum of exponentials $e(\rho)$ that can be expressed as follows:

$$Q \;=\; \mathcal{A}\sum_\beta e(\delta-\beta) \tag{14}$$

where the sum ranges over those β that are sums of positive roots. Since every root is $\mathcal{W}$-conjugate to a dominant root, we only need consider those β so that $\delta-\beta$ has non-negaitve Dynkin coefficients. However, if ρ has any Dynkin coefficient zero, then $\mathcal{A}e(\rho)=0$. To see this, note that if $(\rho,\alpha_i)=0$, then $\sigma_i\rho=\rho$ where σ_i is the reflection in α_i. Then on the one hand $\sigma_i\mathcal{A}e(\rho) = -\mathcal{A}e(\rho)$, and on the other $\sigma_i\mathcal{A}e(\rho) = \mathcal{A}e(\sigma_i\rho) = \mathcal{A}e(\rho)$. Thus $\mathcal{A}e(\rho)=0$. Therefore

$$Q \;=\; \mathcal{A}e(\delta) \tag{15}$$

□

3 The Kostant Partition Formula

If λ is any weight in the root lattice, meaning $\lambda = \sum_{\alpha<0} n_\alpha \alpha$ where $n_\alpha \in \mathbb{Z}$, define

$$P(\lambda) \tag{16}$$

to be the number of ways of expressing λ as a sum of positive roots (not counting multiplicities or order, so for instance $2\alpha + \beta$, $\beta + \alpha + \alpha$, and $\alpha + \beta + \alpha$ count as just one way). Obviously if γ is a positive sum of negative roots, then $P(\gamma) = 0$. One use of the partition function is in expressing the character of a Verma module. We have that $U(\mathfrak{n}^-)$ is (vector-space) isomorphic to the polynomial generated by the y_α for $\alpha \in \Phi^+$, and that $\mathcal{V}^\Lambda$ is a vector space over $U(\mathfrak{n}^-).v^+$ where v^+ is any highest weight vector. Thus

$$\chi_{\mathcal{V}^\Lambda} \;=\; \sum_\mu P(\Lambda - \mu) e(\mu) \;=\; e(\Lambda) \sum_\nu P(\nu)\, e(-\nu) \tag{17}$$

where μ ranges over all points in the weight lattice and ν ranges over all points that are finite sums of roots with non-negative integral coefficients. Define K, which we will call the Kostant character, to be

$$K \;=\; \sum_\nu P(\nu) e(-\nu) \tag{18}$$

so that $\chi_{\mathcal{V}^\Lambda} = e(\Lambda)\, K$.

Lemma 3.1 *We have*

$$K \;=\; \prod_{\alpha>0} (1 \,+\, e(-\alpha) \,+\, e(-2\alpha) \,+ \dots) \tag{19}$$

Pf. If ν is a positive sum of weights, then $e(-\nu)$ is in the product on the right. Further, its coefficient is precisely the number of ways that $e(-\nu)$ can be expressed as a product $e(-\alpha) \dots e(-\zeta)$ for $\alpha, \dots, \zeta \in \Phi^+$. □

The following lemma gives the relation between the Kostant partition function and Q. Namely, K is just Q with a phase shift.

Lemma 3.2 *We have*

$$Q \cdot K \;=\; e(\delta) \tag{20}$$

or

$$Q(x) \cdot K(x) \;=\; e^{2\pi i (\delta,\, x)} \tag{21}$$

Pf. Since

$$Q \;=\; \prod_{\alpha>0} \left(e(\alpha/2) \;-\; e(-\alpha/2)\right) \;=\; e^{\delta} \prod_{\alpha>0} (1 - e(-\alpha)) \tag{22}$$

we have

$$e^{-\delta} Q \cdot K \;=\; \prod_{\alpha>0} (1 - e(-\alpha))\,(1 \;+\; e(-\alpha)\;\ldots) \;=\; 1. \tag{23}$$

□

Therefore $Q \cdot ch_{\mathcal{V}^{\wedge}} \;=\; e(\Lambda + \delta)$.