Lecture 1 - Basic Definitions and Examples of Lie Algebras

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1 Definition

A *Lie algebra* $\mathfrak{l}$ is a vector space V over a base field $\mathbb{F}$, along with an operation $[\cdot, \cdot] : V \times V \to V$ called the *bracket* or *commutator* that satisfies the following conditions:

- Bilinearity: $\alpha[x, y] = [\alpha x, y] = [x, \alpha y]$ for $\alpha \in \mathbb{F}$ and $x, y \in V$
- Antisymmetry: $[x,\, y] \;=\; -[y,\, x]$ for $x, y \in V$
- The Jacobi Identity: $[x,\, [y,\, z]] \,+\, [y,\, [z,\, x]] \,+\, [z,\, [x,\, y]] \;=\; 0$ for $x, y, z \in V$.

This definition works for any base field except those of characteristic 2, in which case the second condition is replaced by $[x, x] = 0$, which is then not equivalent to $[x, y] = -[y, x]$. However most theorems required algebraic completeness, which will generally be assumed. We will also generally ignore the case of finite characteristic, which sometimes requires additional considerations. Thus realistically we are talking about $\mathbb{F} = \mathbb{C}$. In addition, we almost always require that V be finite dimensional.

It should be noted that the Jacobi identity is formally identical to the Leibnitz rule.

A brute-force construction of an arbitrary Lie algebra may begin with selecting a finite-dimensional vector field V, a basis $\{e_i\}_{i=1}^n$, and n^3 many constants $c_{ij}^k \in \mathbb{F}$ subject to the relations

- $c_{ij}^k = -c_{ji}^k$
- $\sum_{k=1}^n \left(c_{is}^u c_{jk}^s \,+\, c_{js}^u c_{ki}^s \,+\, c_{ks}^u c_{ij}^s \right) \;=\; 0.$

One then defines the brackets $[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k$ and extends by linearity.

2 Non-Fundamental Examples

2.1 The abelian case

If V is any vector space with origin o, it has the structure of a Lie algebra where we define

$$[x,\, y] \;=\; o \tag{1}$$

for all $x, y \in V$. Any such example is called an abelian Lie algebra.

2.2 The cross product

In high school we had the cross product on $\mathbb{R}^3$, given by

$$\vec{X} \times \vec{Y} \;=\; \sin(\theta)|\vec{X}||\vec{Y}| \cdot \hat{n}_{\vec{X}\vec{Y}} \tag{2}$$

where $\hat{n}_{\vec{X}\vec{Y}}$ is the unit normal to $\vec{X}$ and $\vec{Y}$ chosen by the right-hand rule, and θ is the angle between the vectors. Orientation (the right-hand rule) requires that

$$\vec{X} \times \vec{Y} \;=\; -\vec{Y} \times \vec{X} \tag{3}$$

and a simple check shows that

$$\vec{X} \times \left(\vec{Y} \times \vec{Z}\right) + \vec{Y} \times \left(\vec{Z} \times \vec{X}\right) + \vec{Z} \times \left(\vec{X} \times \vec{Y}\right) \;=\; 0. \tag{4}$$

2.3 The Heisenberg algebra

In 1-dimensional quantum mechanics, the state of a system is given by an L^2 function $f : \mathbb{R} \to \mathbb{R}$, denoted $f(q)$. The collection of all such states is the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. The coordinate q can itself be considered an operator $q : \mathcal{H} \to \mathcal{H}$, which acts by multiplication: $f(q) \mapsto qf(q)$. Thinking for a moment like a physicist and ingoring the fact that derivatives of L^2 functions do not exist and might not belong to L^2 if they did, a second operator $p : \mathcal{H} \to \mathcal{H}$, called the momentum operator, can be given by

$$p\,(f) \;=\; \sqrt{-1}\hbar \frac{\partial f}{\partial q}. \tag{5}$$

Letting $[\cdot, \cdot]$ be the usual commutator, the action of $[p, q]$ on $f \in \mathcal{H}$ is

$$\begin{aligned} [p,\, q]\; f \;&=\; \sqrt{-1}\,\hbar\, \frac{\partial}{\partial q}\,(q\, f) \;-\; q\left(\sqrt{-1}\,\hbar\, \frac{\partial f}{\partial q}\right) && (6)\\ &=\; \sqrt{-1}\,\hbar\, f, && (7) \end{aligned}$$

or in other words, multiplication by the constant $\sqrt{-1}\,\hbar$. The *Heisenberg algebra* is the Lie algebra on

$$V \;=\; span_{\mathbb{C}}\{p,\, q,\, \sqrt{-1}\,\hbar\} \tag{8}$$

with this bracket operation.

Of course this algebra can be defined abstractly by $V = span_{\mathbb{C}}\{X, Y, Z\}$ where

$$[X,\, Y] \;=\; Z \qquad [X,\, Z] \;=\; 0 \qquad [Y,\, Z] \;=\; 0. \tag{9}$$

The Heisenberg algebra is isomorphic to the concrete example given by

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

and $V = span_{\mathbb{C}}\{X, Y, Z\}$, with the bracket given by the usual matrix commutator.

2.4 The angular momentum operators

In 3D quantum mechanics, we have three position operators q^1, q^2, q^2 and three momentum operators $p_i = \sqrt{-1}\hbar\frac{\partial}{\partial q^i}$, with the usual commutators. From these other useful operators are defined, such as l_x, l_y, l_z, whose eigenvalues record a wavefunction's angular momenta about the the respective axes. They are defined by

$$l_x \;=\; q^2 p_3 \,-\, q^3 p_2 \qquad l_y \;=\; q^3 p_1 \,-\, q^1 p_3 \qquad l_z \;=\; q^1 p_2 \,-\, q^2 p_1. \tag{11}$$

One easily chechs the commutator relations

$$[l_x,\, l_y] \;=\; \sqrt{-1}\,\hbar\, l_z \qquad [l_y,\, l_z] \;=\; \sqrt{-1}\,\hbar\, l_x \qquad [l_z,\, l_x] \;=\; \sqrt{-1}\,\hbar\, l_y. \tag{12}$$

Under the basis $X = \frac{1}{\sqrt{-1}\,\hbar} l_x, Y = \frac{1}{\sqrt{-1}\,\hbar} l_y, Z = \frac{1}{\sqrt{-1}\,\hbar} l_z$, we have

$$[X,\, Y] \;=\; Z \qquad [Y,\, Z] \;=\; X \qquad [Z,\, X] \;=\; Y. \tag{13}$$

This is just the cross product algebra (from above), and is the same as the Lie algebras $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ (below).

2.5 Lie Groups

A Lie group is a differentiable manifold G along with a group structure so that the group operation (multiplication and inversion) are differentiable. Any Lie group M has an associated Lie algebra }, given by derivations at the identity $I \in G$. Specifically, any element a of the tangent space $T_I G$ at the identity can be represented by a path $A : (-\epsilon, \epsilon) \to M$ with $A(0) = I$. Given $a, b \in T_I G$, we define their bracket $[a,\, b]$ to be the vector given by one half the second derivative of the path

$$t \;\mapsto\; A(t)\, B(t)\, A(t)^{-1}\, B(t)^{-1}. \tag{14}$$

If G is an Abelian group, then clearly $\mathfrak{g}$ is an abelian Lie algebra.

2.6 Linear Lie Groups

If G is a linear Lie group, meaning a subgroup of some $L(V)$, then its Lie algebra $\mathfrak{g}$, defined before as derivations at the identity, will have its bracket given precisely by the usual commutator operation on matrices. Indeed, letting the paths $A(t), B(t)$ represent the vectors a, b, then using the definition we compute

$$\begin{aligned}
[a,\, b] &= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} A(t)B(t)A(t)^{-1}B(t)^{-1} && (15)\\
&= \left(\frac{dA}{dt}\frac{dB}{dt} + \frac{dA^{-1}}{dt}\frac{dB^{-1}}{dt} + \frac{dA}{dt}\frac{dB^{-1}}{dt} + \frac{dB}{dt}\frac{dA^{-1}}{dt}\right)\Big|_{t=0} && (16)\\
&= \dot{A}(0)\dot{B}(0) - \dot{B}(0)\dot{A}(0) = ab - ba && (17)
\end{aligned}$$

where we have abbreviated $\dot{A}(\tau) = \frac{d}{dt}\big|_{t=\tau} A(t)$.

Example. Consider the *special Linear group* $SL(n, \mathbb{C})$, defined to be group of matrices with unit determinant. If $a(t)$ is a path in $SL(n, \mathbb{C})$ with $det(a(t)) = 1$ and $a(0) = I$, then

$$0 = \frac{d}{dt}\Big|_{t=0} \det(a(t)) = \text{Trace}\left(\frac{da}{dt}\Big|_{t=0}\right) \tag{18}$$

so the associated Lie algebra $\mathfrak{g}$ is the *special linear algebra*, or $\mathfrak{sl}(n, \mathbb{C})$, which consists of the trace-free matrices. (Of course we only proved containment—actual equality comes from exponentiation along with $det(e^A) = e^{Tr(A)}$.)

Example. Consider the *orthogonal group*, $O(n)$, consisting of $n \times n$ matrices A such that $A^T A = I$. Taking derivatives at the origin, we see that if $A(t)$ is a path through $I \in O(n)$ with $a = \dot{A}(0)$, we have

$$0 = \frac{d}{dt}\Big|_{t=0} \left(A(t)^T A(t)\right) = a^T + a. \tag{19}$$

The Lie algebra therefore associated with $O(n)$ is the *orthogonal algebra* $\mathfrak{o}(n)$, consisting of antisymetric $n \times n$ matrices.

Example. The third main examples is the symplectic group on V. Assume V has dimension $2n$, and, after choosing a basis, let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{20}$$

The symplectic group $Sp(2n, \mathbb{C})$ consists of matrices A so that $A^T J A = I_{2n}$. Taking derivatives, we see that if $A(t)$ is a path in $Sp(2n, \mathbb{C})$ with $A(0) = I_{2n}$ and $\dot{A}(0) = a$, then

$$0 = \frac{d}{dt}\Big|_{t=0} \left(A^T J A\right) = a^T J + J\, a. \tag{21}$$

The *symplectic algebra* $\mathfrak{sl}(2n, \mathbb{C})$ is the set of complex matrices a so that $a^T J = -Ja$. (In a bit of unfortunate notation, the groups $Sp(2n, \mathbb{C})$ are not the same as the groups $Sp(n)$.)

3 The Fundamental Examples

3.1 The general linear algebra, $\mathfrak{gl}$

Let V be any vector space, with $L(V)$ its linear group ($End(V)$ is equivalent notation). We know $L(V)$ is a vector space, and has the structure of an associative algebra under the usual operation of composition. It also carries the structure of a Lie algebra, denoted $\mathfrak{gl}(V)$, where the bracket is the usual commutator

$$[x,\, y] \;=\; x \circ y \;-\; y \circ x. \tag{22}$$

(Exercise: Verify the Jacobi identity). The Lie algebra $\mathfrak{gl}(V)$ should not be confused with the general linear group $GL(V)$ (the subgroup of $L(V)$ of invertible transformations); in particular $GL(V)$ is not a vector space so cannot be a Lie algebra.

Any subspace of any $gl(V)$ that is closed under the commutator operation is known as a linear Lie algebra.

3.2 Series A, B, C, and D

Cartan's notation for the special linear algebras was A_l, which is defined to be simply $\mathfrak{sl}(l+1, \mathbb{C})$. Likewise, the C-series algebras are precisely the symplectic algebras defined above: $C_l = \mathfrak{sp}(2l, \mathbb{C})$. For reasons that are certainly not clear at present, the orthogonal algebras are divided into two series, with the B-series being the odd dimensional and the D-series being the even dimensional orthogonal algebras. Specifically, Humphreys defines B_l to be the algebra of $(2l+1) \times (2l+1)$ matrices (with complex entries) a so that $a^T \Omega + \Omega\, a = 0$, where

$$\Omega \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}. \tag{23}$$

Of course this is not the definition of the orthogonal algebra that was given above, but B_l and $\mathfrak{o}(2l+1) \otimes \mathbb{C}$ are isomorphic by conjugacy. Finally the algebra D_l consists of the $2l \times 2l$ matrices (again with complex matrices) a so that $a^T \Omega + \Omega\, a = 0$, where now

$$\Omega \;=\; \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}. \tag{24}$$

Again, the algebras D_l and $\mathfrak{o}(2l) \otimes \mathbb{C}$ are isomorphic via conjugacy.