Math 651 Homework 2 - Clifford Algebras and Weight Schemes Due 3/15/2013

Letting $(V, (\,,\,))$ be a real inner product space of signature (r, s), let $I_{r,s} \subset TV$ be the ideal generated by elements of the form $x \otimes y + y \otimes x + (x,\, y)$, and recall that the Clifford algebra $Cl_{r,s}$ is

$$Cl_{r,s} \;=\; TV \;/\; I_{r,s}. \tag{1}$$

As mentioned in class, it is common to start with an orthonormal basis

$$e_1,\, \dots,\, e_r,\, e_{r+1},\, \dots,\, e_{r+s} \tag{2}$$

of V, and then produce $Cl_{r,s}$ as the span of ordered monomials

$$e_{i_1} e_{i_2} \dots e_{i_p}, \qquad i_1 \,<\, i_2 \,<\, \dots \,<\, i_p \tag{3}$$

1) Three involutions can always be defined on $Cl_{r,s}$. The *parity involution* $\alpha : Cl_{r,s} \to Cl_{r,s}$ is given by sending $1 \mapsto 1$ and $e_i \mapsto -e_i$, then extending to the rest of the algebra. The *Clifford transpose* $(\cdot)^t$ is the map that reverses the order of factors in each monomial, so for instance

$$e_1 e_2 e_3 \in Cl_3 \;\longmapsto\; e_3 e_2 e_1 \;=\; -e_1 e_2 e_3. \tag{4}$$

Note that this is not a homomorphism but an anti-homomorphism, in the sense that for $a, b \in Cl_{r,s}$ we have $(ab)^t = b^t a^t$. Finally define *Clifford conjugation* to be

$$a \in Cl_{r,s} \;\longmapsto\; \alpha(a)^t \tag{5}$$

which is another anti-homomorphism.

a) Show that α, $(\cdot)^t$, and $\alpha(\cdot)^t$ are well-defined. (Hint: first show they are well defined on TV.)

b) We saw that $Cl_1 \approx \mathbb{C}$ and $Cl_2 \approx \mathbb{H}$. Show that Clifford conjugation corresponds to complex and quaternionic conjugation, respectively.

c) Recall the $\mathbb{Z}_2$-grading $Cl_{r,s} = Cl^0_{r,s} \oplus Cl^1_{r,s}$. Show that

$$\pi^0 = \frac{1}{2}\,(1+\alpha) : Cl_{r,s} \to Cl_{r,s} \tag{6}$$

is a vector space idempotent (it is not an algebra homomorphism). Show that $\pi^0 : Cl^0_{r,s} \to Cl^0_{r,s}$ is the identity and that $Ker\, \pi^0 \approx Cl^1_{r,s}$.

2) Define the *Clifford volume element* in $Cl_{r,s}$ to be the element

$$\omega \;=\; e_1 \,\ldots\, e_r\, e_{r+1} \,\ldots\, e_{r+s} \tag{7}$$

(Depending on the context, ω is sometimes called the chirality operator, or the pseudoscalar).

a) Show that $\omega^2 \;=\; (-1)^{\frac{(r+s)(r+s+1)}{2}\,-\,s}$.

b) Show that ω is central in $Cl_{r,s}$ if and only if $r+s$ is odd.

c) Set

$$\begin{aligned} \pi^+ \;&=\; \frac{1}{2}\,(1\,+\,\omega) \\ \pi^- \;&=\; \frac{1}{2}\,(1\,-\,\omega) \end{aligned} \tag{8}$$

The elements $\pi^\pm \in Cl_{r,s}$ can be thought of as maps $Cl_{r,s} \to Cl_{r,s}$ via left-multiplication. If $\omega^2 = +1$, show that $\pi^\pm$ are idempotents, so we obtain a vector space splitting

$$Cl_{r,s} \;=\; Cl^+_{r,s} \oplus Cl^-_{r,s}. \tag{9}$$

If in addition ω is central, show that (9) is an *algebra splitting.*

d) If $n \,\equiv\, 3\,(mod\,4)$, show that $Cl_n \,\approx\, Cl_n^+ \oplus Cl_n^-$ is an algebra splitting. Show that Clifford conjugation preserves the factors, and that Clifford transposition exchanges the factors.

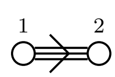
e) Explicitly show the splitting $Cl_3 = Cl_3^+ \oplus Cl_3^-$, showing that naturally

$$Cl_3 \;\approx\; \mathbb{H} \oplus \overline{\mathbb{H}}^t. \tag{10}$$

Show that the subalgebra $Cl_3^0 \approx Cl_2 \approx \mathbb{H}$ is the diagonal.

3) In class we stated that every Dynkin coefficient of the Weyl vector δ is unity. Prove this. (Hint: This can be deduced from two facts: first that any generator σ_α, $\alpha \in \triangle$, of the Weyl group permutes all positive roots except α, which is sends to its negative, and second that the Weyl group acts as orthogonally.)

4) Consider the Lie algebra $\mathfrak{g}_2$, with Dynkin diagram

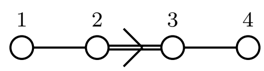


and Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \tag{11}$$

Construct the weight schemes for the $\mathfrak{g}_2$-modules of highest weight $(1, 0)$ and $(0, 1)$. Note that $\mathfrak{g}_2$ has highest root $2\alpha_1 + 3\alpha_2$, of length 5. What is its relation to $V^{(1,0)}$?

4) Notice that the $\mathfrak{g}_2$ adjoint representation has within it a second dominant integral weight (namely $(0, 1)$) which, of course, is not maximal. The root vector for this root is x_α, where α is the "maximal short root" (as defined by Humphreys, §10, exercise 11). A similar phenomenon occurs in $\mathfrak{f}_4$. The Dynkin diagram and Cartan matrix are



$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \tag{12}$$

and the highest root, of height 11, is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. Determine the highest weight of the adjoint representation. Show that the single root of height 8 is dominant integral, and is a short root.