Math 170 Extra Credit #2 Name

Section

Everybody likes money, but being good with it requires being good with numbers. The first three problems analyze, numerically, a money question which most Americans face at some point in their lives: what kind of loan is best for a home purchase?

1) Home loans are typically *amortized* loans, in which the borrowed money is paid back in equal installments over a fixed period of time. Given a principle balance P, an annual interest rate r (compounded monthly), and a term length N, the question is the following: what is your monthly payment?

   To set up this problem, let P be the initial principle balance of the loan, let r be the annual interest rate (compounded monthly), and let N be the term of the loan in months. Given this data, the unknown is x, the monthly payment.

   a) (10 points) Let P_i be the loan balance at the end of the i^{th} month, and let $P_0 = P$ be the initial loan balance.

      Each month, precisely two things happen: that month's interest is charged to your loan, and then your monthly payment is subtracted from the balance. Then we have a recursive formula for P_i:

$$\begin{aligned} P_0 &= P \\ P_i &= P_{i-1}\left(1 + \frac{r}{12}\right) - x \quad for \quad i \in \{1, \ldots, N\} \end{aligned} \tag{1}$$

      For $i \in \{0, 1, 2, 3\}$, give explicit (as opposed to recursive) expressions for P_i.

   b) (10 points) Prove that

$$P_i = P \cdot \left(1 + \frac{r}{12}\right)^i - x \cdot \frac{12}{r} \cdot \left(\left(1 + \frac{r}{12}\right)^i - 1\right) \tag{2}$$

   c) (10 points) Since the loan balance must be zero after precisely N months, it is necessary that $P_N = 0$. Using this, show that your monthly payment, x, is

$$x = P \cdot \frac{r}{12} \cdot \frac{\left(1 + \frac{r}{12}\right)^N}{\left(1 + \frac{r}{12}\right)^N - 1}. \tag{3}$$

2) As of 12/2/2013, home-loan interest rates (for borrowers with good credit and a 20% down payment) are 4.35% on 30-year loans and 3.41% on 15-year loans. Both are near historic lows.

   a) (10 points) Using your results from Problem 1, what is the monthly payment on a loan with an intial balance of $400,000 and an amortization period of 15 years? With an amortization period of 30 years?

   b) (10 points) Using your answer from Problem 2, what is the total amount you will repay to the bank over the life of the 15-year loan? Over the life of the 30-year loan?

3) (10 points) In this problem, we examine the financial wisdom of obtaining a 15-year versus a 30-year home loan. From the previous problem, it might seem "obvious" that the 15-year loan is better, assuming the higher monthly payments are affordable. However an analysis that incorporated further information shows this may not be the case. The fundamental issue is with interest rates: since they are currently very low, possibly it is wiser to lock in today's rates for as long as possible.

   In this scenario, you require a \$400,000 loan for a home purchase, and you have \$3000/month free cash available. According to your previous calculations, you clearly have more cash available to put into savings if you take out the 30 year loan.

   Currently, interest rates on savings accounts are virtually zero. Historically, you could get closer to 6% on safe investments. Assume that, after taking the loan, interest rates immediately rise to the historically reasonable 6%, where they stay for the next 30 years.

   a) (5 points) In the case you took out the 15-year loan, how much is available to place into savings each month?
   b) (5 points) In the case you took out the 30-year loan, how much is available to place into savings each month?
   c) (10 points) In the case you took the 15-year loan, how much do you have in savings at the end of the first 15 years? How much do you have in the case that you took the 30-year loan?
   d) (10 points) In the case you took the 15-year loan, how much do you have in savings after 30-years? (To compute this, assume that, after year 15 you deposit the entire \$3000 into your bank account.) How much do you have in the case that you took the 30-year loan?
   e) (10 points) Despite the complexity of this problem, still more complexity lies in real-life situations. For instance, if home prices fall, it may be wiser to wait some years before making a purchase and thereby obtaining a smaller loan, despite interest rates possibly being less favorable. What other complexities you can envision? Name at least three.

4) (10 points) If you flip a coin 20 times, what are the odds that, at some point, you get a run of 5 or more heads in a row? (Moral: randomness is streaky.)

5) (10 points) If you flip a coin 20 times, what are the odds that, at some point, you get a run of 10 or more heads in a row? (Moral: randomness is not *that* streaky.)

6) (10 points) A monkey sits down to a typewriter with 32 keys (26 letters all lower-case, a space bar, and the five symbols : ; , . ?) and begins hitting keys randomly. Determine the probability that the monkey's first 42 keystrokes spell out the phrase

   $$to\ be,\ or\ not\ to\ be,\ that\ is\ the\ question:$$

   from beginning to end, with no mistakes. Justify your answer. (The correct answer is a bit better than one chance in a billion billion billion billion billion billion billion.)

7) Complete the survey on pages 684-685 of the book.