Lecture 15 - A Few Odds and Ends

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1 Completion of the Harish-Chandra Theorem

1.1 The Setup

A semisimple Lie algebra $\mathfrak{g}$ of rank n with choice of CSA $\mathfrak{h}$, leading to a Borel subalgebra $\mathfrak{b}$ and corresponding positive an negative nilpotent algebras $\mathfrak{n}^+$ and $\mathfrak{n}^-$. Then we have a decomposition of the universal enveloping algebra

$$U(\mathfrak{g}) \;=\; U(\mathfrak{h}) \oplus \big(U(\mathfrak{h})\mathfrak{n}^+ \;+\; \mathfrak{n}^- U(\mathfrak{h})\big)\,. \tag{1}$$

Note that $U(\mathfrak{h})$ is a polynomial algebra on n generators.

1.2 The Maps

Our first maps are the central characters $\chi_\Lambda : \mathfrak{Z} \to \mathfrak{C}$, defined implicitly by $z.v = \gamma(z)v$ where $v \in V^\Lambda$ is any vector in the irreducible representation of highest weight Λ. Then we have the twisted character

$$\widetilde{\chi}_\Lambda \;\triangleq\; \chi_{\Lambda-\delta}. \tag{2}$$

The following maps were introduced, initially, to help evaluate the characters. These are the projection $\gamma : \mathfrak{Z} \to U(\mathfrak{h})$, and the twisted projection $\widetilde{\gamma} : \mathfrak{Z} \to U(\mathfrak{h})$ formed by composing the projection with the map σ defined on generators of $U(\mathfrak{h})$ by $\sigma(h) = h - \delta(h)1$.

The homomorphism $\widetilde{\gamma}$ is called the Harish-Chandra map. Our four maps are related as such: if $v^+ \in V^\Lambda$ is a highest weight vector, then

$$\begin{aligned} \chi_\Lambda(z) \;&=\; \gamma(z)(\Lambda) \\ \widetilde{\chi}_\Lambda(z) \;&=\; \widetilde{\gamma}(z)(\Lambda) \end{aligned} \tag{3}$$

where $\gamma(z)$, $\widetilde{\gamma}(z)$, as polynomials in $U(\mathfrak{h}) \approx S(\mathfrak{h})$, are considered maps $\mathfrak{h}^* \to \mathbb{C}$.

1.3 The Theorems

Now the Weyl group acts on $U(\mathfrak{h})$, and the Lie algebra $\mathfrak{g}$ acts on $U(g)$. Our first theorem was the vector space isomorphism between the polynomial modules $S(\mathfrak{h})$ and $S(\mathfrak{g})$:

$$\rho : S(\mathfrak{h})^{\mathcal{W}} \ \rightarrow \ S(\mathfrak{g})^{\mathfrak{g}} \tag{4}$$

where ρ is simply the restriction map. The proof of injectivity was relatively direct, via using the expression of Weyl-group generators as elements of $Int(\mathfrak{g})$. Surjectivity was proved by considering the symmetric invariant polynomials obtained via irreducible representations.

This is useful because $\mathfrak{Z} \subseteq U(g)$ is precisely the set of $\mathfrak{g}$-invariant elements, so we now have

$$S(\mathfrak{h})^{\mathcal{W}} \ \approx \ \mathfrak{Z} \tag{5}$$

as a vector space isomorphism.

Our next theorem was the fact that the twisted character is $\mathcal{W}$-invariant:

$$\widetilde{\chi}_{\sigma\Lambda} \ = \ \widetilde{\chi}_{\Lambda} \tag{6}$$

so that, as a consequence, we have the Harish-Chandra (algebra) homomorphism

$$\widetilde{\gamma} : \mathfrak{Z} \rightarrow U(\mathfrak{h})^{\mathcal{W}}. \tag{7}$$

Now we have vector space homomorphisms

$$S(\mathfrak{h})^{\mathcal{W}} \ \approx \ S(\mathfrak{g})^{\mathfrak{g}} \ \approx \ \mathfrak{Z} \xrightarrow{\gamma} U(\mathfrak{h}). \tag{8}$$

The only central element that acts as zero on all representations is zero, so γ has no kernel. Thus the Harish-Chandra map $\widetilde{\gamma}$ has no kernel as we have vector-space homomorphisms

$$\begin{tikzcd} U(\mathfrak{h})^{\mathcal{W}} \arrow[r, hook, "\rho"] & U(\mathfrak{g})^{\mathfrak{g}} \arrow[r, "\widetilde{\gamma}"] \arrow[d, "\approx"'] & U(\mathfrak{h})^{\mathcal{W}} \\ & \mathfrak{Z} & \end{tikzcd} \tag{9}$$

That $\widetilde{\gamma}$ is surjective can be seen by noting that $U(\mathfrak{h})^{\mathcal{W}}$ passes through with no kernel, as follows. Degree k polynomials are conjugate to degree k polynomials under $\mathcal{W}$, so $U(\mathfrak{h})^{\mathcal{W}}$ is generated by homogeneous polynomials. Such a degree k generator maps to a degree non-homogeneous polynomial under $\widetilde{\gamma}$, with the same degree k part. By throwing away lower-degree terms and using that $\widetilde{\gamma}$ is an injection, the resulting map is simply γ, the projection, and a vector-space isomorphism. Thus $\widetilde{\gamma}$ is an algebra isomorphism.

2 Subalgebras

We have a good deal of knowledge concerning maximal solvable subalgebras of simple Lie algebras. Namely these are the Borel subalgebras, and they are all conjugate via inner automorphisms. What about maximal semi-simple subalgebras? One way to obtain a maximal subalgebra is simply to delete a dot from a Dynkin diagram.

There is a trick due to Dynkin that lets us find many more maximal semisimple subalgebras. Given $\mathfrak{g}$ with base $\triangle$, set $\overline{\triangle} = \triangle \cup \{\gamma^-\}$ where γ^- is the maximally negative root (that is, the maximal root of the base $-\triangle$). Now consider sums of the form $\alpha - \beta$ where $\alpha, \beta \in \overline{\triangle}$. If $\alpha, \beta \in \triangle$, then $\alpha - \beta$ is not a root. If $\alpha = \gamma^-$, then by maximal negativity $\alpha - \beta$ is also not a root. If $\beta = \gamma^-$ then $-(\alpha - \beta)$ is not a root so $\alpha - \beta$ is also not a root.

The A_n algebras are a bit pathological, or perhaps pathalogically symmetric. Appending a maximal root turns the Dynkin diagram into a cycle, and removing a dot produces A_n back again. Removing two dots gives a maximal semi-simple subalgebra, but not a maximal subalgebra.

Thus $\overline{\triangle}$ satisfies the root-system axioms, except for linear independence. Deleting any one root from $\overline{\triangle}$ resolves this problem.

This process always produces a subalgebra. However in the cases of F_4, E_7 and E_8, these can fail to be maximal. For instance $A_3 \oplus A_1 \subset B_4 \subset F_4$.

In addition, some subalgebras are completely hidden. For instance $D_4 \subset B_4 \subset F_4$, and B_4 is a maximal subalgebra of F_4. Whereas $D_6 \oplus A_1$ is a maximal subalgebra of E_7, and D_8 is a maximal subalgebra of E_8.

There is much more to say about subalgebras, but we don't have time to go any further.

3 Transformations of the Octonions

Are there any automorphisms of the octonions? Due to non-associativity, we cannot expect the adjoint maps φ_q^1 or φ_q^2 to be automorphisms, except in the trivial case where $q \in \mathbb{R}$. Should any automorphisms exist, the group they form is called G_2.

Let φ be any automorphism. As before, it fixes the real line, and therefore the trace, and $\varphi(\overline{x}) = \overline{\varphi(x)}$. Thus it fixes the norm, and so $G_2 \subseteq O(7)$. Given points $x, y \in \mathbb{O}$ we can define a function $f_{x,y} : O(7) \to \mathbb{R}^8$ by $f_{x,y}(A) = A(xy) - A(x)A(y)$. Since G_2 is the set of common zeros of the continuous functions $f_{x,y}$, we know that G_2 is a compact subgroup of $O(7)$.

Basic Triples: Any ordered triple of purely imaginary octonions e_1, e_2, e_3 that are mutually perpendicular and so that e_3 is perpendicular to e_1e_2. Then $1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, (e_1e_2)e_3$

is an orthonormal basis for the octonions. Thus any map taking a basic triple to a basic triple is in G_2, and vice-versa. This is clearly connected. The dimension is 14: we have a 6-sphere of unit purely-imaginary quaternions for the first choice, so 5 dimensions for the second choice. The third choice must be perpendicular to the 3-dimensional quaternionic subspace so generated, so there are 3 dimensions remaining. 6+5+3=14.

We can find a copy of $SU(3)$, which is a maximal subgroup. Fix a purely imaginary quaternion, say l, and consider its isotropy group in G_2. This group permutes the six-dimensional space V orthogonal to 1 and l. Now if $v \in V$, then

$$\langle lv,\, 1\rangle \;=\; \left\langle v, \bar{l}\right\rangle \;=\; 0 \tag{10}$$

$$\langle lv,\, l\rangle \;=\; \left\langle v, \bar{l}l\right\rangle \;=\; 0 \tag{11}$$

$$\tag{12}$$

so l leaves V invariant. Choosing $i, j \in V$ so that i, j, l is a basic triple, we have a basis

$$i, j, ij, li, lj, l(ij). \tag{13}$$

Since $l^2 = -1$ we can consider V to be the vector space $\mathbb{C}^3$ with scalars of the form $a + bl$. An hermitian metric $(\cdot, \cdot)$ is usually had by defining $(v, w) = \langle v, \bar{w}\rangle$. This is a bit awkward in our situation, but we can effect the same thing by setting

$$(v,\, w) \;=\; \langle v,\, w\rangle \;+\; l\,\langle lv,\, w\rangle\,. \tag{14}$$

Since l is fixed by definition, and $\langle , \rangle$ is fixed by anything in G_2, we have that the isotropy group is a subgroup of $U(3)$. On the other hand, the element of $U(3)$ that generates the commuting circle-action is multiplication by $e^{\theta l}$, which is not a part of G_2 unless $\theta = \pi n$. Thus the isotropy group is a subset of $SU(3)$. It is a compact subgroup, and therefore reductive, so that possibilities are $\mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1$, $SU(2)$, $U(2)$, and $SU(3)$. There is no commuting subgroup, as it would pass to a commuting subgroup of the automorphisms of the quaternions. Thus the possibilities are $SU(2)$ and $SU(3)$. However $SU(2)$ has dimension 3, and there the automorphism group must strictly contain the automorphism group of the quaternions, because there is more than one basic pair in $\mathbb{C}^2 \subset \mathbb{O}$ we are considering. Therefore the automorphism group is $SU(3)$.

The group G_2 has no center, so it is semisimple. It strictly contains $SU(3)$, and since $dim(G_2) = dim(SU(3)) + 6$, we have three possibilities: G_2 is simple, $G_2 \subset SU(3) \oplus SO(4)$, $G_2 \subset SU(3) \oplus Spin(4)$. The second and third possibilities are ruled out by realizing that the action of $SU(3)$ on its compliment in G_2 is non-trivial. Equivalently (by complete reducibility of the representation), $Spin(4)$ has no non-trivial action on $\mathbb{R} \oplus \mathbb{R}l$.

Therefore G_2 is simple.

We have seen that quaternionic multiplication can be obtained via the cross product, and vice-versa. In addition, the associator can be obtained by partially dualizing the cross product. Thus G_2 may be described as the group that preserves the quaternionic cross product, and the group that preserves the quaternionic associator.

As a final note, there *is* an $SO(4)$ subgroup of G_2. This comes from the $Sp(1) \times SP(1)$ action on $\mathbb{H} \oplus \mathbb{H}$, as follows. Note that $Sp(1) \approx \mathbb{S}^3$, the unit sphere of quaternions. If (s,t) is an octonion and $(q_1, q_2) \in Sp(1) \times Sp(1)$ then set

$$(q_1, q_2).(s,t) \;=\; (q_1 s \overline{q}_1,\; q_2 t \overline{q}_1) \tag{15}$$

(notice that this is not the adjoint representation). The Kernel is ± 1, and we have an action of

$$Sp(1) \times Sp(1)/\pm 1 \;\approx\; Spin(4)/\pm 1 \;\approx\; SO(4) \tag{16}$$

on the octonions. One easily checks that this preserves the Cayley-Dickson product. Thus we can regard G_2 as $SU(3) + SO(4)$ with a twisted structure.