Lecture 2 - Fundamental definitions, and Engel's Theorem

September 11, 2012

1 Basic Definitions

A *representation* of a Lie algebra L is a homomorphism φ of L into the Lie algebra $\mathfrak{gl}(V)$ for some vector space V over $\mathbb{F}$. Every Lie algebra has at least one representation, the adjoint representation $ad : L \to End(V)$.

A *subalgebra* K of L is a subspace that is closed under the bracket.

An *ideal* I of K is a subalgebra so that $x \in L$, $y \in I$ implies $[x, y] \in I$.

If I and J are ideals, so is $I + J$, defined to be the set of all elements $c_1x + c_2y$ where $c_1, c_2 \in \mathbb{F}$, $x \in I$, $y \in J$.

If I and J are ideals, so is $[I,]\, J]$, which is defined to be the vector space spanned by elements of the form $[x,\, y]$ where $x \in J$ and $y \in J$.

Two ideals possessed by any Lie algebra are the *derived algebra* $[L,\, L]$, and the *center* $Z(L)$ or $C(L)$, defined to be the set of elements $x \in L$ so that $[x, y] = 0$ for all $y \in L$. Either of these algebras may be trivial or may equal L itself.

Proposition 1.1 (Humphreys 2.2) *Assuming $I, J \subset L$ are ideals. Then $(I + J)/J$ is canonically isomorphic to $I/(I \cap J)$, and if $I \subseteq J$ then K/J is canonically isomorphic to $(K/I)/(J/I)$*

If $K \subseteq L$ is a subalgebra, we define the *normalizer* of K in L

$$N_L(K) \;=\; \{\, x \in L \;\big|\; [x, K] \in K \,\}. \tag{1}$$

Using the Jacobi identity, $N_L(K)$ can be seen to be a subalgebra of L. It is the largest subsalgebra that contains K as an ideal. If $N_L(K) = L$ then K is an ideal. If $N_L(K) = K$ then K is said to be self-normalizing.

If $K \subseteq L$ is a subalgebra, we define the *cetralizer* of K in L to be

$$C_L(K) \;=\; \{\, x \in L \;\big|\; [x, K] = 0 \,\}. \tag{2}$$

Note that K is usually not contained in $C_L(K)$. Also, $C_L(L) = C(L)$.

A *derivation* of a Lie algebgra L is a linear map $\delta : L \to L$ so that $\delta[x, y] = [\delta x, y] + [x, \delta y]$. The vector space of derivations $Der(L)$ is in fact a Lie algebra; it is easily checked that if $\delta, \delta' \in Der(L)$ then $[\delta, \delta'] \triangleq \delta\delta' - \delta'\delta$ is also a derivation. By the Jacobi identity, the adjoint map can be thought of as $ad : L \to Der(L)$.

If $[L, L] = \{0\}$ then L is called *abelian*. If L has no nontrivial proper ideals, then L is called *simple*.

If L, N are Lie algebras, a *homomorphism* $\varphi : L \to N$ is a linear map that commutes with the backets; that is

$$\varphi\left([x,\, y]\right) \;=\; [\varphi(x),\, \varphi(y)] \tag{3}$$

whenever $x, y \in L$. The *Kernel* of φ, denote $Ker(\varphi)$ is the vector space kernel of the map φ. It is easily checked that $Ker(\varphi)$ is an ideal.

A homomorphism $\varphi : L \to N$ is called a *monomorphism* if $Ker(\varphi)$ is the trivial subspace. It is called an epimorphism if its image is N. It is called an isomorphism if it is a monomorphism and an epimorphism.

Proposition 1.2 (Humphreys 2.2) *If $\varphi : L \to N$ is a homomorphism, then $Im(\varphi)$ is canonically ismorphic to $L/Ker(\varphi)$.*

A Lie algebra L is called a linear Lie algebra if it is a subalgebra of $\mathfrak{gl}(V)$ for some finite dimensional vector space V.

Proposition 1.3 *Any simple Lie algebra is isomorphic to a linear Lie algebra*

2 Solvable Lie algebras

Let L be a Lie algebra. We can define its derived series $L^{(0)}$, $L^{(1)}$, ... by $L^{(0)} = L$ and

$$L^{(k)} \;=\; \left[L^{(k-1)}, L^{(k-1)}\right]. \tag{4}$$

We call L *solvable* if $L^{(n)} = \{0\}$ for some n.

Proposition 2.1 (Humphreys 3.1) *Let L be a Lie algebra.*

a) If L is solvable, so are all subalgebras and all homomorphic images.

b) If $I \subseteq L$ is a solvable ideal and L/I is solvable, then L is solvable.

c) If $I, J \subseteq L$ are solvable ideals, then $I + J$ is a solvable ideal.

Pf. Easy. □

3 Nilpotent Lie algebras

Let L be a Lie algebra. We can define its descending central series L^0, L^1, ... by $L^0 = L$ and

$$L^k = \left[L, L^{k-1}\right]. \tag{5}$$

We call L *nilpotent* if $L^n = \{0\}$ for some n.

Proposition 3.1 (Humphreys 3.2) *Let L be a Lie algebra.*

a) If L is nilpotent, so are all subalgebras and all homomorphic images.

b) If $L/Z(L)$ is nilpotent, so is L.

c) If L is nilpotent, then $Z(L)$ is not trivial.

Pf. a) Easy
b) If $L/Z(L)$ is nilpotent then $L^n \subseteq Z(L)$ for some L. Then $L^{n+1} = \{0\}$.
c) There is some n so that $L^n = \{0\}$ but $L^{n-1} \neq \{0\}$. Clearly $\{0\} = L^n = [L, L^{n-1}]$ implies $L^{n-1} \subseteq Z(L)$. □

<u>Definition</u> An element $x \in L$ is called *ad-nilpotent* if $(adx)^n = 0$ for some n. A Lie algebra L is called *ad-nilpotent* if every element of L is ad-nilpotent.

If $x, y \in L$, then $(adx)^n y = [x, [x, \ldots, [x, y] \ldots],] \in L^{n+1}$. Thus if L is nilpotent, it is ad-nilpotent.

Theorem 3.2 (Engel's Theorem) *If L is ad-nilpotent, it is nilpotent*

Theorem 3.3 *If L is a subalgebra of $\mathfrak{gl}(V)$ (V finite dimensional) and every $x \in L$ is a nilpotent transformation (meaning given x there is some $n \in \mathbb{N}$ so that $x^n.v = 0$ whenever $v \in V$), then there is some $v \in V$ so that $x.v = 0$ for all $x \in L$.*

Pf. Induction on the dimension of L. The theorem is clearly true for all L with $dim(L) = 1$. This is because $x \in L$ implies $x^n.v = 0$ for some n, so that there is a largest $i \in \ltimes$ with $x^i.v \neq 0$ but $x^{i+1}.v = 0$, in which case $v_i = x^i.v$ is a zero eigenvector.

Assume the theorem is true for all Lie algebras K with $dim(K) < dim(L)$. Let K be any maximal subalgebra of L—clearly subalgebras exist, for instance 1-dimensional subalgebras. We will prove first that K has codimension 1, and since the theorem is true for the action of K on V we are left just a single basis element whose action must be checked.

To prove K has codimension 1, consider the adjoint action of K on L/K (of course L/K is not a Lie algebra but only a vector space; still the action of K is well-defined (check)). By the inductive hypothesis, there is some vector $z \in L$ so that $z + K \in L/K$ is a zero-eigenvector for every element of K. This means that $K + \mathbb{F}z$ is also a subalgebra that strictly contains K, implying that either K was not maximal (which it is) or that $K + \mathbb{F}z$ is in fact L, verifying that K has codimension 1. Since $L = K + \mathbb{F}z$ and we showed$[z, k] \in K$ when $k \in K$, K is an ideal.

Now let $W \subseteq V$ be the subspace consisting of all zero-eigenvectors for K, or

$$W \;=\; \{\, w \in V \mid k.w = 0 \; when \; k \in K \,\}. \tag{6}$$

To see that L fixes W, let $w \in W$, $y \in L$, and $k \in K$. Since

$$k.y.w \;=\; y.k.w \;+\; [k,\, y].w \tag{7}$$

Because $k, [k, y] \in K$, the right-side is zero. Therefore $y.w \in W$. Using again $L = K + \mathbb{F}z$ and knowing that z has a nilpotent action on V and therefore on W, there must be a zero-eigenvector w' for z in W. Thus w' is a zero-eigenvector for all of L. $\square$

Proof of Engel's theorem. Again we argue inductively on the dimension of L, assuming Engel's theorem holds for all Lie algebras K with $dim(K) < dim(L)$. The adjoint action $ad : L \to \mathfrak{gl}(L)$ expresses L as an algebra of nilpotent endomorphisms, which therefore have a common zero-eigenvector. Thus $Z(L)$ is non-trivial. Since $L/Z(L)$ is ad-nilpotent, and therefore nilpotent by the induction hypothesis, L is also nilpotent. $\square$