Lecture 14 - Isomorphism Theorem of Harish-Chandra

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This lectures shall be focused on central characters and what they can tell us about the universal enveloping algebra of a semisimple Lie algebra.

1 Invariant Polynomials

In our discussion last time of formal characters, we showed that the formal character of a representation is $\mathcal{W}$-invariant, and further that any $\mathcal{W}$-invariant character was a sum of characters of representations. The same statement holds true for central characters. Obviously we must first determine how a central character is obtained from a representation. This leads us to the study of invariant polynomials.

A polynomial on $\mathfrak{g}$ is an element of $S(\mathfrak{g}^*)$. Given a monomial $P = g_{i_1}^* \dots g_{i_k}^*$, it acts with homogeneity k on $\mathfrak{g}$ by $(g_{i_1}^* \dots g_{i_k}^*)(X) = g_{i_1}^*(X) \dots g_{i_k}^*(X)$. If $P \in S(\mathfrak{g}^*)$ is a homogeneous polynomial (of order k, say), we can *polarize* (symmetrize) it to obtain a function of k variables. If $P = g_1^* \dots g_k^*$, it acts on $\bigotimes^k \mathfrak{g}$ by

$$P(X_1, \dots, X_k) \;=\; \frac{1}{k!} \sum_{\pi \in S_k} g_1^*(X_{\pi 1}) \dots g_k^*(X_{\pi k}) \tag{1}$$

for $X_1, \dots, X_n \in \mathfrak{g}$. Note that $P(X) = P(X, \dots, X)$. Notice that many non-trivial k-homogeneous polynomials P have homegeity k action being precisely zero: $P(X) = 0$ for all X, although $P \neq 0$. Nevertheless, polynomials with non-zero homogeneity-k actions play a distinguished role in or study.

Given any homogeneity-k polynomial $P \in S(\mathfrak{g}^*)$ and $g \in \mathfrak{g}$, we have the contragredient action of g on P, given by

$$(g.P)(X_1, \dots, X_k) \;=\; -\sum_i P(X_1,\, \dots,\, [g, X_i],\, \dots,\, X_k) \tag{2}$$

A polynomial P for which $g.P = 0$, all $g \in \mathfrak{g}$ is called *invariant*. The algebra of invariant polynomials is denoted $S(\mathfrak{g}^*)^{\mathfrak{g}}$.

Let $\tau : \mathfrak{g} \to V$ be a finite-dimensional representation, and consider the homogeneity k map

$$P_{\tau,k}(X) \;=\; tr\,(\tau(X))^k. \tag{3}$$

Polarizing, we obtain

$$P_{\tau,k}(X_1,\ldots,X_k) \;=\; \frac{1}{k!}\sum_{\pi\in S_k} tr\,(\tau(X_{\pi 1})\ldots\tau(X_{\pi k})) \tag{4}$$

for $X_1,\ldots,X_k \in \mathfrak{g}$. Breaking apart this sum, consider a single monomial F, where say

$$F(X_1,\ldots,X_k) \;=\; tr(\tau(X_1)\ldots\tau(X_k)). \tag{5}$$

With $\tau([g,X]) = \tau(g)\tau(X) - \tau(X)\tau(g)$ we obtain

$$\begin{aligned}
&\sum_i F(X_1,\,\ldots,\,[g,X_i],\,X_k)\\
&\quad=\;\sum_i tr(\tau(X_1)\ldots\tau(g)\tau(X_i)\tau(X_k)) - \sum_i tr(\tau(X_1)\ldots\tau(X_i)\tau(g)\tau(X_k))\\
&\quad=\; tr(\tau(g)\tau(X_1)\ldots\tau(X_k)) - tr(\tau(X_1)\ldots\tau(X_k)\tau(g)).
\end{aligned} \tag{6}$$

which equals zero by the cyclic property of traces. Thus polynomials of the form $P_{\tau,k}$ are invariant.

Now an invariant polynomial on $\mathfrak{g}$ restricts to $\mathfrak{h}$, where in fact it is $\mathcal{W}$-invariant; the algebra of $\mathcal{W}$-invariant polynomials on $\mathfrak{h}$ is denoted $S(\mathfrak{h}^*)^{\mathcal{W}}$. To see this, simply note that P_τ is invariant under $\sigma_\alpha = e^{ad_x}e^{-ad_y}e^{ad_x}$ where $span_{\mathbb{C}}\{x,y,h\}$ is the $\mathfrak{sl}_2$-subspace of $\mathfrak{g}$ corresponding to the root α. We have already see that the contragredient action of σ_α on $\mathfrak{h}^*$ is the Weyl reflection in α. Thus we have a restriction map

$$\rho : S(\mathfrak{g}^*)^{\mathfrak{g}} \;\to\; S(\mathfrak{h}^*)^{\mathcal{W}}. \tag{7}$$

Proposition 1.1 *The restriction map ρ is an isomorphism. Indeed $S(\mathfrak{h}^*)^{\mathcal{W}}$ is spanned by polynomials of the form $P_{\tau,k}$ as τ ranges over all finite dimensional irreducible representations and k over all non-negative integers.*

Pf. If λ is a weight, note that $\lambda^k \in S(\mathfrak{h}^*)$. We can average λ^k over the action of $\mathcal{W}$ to obtain $A\lambda^k \in S(\mathfrak{h}^*)^{\mathcal{W}}$. Further, we can assume λ is dominant. Let M_λ be the set of dominant weights λ' so that $\lambda' < \lambda$. Now if τ has highest weight λ, then $P_{\tau,k}$ is a polynomial in terms of $A\lambda^k$ and lower weights. In fact $P_{\tau,k} - A\lambda_k$ is a combination of the $A(\lambda')^k$ for $\lambda' \in M_\lambda$. Now an induction argument can proceed. $\square$

We make a final note before moving on. The killing form κ produces an isomorphism $\mathfrak{g}^* \to \mathfrak{g}$, which extends to $S(\mathfrak{g}^*) \to S(\mathfrak{g})$. Due to the associativity of κ, we have in fact $\kappa : S(\mathfrak{g}^*)^{\mathcal{W}} \to S(\mathfrak{g})^{\mathcal{W}}$

Now we have a vector-space isomorphism

$$\mathfrak{B} : S(\mathfrak{g}) \to U(\mathfrak{g}) \tag{8}$$

which, among other possibilities, can be given by

$$X_1 \odot \cdots \odot X_k \;\mapsto\; \frac{1}{k!} \sum_{\pi \in S_k} X_{\pi 1} \ldots X_{\pi k} \tag{9}$$

Now if $P \in S(\mathfrak{g})^{\mathfrak{g}}$, then in fact $\mathfrak{B}(P)$ is central. This is clear from the commutator relation

$$\mathfrak{B}([g,\, X_1 \odot \cdots \odot X_k]) \;=\; [g, \mathfrak{B}(X_1 \odot \cdots \odot X_k)] \tag{10}$$

and that the left side equals zero. Lastly, if a monomial $X_1 \ldots X_k \in U(\mathfrak{g})$ is central, it is $\mathfrak{g}$-invariant, meaning

$$\begin{aligned} 0 \;&=\; [g,\, X_1 \ldots X_k] \\ &=\; [g,\, X_1]X_2 \ldots X_k \;+\; \ldots \;+\; X_1 \ldots X_{k-1}[g, X_k] \\ &=\; \mathfrak{B}\left([g,\, X_1], X_2, \ldots, X_k\right) \;+\; \ldots \;+\; \mathfrak{B}(X_1, \ldots, X_{k-1}[g, X_k]) \; mod\, S^{k-1}(\mathfrak{g}) \end{aligned} \tag{11}$$

Similarly for a sum of monomials in $U(\mathfrak{g})$. The lower-order terms are $\mathcal{W}$-invariant as well, so we see that

$$\mathfrak{B} : S(\mathfrak{g})^{\mathfrak{g}} \;\to\; U(\mathfrak{g})^{\mathfrak{g}} \;\approx\; \mathfrak{Z} \tag{12}$$

is a vector space isomorphism.

2 The Harish-Chandra map

2.1 Evaluation of characters

Given a central character χ, how do we determine its value at a point $\lambda \in \mathcal{L}$? As usual, set

$$\begin{aligned} \mathfrak{n}^+ \;&=\; \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \\ \mathfrak{n}^- \;&=\; \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \end{aligned} \tag{13}$$

Consider the decomposition of the enveloping algebra

$$U\mathfrak{g} \;=\; U\mathfrak{h} \oplus \left(U(\mathfrak{g})\mathfrak{n}^+ \;+\; \mathfrak{n}^- U(\mathfrak{g})\right) \tag{14}$$

which we know is possible due to Poincare-Birkhoff-Witt. Notice that $U(\mathfrak{g})\mathfrak{n}^+$ and $\mathfrak{n}^- U(\mathfrak{g})$ are not disjoint. Now if $z \in \mathfrak{Z}$ is central, then in fact

$$z \in U\mathfrak{h} \oplus \left(U(\mathfrak{g})\mathfrak{n}^+ \;\cap\; \mathfrak{n}^- U(\mathfrak{g})\right). \tag{15}$$

Define the map γ to be the projection onto the first factor. Note that $\mathfrak{h}$ is commutative, so $U(\mathfrak{h}) = S(\mathfrak{h})$. One easily sees γ is an algebra homomorphism. We therefore have the algebra homomorphism

$$\gamma : \mathfrak{Z} \ \rightarrow \ S(\mathfrak{h}). \tag{16}$$

How exactly does this help? Consider the irreducible highest-weight module V^Λ of highest weight Λ, and suppose v^+ is a highest-weight module. Since $\mathfrak{n}^+$ kills v^+, we have

$$z.v^+ \ = \ \gamma(z).v^+ \ = \ \Lambda(\gamma(z))v^+ \tag{17}$$

where the weight Λ acts on the polynomial $\gamma(z)$ in the obvious way. Considering $S(\mathfrak{h})$ to be the polynomial algebra on $\mathfrak{h}^*$, we can write $\Lambda(\gamma(z)) = \gamma(z)(\Lambda)$, which is probably more natural.

We have shown that

$$\chi_\Lambda(z) \ = \ \gamma(z)(\Lambda). \tag{18}$$

2.2 Twisted Chracters

This is all very nice, but, as it happens, unsymmetrical. Consider the *twisted character* $\tilde{\chi}_\Lambda$ defined by

$$\tilde{\chi}_\Lambda \ = \ \chi_{\Lambda-\delta} \tag{19}$$

where δ is the Weyl vector.

Consider also the twist map $\tau : S(\mathfrak{h}) \to S(\mathfrak{h})$ given on $\mathfrak{h} \subset U(\mathfrak{h})$ by $\tau(h) = h \ - \ \delta(h)1$. Define the *Harish-Chandra map* $\tilde{\gamma} : \mathfrak{Z} \to S(\mathfrak{h})$ by

$$\tilde{\gamma} \ = \ \tau \circ \gamma. \tag{20}$$

Then we have $\tilde{\chi}_\Lambda(z) \ = \ \tilde{\gamma}(z)(\Lambda)$.

$$\begin{aligned} \tilde{\chi}_\Lambda(z) \ &= \ \chi_{\Lambda-\delta}(z) \\ &= \ \gamma(z)(\Lambda - \delta) \\ &= \ \tau \circ \gamma(z)(\Lambda) \\ &= \ \tilde{\gamma}(z)(\Lambda) \end{aligned} \tag{21}$$

Proposition 2.1 *The twisted character is Weyl-invariant:* $\tilde{\chi}_{\sigma\Lambda} = \tilde{\chi}_\Lambda$ *for any* $\sigma \in \mathcal{W}$.

Pf. Consider the Verma module $\mathcal{V}^{\Lambda-\delta}$. Recall that this is an infinite dimensional module with a maximal submodule so that the quotient is the irreducible module of highest weight $\Lambda - \delta$. Let v^+ be a highest weight vector, so that

$$\tilde{\gamma}(z)(\Lambda)v^+ \ = \ \tilde{\chi}_\Lambda(z)v^+ \ = \ \chi_{\Lambda-\delta}(z)v^+ = \lambda(z).v^+ \tag{22}$$

Now consider the $\mathfrak{sl}_2$ subalgebra $\{x, y, h\}$ corresponding to the simple root α_i, and let r be the Dynkin coefficient $r = \langle \alpha_i, \Lambda \rangle$. Note that the i^{th} Dynkin coefficient of $\Lambda - \delta$ is $r - 1$. Therefore

$$x.y^r.v^+ \ = \ 0 \tag{23}$$

and because $[x_j, y] = 0$ for any other x_j in the root-space $\mathfrak{g}_{\alpha_j}$ (simple root α_j), we have that $x_j.y^r.v^+ = y^r.x_j.v^+ = 0$. Thus v^+ is killed by $\mathfrak{n}^+$. This means that

$$z.y^r.v^+ \ = \ \gamma(z).y^r.v^+ \ = \ \gamma(z)(\Lambda - r_i\alpha - \delta)y^r.z \ = \ \tilde{\gamma}(z)(\sigma_i\Lambda)y^r.z \tag{24}$$

where σ_i is the Weyl reflection in α_i. But

$$z.y^r.v^+ \ = \ y^r.z.v^+ \ = \ y^r.\gamma(z).v^+ \ = \ \tilde{\gamma}(z)(\Lambda)v^+ \tag{25}$$

Therefore $\tilde{\gamma}(z)(\sigma_i\Lambda) = \tilde{\gamma}(z)\Lambda$, which means

$$\tilde{\chi}_\Lambda \ = \ \tilde{\chi}_{\sigma_i\Lambda}. \tag{26}$$

□

Corollary 2.2 *The Harish-Chandra map $\tilde{\gamma}$ is a homomorphism $\tilde{\gamma} : \mathfrak{Z} \to S(\mathfrak{h})^{\mathcal{W}}$.*

3 The Harish-Chandra Isomorphism

To see the bijectivity of $\tilde{\gamma}$, first recall that we have the following vector space isomorphisms

$$\begin{aligned} S(\mathfrak{h})^{\mathcal{W}} \ &\approx \ U(\mathfrak{h})^{\mathcal{W}} \\ S(\mathfrak{g})^{\mathfrak{g}} \ &\approx \ U(\mathfrak{g})^{\mathfrak{g}} \end{aligned} \tag{27}$$

and vector space maps

$$S(\mathfrak{h})^{\mathcal{W}} \ \hookrightarrow \ S(\mathfrak{g})^{\mathfrak{g}} \ \approx \ \mathfrak{Z} \xrightarrow{\gamma} U(\mathfrak{h}). \tag{28}$$

The un-twisted map $\gamma : \mathfrak{Z} \to U(\mathfrak{h})$ is a projection, and while not an algebra homomorphism, it has no kernel on elements of $U(\mathfrak{h})^{\mathcal{W}}$. Since τ is a vector space isomorphism and $\tilde{\gamma} = \sigma \circ \gamma$ is an algebra homomorphism on $U(\mathfrak{h})^{\mathcal{W}}$, we get a composition of vector space isomorphisms

$$S(\mathfrak{h})^{\mathcal{W}} \ \hookrightarrow \ S(\mathfrak{g})^{\mathfrak{g}} \ \approx \ \mathfrak{Z} \xrightarrow{\tilde{\gamma}} U(\mathfrak{h})^{\mathcal{W}}. \tag{29}$$

so that

$$U(\mathfrak{g})^{\mathfrak{g}} \ \approx \ \mathfrak{Z} \xrightarrow{\tilde{\gamma}} U(\mathfrak{h})^{\mathcal{W}}. \tag{30}$$

is an algebra isomorphism.