Lecture 7 - Complete Reducibility of Representations of Semisimple Algebras

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1 New modules from old

A few preliminaries are necessary before jumping into the representation theory of semisimple algebras. First a word on creating new $\mathfrak{g}$-modules from old. Any Lie algebra $\mathfrak{g}$ has an action on a 1-dimensional vector space (or $\mathbb{F}$ itself), given by the trivial action. Second, any action on spaces V and W can be extended to an action on $V \otimes W$ by forcing the Leibnitz rule: for any basis vector $v \otimes w \in V \otimes W$ we define

$$x.(v \otimes w) \;=\; x.v \,\otimes\, w \,+\, v \,\otimes\, x.w \tag{1}$$

One easily checks that $x.y.(v \otimes w) - y.x.(v \otimes w) = [x, y].(v \otimes w)$. Assuming $\mathfrak{g}$ has an action on V, it has an action on its dual V^* (recall V^* is the vector space of linear functionals $V \to \mathbb{F}$), given by

$$(v.f)(x) \;=\; -f(x.v) \tag{2}$$

for any functional $f : V \to \mathbb{F}$ in V^*. This is in fact a version of the "forcing the Leibnitz rule." That is, recalling that we defined $x.(f(v)) = 0$, we define $x.f \in V^*$ implicitly by

$$x.\,(f(v)) \;=\; (x.f)(v) \,+\, f(x.v). \tag{3}$$

For any vector spaces V, W, we have an isomorphism

$$Hom(V,\, W) \;\approx\; V^* \otimes W, \tag{4}$$

so $Hom(V, W)$ is a $\mathfrak{g}$-module whenever V and W are. This can be defined using the above rules for duals and tensor products, or, equivalently, by again forcing the Leibnitz rule: for $F \in Hom(V, W)$, we define $x.F \in Hom(V, W)$ implicitly by

$$x.(F(v)) \;=\; (x.F)(v) \,+\, F(x.v). \tag{5}$$

2 Schur's lemma and Casimir elements

Theorem 2.1 (Schur's Lemma) *If $\mathfrak{g}$ has an irreducible representation on $\mathfrak{gl}(V)$ and if $f \in End(V)$ commutes with every $x \in \mathfrak{g}$, then f is multiplication by a constant.*

Pf. The operator f has a complete eigenspace decomposition, which is preserved by every $x \in \mathfrak{g}$. Namely if $v \in V$ belongs to the generalized eigenspace with eigenvector λ, meaning $(f - \lambda I)^k.v = 0$ for some k, then

$$(f - \lambda I)^k.x.v \;=\; x.(f - \lambda I)^k.v \;=\; 0. \tag{6}$$

Thus the generalized λ-eigenspace is preserved by $\mathfrak{g}$ and is therefore a sub-representation. By irreducibility, this must be all of V. Clearly then $f - \lambda I$ is a nilpotent operator on V that commutes with $\mathfrak{g}$. Thus $V_0 = \{v \in V \mid (f - \lambda I).v = 0\}$ is non-trivial. But V_0 is preserved by $\mathfrak{g}$, so must equal V. Therefore $f = \lambda I$. □

Now assume V is a $\mathfrak{g}$-module, or specifically that a homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ exists. As with the adjoint representation we can establish a bilinear form $B_\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$

$$B_\varphi(x, y) \;=\; Tr\,(\varphi(x)\,\varphi(y)). \tag{7}$$

If φ is the adjoint map, of course this is the Killing form. Clearly

$$B_\varphi([x, y], z) \;=\; B_\varphi(x, [y, z]) \tag{8}$$

so that the radical of B_φ is an ideal of $\mathfrak{g}$. Also, the Cartan criterion implies that the image under φ of the radical of B_φ is solvable.

Thus if φ is a faithful representation of a semisimple algebra, B_φ is non-degenerate. Letting $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$, a (unique) dual basis $\{y_i\}_{i=1}^n$ exists, meaning the y_i satisfy

$$B_\varphi(x_i, y_j) \;=\; \delta_{ij}. \tag{9}$$

We define the *casimir element* c_φ of such a representation by

$$c_\varphi \;=\; \sum_{i=1}^n \varphi(x_i)\varphi(y_i) \;\in\; End\,V. \tag{10}$$

Lemma 2.2 *Given a faithful representation φ of a semisimple Lie algebra, the casimir element commutes with all endomorphisms in $\varphi(\mathfrak{g})$.*

Pf. Let $x \in \mathfrak{g}$ be arbitrary, and define constants

$$\begin{aligned} [x, x_i] \;&=\; a_{ij}\,x_j \\ [x, y_i] \;&=\; b_{ij}\,x_j \end{aligned} \tag{11}$$

We have

$$- b_{ji} \;=\; -\sum_{k=1}^{n} b_{jk}\delta_{ik} \;=\; -B_\varphi(x_i, [x, y_j]) \;=\; B_\varphi([x, x_i], y_k) \;=\; \sum_{k=1}^{n} a_{ij}\delta_{jk} \;=\; a_{ij} \qquad (12)$$

Therefore

$$\begin{aligned} [\varphi(x),\, c_\varphi] \;&=\; \sum_{i=1}^{n} [\varphi(x),\, \varphi(x_i)\varphi(y_i)] \\ &=\; \sum_{i=1}^{n} [\varphi(x),\, \varphi(x_i)]\, \varphi(y_i) \;+\; \sum_{i=1}^{n} \varphi(x_i)\, [\varphi(x),\, \varphi(y_i)] \\ &=\; \sum_{i=1}^{n} \varphi\left([x,\, x_i]\right) \varphi(y_i) \;+\; \sum_{i=1}^{n} \varphi(x_i)\varphi\left([x,\, y_i]\right) \\ &=\; \sum_{i,j=1}^{n} a_{ij}\varphi\left(x_j\right) \varphi(y_i) \;+\; \sum_{i,j=1}^{n} b_{ij}\varphi(x_i)\varphi\left(y_j\right) \\ &=\; 0 \end{aligned} \qquad (13)$$

□

Lemma 2.3 *If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is an irreducible, faithful representation of the semisimple Lie algebra $\mathfrak{g}$, then the Casimir endomorphism c_φ acts by constant multiplication, with the constant equal to $dim(\mathfrak{g})/dim(V)$.*

Pf. That c_φ acts by constant multiplication by some $\lambda \in \mathbb{F}$ follows from Schur's lemma. We see that

$$Tr(c_\varphi) \;=\; \sum_{i=1}^{dim(\mathfrak{g})} Tr\left(\varphi(x_i)\varphi(y_i)\right) \;=\; \sum_{i=1}^{dim(\mathfrak{g})} B_\varphi(x_i,\, y_i) \;=\; dim(\mathfrak{g}) \qquad (14)$$

and also that $Tr(c_\varphi) = \lambda \cdot dim(V)$. Thus $\lambda = dim(\mathfrak{g})/dim(V)$. □

3 Weyl's Theorem

Lemma 3.1 *If $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation and $\mathfrak{g}$ is semisimple, then $\varphi(\mathfrak{g}) \subseteq \mathfrak{sl}(V)$.*

Pf. Because $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have $[\varphi(\mathfrak{g}), \varphi(\mathfrak{g})] = \varphi([\mathfrak{g}, \mathfrak{g}]) = \varphi(\mathfrak{g})$. □

Theorem 3.2 (Weyl) *Let $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation[1] of a semisimple Lie algebra. Then φ is completely reducible.*

[1] under the usual conditions: $\mathfrak{g}$ and V are finite dimensional, and the field is algebraically closed and of characterstic 0.

Pf. First, we can assume φ is faithful, for $Ker(\varphi)$ consists of summands on $\mathfrak{g}$, and we can quotient $\mathfrak{g}$ by these summands without affecting the reducibility of the representation.

Step I: *Case of an irreducible codimension 1 submodule.* Assume φ is a representation of $\mathfrak{g}$ on V, and assume $W \subset V$ is an irreducible codimension 1 submodule. The representation on V, being faithful, has a Casimir operator c_φ, which acts by constant multiplication on W (because W is irreducible). In fact $Tr(c_\varphi) = dim(\mathfrak{g}) > 0$. Since V/W is a 1-dimensional module and since $\varphi\mathfrak{g} = [\varphi\mathfrak{g}, \varphi\mathfrak{g}]$ (by the lemma), we have that V/W is a trivial $\mathfrak{g}$-module, so c_φ also acts on V/W by multiplication by 0. All this means that $c_\varphi : V \to V$ has a 1-dimensional Kernel that trivially intersects W, so

$$V \;=\; W \oplus Ker(c_\varphi). \tag{15}$$

Since c_φ commutes with $\varphi(\mathfrak{g})$, we have that $Ker(c_\varphi)$ is indeed a (trivial) $\mathfrak{g}$-module.

Step II: *Case of a general codimension 1 irreducible submodule.* Let $W \subset V$ be an arbitrary codimension 1 submodule of $\mathfrak{g}$. If W is not irreducible, there is another submodule $W_1 \subset W$, which we can assume to be maximal. Then W/W_1 is an irreducible submodule of V/W_1, and still has codimension 1. Thus by step I, we have

$$V/W_1 \;\;=\;\; W/W_1 \oplus V_1/W_1, \tag{16}$$

where V_1/W_1 is a 1-dimensional submodule of V/W_1. Because $dim(W) \neq 0$, we have $dim(V_1) < dim(V)$. We also have that W_1 is a codimension 1 submodule of V_1.

Since $dim(V_1) < dim(V)$, an induction argument lets us assert V_1 that $V_1 = W_1 \oplus \mathbb{F}z$, for some $z \in V_1$, as $\mathfrak{g}$-modules. Note that $\mathbb{F}z \cap W = \{0\}$, so $V = W \oplus \mathbb{F}z$ as vector spaces; the question is whether this is a $\mathfrak{g}$-module decomposition. However because $V/W_1 = (W/W_1) \oplus (V_1/W_1)$, we have $\mathfrak{g}.W \subseteq W$, so indeed $W \oplus \mathbb{F}z$ is a $\mathfrak{g}$-module decomposition.

Step III: *The general case.* Assume $W \subset V$ is submodule of strictly smaller dimension, and let $\mathcal{V} \subset Hom(V, W)$ be the subspace of $Hom(V, W)$ consisting of maps that act by constant multiplication on W. Let $\mathcal{W} \subset \mathcal{V}$ be the subset of maps that act as multiplication by zero on W. Moreover, $\mathcal{W} \subset \mathcal{V}$ has codimension, as any element of $\mathcal{V}/\mathcal{W}$ is determined by its scalar action on W.

However we can prove that $\mathcal{V}$ and $\mathcal{W}$ are $\mathfrak{g}$-modules. Letting $F \in \mathcal{V}$, $w \in W$, and $x \in \mathfrak{g}$, we have that $F(w) = \lambda w$ for some $\lambda \in \mathbb{F}$ and, since $x.w \in W$ also $F(x.w) = \lambda x.w$. Thus

$$(x.F)(w) = x.(F(w)) \;-\; F(x.w) \;=\; x.(\lambda w) - \lambda(x.w) \;=\; 0. \tag{17}$$

Thus all operators in $\mathfrak{g}$ take $\mathcal{V}$ to $\mathcal{W}$, so in particular they are both $\mathfrak{g}$-modules.

By Step II above, there is a $\mathfrak{g}$-submodule in $\mathcal{V}$ complimentary to $\mathcal{W}$, spanned by some operator F_1. Scaling F_1 we can assume $F_1|_W$ is multiplication by 1. Because F_1 generates a 1-dimensional submodules and $\mathfrak{g}$ acts as an element of $\mathfrak{sl}(1, \mathbb{C}) \approx \{0\}$, we have $\mathfrak{g}.F_1 = 0$. Thus we have that $x \in \mathfrak{g}$, $v \in V$ implies

$$0 \;=\; (x.F_1)(v) \;=\; x.(F_1(v)) \;-\; F_1(x.v). \tag{18}$$

This is the same as saying F_1 is a $\mathfrak{g}$-module homomorphism $V \to W$. Its kernel is therefore a $\mathfrak{g}$ module, and, since F_1 is the identity on V and maps V to W, must be complimentary as a vector space to W. Therefore

$$V \;=\; W \,\oplus\, Ker(F_1) \tag{19}$$

as $\mathfrak{g}$-modules. □