Homework 8 Worksheet

Remember, no credit will be given for answers without justification.

1 Hamiltonian Systems

1) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 2xy \\ \frac{dy}{dt} &= -3x^2 - y^2 + 3.\end{aligned}$$

   a) Verify that this is a Hamiltonian system.

   b) Verify that the Hamiltonian H is

$$H(x, y) = x\left(x^2 + y^2 - 3\right)$$

   that the gradient of H is

$$\nabla H = \left(3x^2 + y^2 - 3,\, 2xy\right),$$

   and that the Hessian of H is

$$Hess\, H = \begin{pmatrix} 6x & 2y \\ 2y & 2x \end{pmatrix}$$

   c) Find all critical points of H, and identify what type each critical point is.

   d) There are two centers. Are they local maxima or local minima? Justify.

   e) On a large graph, plot a few level sets of H, as best you can. Plot all x- and y-nullclines. Identify any separatrices. Identify all closed regions.

2) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 4y(y^2 - x^2 - 1) \\ \frac{dy}{dt} &= 4x(y^2 - x^2 + 1)\end{aligned}$$

   a) Verify that this is a Hamiltonian system.

   b) Verify that the Hamiltonian H is

$$H(x, y) = \left((x+y)^2 - 1\right)\left((x-y)^2 - 1\right),$$

   that the gradient of H is

$$\nabla H = \left(4x\left(x^2 - y^2 - 1\right), 4y\left(y^2 - x^2 - 1\right)\right),$$

   and that the Hessian of H is

$$Hess\, H = \begin{pmatrix} 4(3x^2 - y^2 - 1) & -8xy \\ -8xy & 4(3y^2 - x^2 - 1) \end{pmatrix}$$

   c) Find all critical points of H. Using the Hessian and the second derivative test discussed in class, determine whether each point is a saddle or a center.

   d) There is a critical point at $(x, y) = (0, 0)$; is this a local maximum, mininimum, or saddle point of H? Justify.

e) On a large graph, plot a few level sets of H, as best you can. Plot all x- and y-nullclines. Identify any separatrices. Identify all closed regions.

3) (A compound saddle) Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -2xy \\ \frac{dy}{dt} &= -3x^2 + y^2.\end{aligned}$$

a) Prove the system is Hamiltonian.

b) Find the Hamiltonian, find its gradient, and show that

$$Hess\, H = \begin{pmatrix} 6x & -2y \\ -2y & -2x \end{pmatrix}$$

c) The only critical point is $(x, y) = (0, 0)$. One normally uses the second derivative test to determine if this is a saddle or center; show that the second derivative test fails.

d) On a large graph, plot the level set $H = 0$. Plot approximate trajectories of phase curves in the six regions that you find.

2 Dissipative Systems

4) Consider the system

$$\begin{aligned}\dot{x} &= y^3 \\ \dot{y} &= -x^3.\end{aligned}$$

a) This system is Hamiltonian; find a Hamiltonian function.

b) Plot the phase plane.

c) Consider the modified system

$$\begin{aligned}\dot{x} &= y^3 - 0.01x \\ \dot{y} &= -x^3 - 0.01y.\end{aligned}$$

Verify that this system is dissipative.

d) Plot the phase plane for the new system.

5) Assume the system from problem (1) is changed to

$$\begin{aligned}\frac{dx}{dt} &= 2xy \\ \frac{dy}{dt} &= -3x^2 - y^2 + 3 - 0.01xy.\end{aligned}$$

a) Verify that this system is dissipative.

b) How does the phase plane change? Make an approximate graph of the phase plane for this system.