Lecture 14 - $\mathfrak{o}(4)$ and $\mathfrak{g}_2$

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In this lecture we take a closer look at the orthogonal algebras.

1 Example: $\mathfrak{o}(4)$

1.1 Identification with an alternating algebra

Given a Riemannian metric $g(\cdot,\,\cdot)$ on any vector space V, there is are two a bilinear maps

$$\begin{aligned} G : V^{\otimes 2}\otimes V^{\otimes 2} &\rightarrow \mathbb{C} \\ H : V^{\otimes 2}\otimes V^{\otimes 2} &\rightarrow V^{\otimes 2}. \end{aligned} \tag{1}$$

The "metric" G is directly inherited from the metric on V. Namely, on basis elements

$$G(e_i\otimes e_j,\, e_k\otimes e_l) \;=\; g(e_i,\, e_k)\, g(e_j,\, e_l) \tag{2}$$

On $\bigwedge^2 V$, it is conventional to divide by 2:

$$\begin{aligned} G(e_i\wedge e_j,\; e_k\wedge e_l) &= \frac{1}{2}G(e_i\otimes e_j - e_j\otimes e_i,\; e_k\otimes e_l - e_l\otimes e_k) \\ &= g(e_i,\, e_k)g(e_j, e_l) \;-\; g(e_i,\, e_l)g(e_j, e_k). \end{aligned} \tag{3}$$

The second map, H is given by contraction on middle terms:

$$H\left(e_i\otimes e_j,\, e_k\otimes e_l\right) \;=\; e_i\otimes e_l\cdot g\left(e_j,\, e_k\right). \tag{4}$$

This passes to $\bigwedge^2 V$, which becomes a Lie algebra under the bracket:

$$[e_i\wedge e_j,\, e_k\wedge e_l] \;\;=\;\; H(e_i\wedge e_j,\, e_k\wedge e_l) \;-\; H(e_k\wedge e_l,\, e_i\wedge e_j). \tag{5}$$

If $V = span_{\mathbb{C}}\left\{e_1,\dots,e_n\right\}$ is $\mathbb{R}^n$, then $\bigwedge^2 V$, with this bracket, is isomorphic to $\mathfrak{sl}(n,\mathbb{C})$.

1.2 The 4-dimensional case

The 4-dimensional case is special, because there exists a second non-degenerate, bilinear, symmetric form. If $V = \{e_1, e_2, e_3, e_4\}$ is $\mathbb{R}^4$, then define

$$B : \bigwedge^2 V \otimes \bigwedge^2 V \to \mathbb{C} \tag{6}$$

on homogeneous elements by

$$B(a_i \wedge a_j,\, a_k \wedge a_l) \;=\; \frac{Det(A_{ij})}{\sqrt{|Det(E_{ij})|}} \tag{7}$$

where $A_{ij} = g(e_i, a_j)$ and $B_{ij} = g(e_i, e_j)$. One clearly sees that this definition is bilinear, symmetric, and independent of the choice of basis, as long as the change retains the orientation. If e_1, e_2, e_3, e_4 is an orthonormal basis, we can abuse notation and set

$$B(a_i \wedge a_j,\, a_k \wedge a_l) \;=\; \frac{a_1 \wedge a_2 \wedge a_3 \wedge a_4}{e_1 \wedge e_2 \wedge e_3 \wedge e_4}. \tag{8}$$

It is easy to verify non-degeneracy; since $\bigwedge^2 V$ is 6-dimensional, one can check this on a basis.

Thus a unitary linear operator $* : \bigwedge^2 V \to \bigwedge^2 V$, known as the duality operator or Hodge star, can be defined implicitly by

$$B\,(\,v \wedge w,\, *(v \wedge w)\,) \;=\; G(\,v \wedge w\,,\; v \wedge w\,). \tag{9}$$

By the bilinearity of both factors, we have

$$**\;=\; Id \,:\, \bigwedge^2 V \;\to\; \bigwedge^2 V. \tag{10}$$

Note that if e_1, e_2, e_3, e_4 is an ordered, orthonormal basis, then we have as usual

$$\begin{array}{llll} *(e_1 \wedge e_2) = e_3 \wedge e_4 & *(e_3 \wedge e_4) = e_1 \wedge e_2 \\ *(e_1 \wedge e_3) = -\, e_2 \wedge e_4 & *(e_2 \wedge e_4) = -e_1 \wedge e_3 \\ *(e_1 \wedge e_4) = e_2 \wedge e_3 & *(e_2 \wedge e_3) = e_1 \wedge e_4. \end{array} \tag{11}$$

We have thus established a map

$$* \,:\, \mathfrak{o}(4) \;\to\; \mathfrak{o}(4) \tag{12}$$

with

$$**\;=\;1. \tag{13}$$

The possible eigenvalues of $*$ are therefore ± 1. These can denoted by

$$\begin{array}{l} \bigwedge^+ V \;=\; \mathfrak{o}^+(4) \;=\; +1 \text{ eigenspace of } * \\ \bigwedge^- V \;=\; \mathfrak{o}^-(4) \;=\; -1 \text{ eigenspace of } * \end{array} \tag{14}$$

Further, it can be proved that

$$* [v \wedge w,\, a \wedge b] \;=\; [*(v \wedge w),\, a \wedge b]. \tag{15}$$

From this and the semi-simplicity of $\mathfrak{o}(4)$ it follows that

$$\begin{aligned} \big[\, \mathfrak{o}^{+}(4),\, \mathfrak{o}^{+}(4) \,\big] \;&=\; \mathfrak{o}^{+}(4) \\ \big[\, \mathfrak{o}^{-}(4),\, \mathfrak{o}^{-}(4) \,\big] \;&=\; \mathfrak{o}^{-}(4) \\ \big[\, \mathfrak{o}^{+}(4),\, \mathfrak{o}^{-}(4) \,\big] \;&=\; \{0\}. \end{aligned} \tag{16}$$

In particular $\mathfrak{o}(4)$ is not simple:

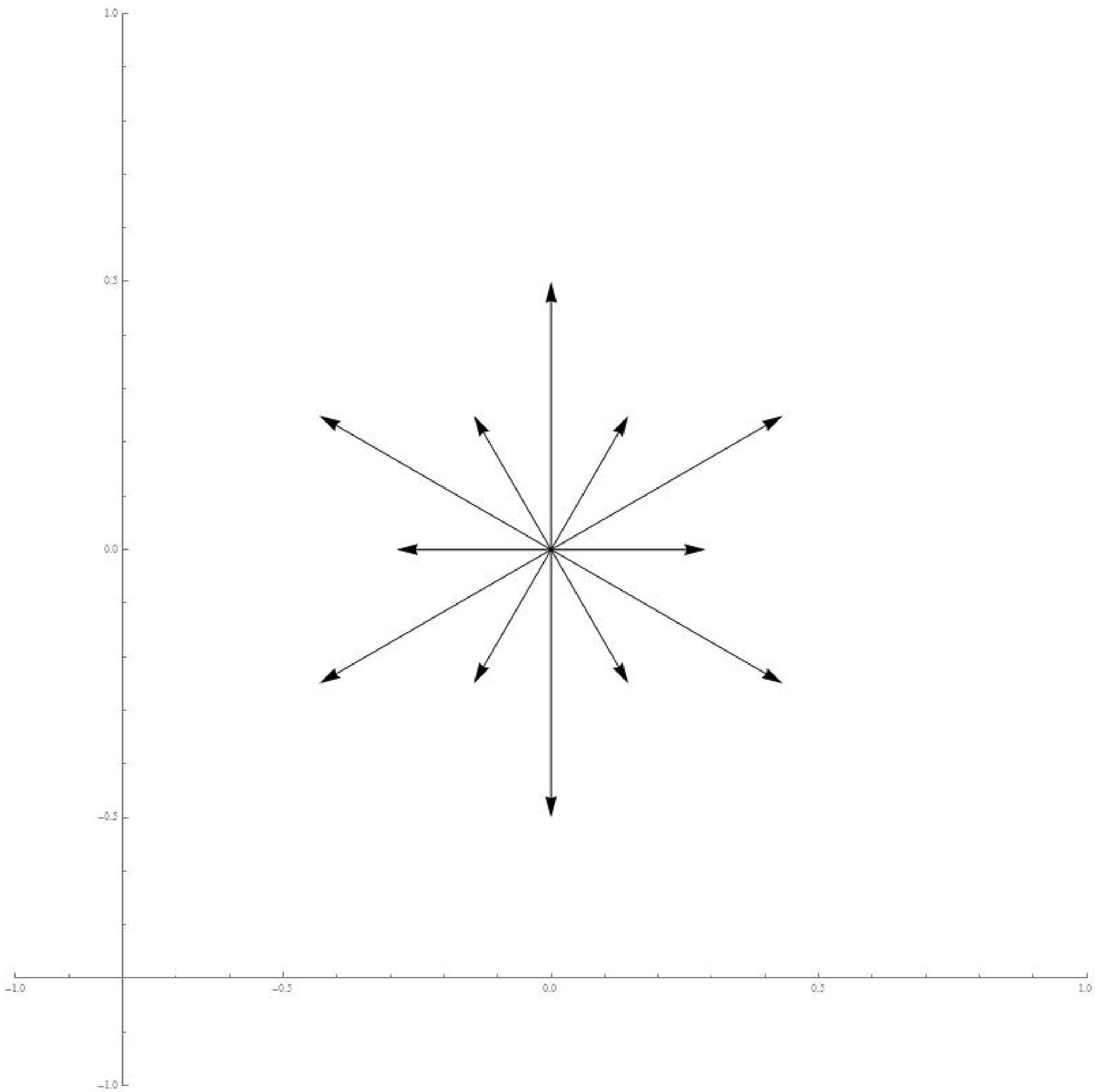
$$\mathfrak{o}(4) \;=\; \mathfrak{o}^{+}(4) \;\oplus\; \mathfrak{o}^{-}(4). \tag{17}$$

2 Example: $\mathfrak{g}_2$

There is a single Lie algebra of rank 2: $\mathfrak{sl}_2 \approx \mathfrak{sp}_2 \approx \mathfrak{o}_3$.

There are four semisimple Lie algebras of rank 2: $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \approx \mathfrak{o}_4$, $\mathfrak{sl}_3$, $\mathfrak{sp}_4 \approx \mathfrak{o}_5$, and $\mathfrak{g}_2$.

The only other simple Lie algebra that has a maximal toral subalgebra of dimension less than three is $\mathfrak{g}_2$. This Lie algebra can be defined as the Lie algebra of derivations on the purely imaginary octonions. It has the following root system:


1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0

3 Example: $\mathfrak{g}_2$

The smallest representation as a matrix group is by 7×7 matrices. We have $\mathfrak{g}_2 \subset \mathfrak{o}(7)$. A basis for a maximal toral subalgebra can be taken to be

$$n_1 = \frac{\sqrt{-3}}{12} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{18}$$

$$n_2 = \frac{\sqrt{-1}}{4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

The rest of the Lie algebra is given by the following matrices

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & -2 & 2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_1 = \begin{pmatrix} 0 & 0 & 0 & -2 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{20}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{21}$$

$$x_3 = \begin{pmatrix} 0 & 2 & 2i & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 \end{pmatrix} \quad y_3 = \begin{pmatrix} 0 & 2 & -2i & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 \end{pmatrix} \tag{22}$$

$$
x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -i & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & i & 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}
$$

$$
x_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2i & 2 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad y_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -2i & 2 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 \\ 0 & 1 & i & 0 & 0 & 0 & 0 \\ 0 & i & -1 & 0 & 0 & 0 & 0 \\ 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{24}
$$

$$
x_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -i & 0 & 0 \end{pmatrix} \quad y_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i \\ 0 & 0 & 0 & -i & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & i & 0 & 0 \end{pmatrix} \tag{25}
$$

The matrices representing the adjoints of n_1 and n_2 are diagonal, and we have

$$
\begin{aligned}
ad\, n_1 &= diag\left(0, 0, \frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}, \frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{12}, -\frac{\sqrt{3}}{12}, 0, 0, -\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{12}, -\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{4}\right) \\
ad\, n_2 &= diag\left(0, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right)
\end{aligned} \tag{26}
$$

in the $n_1, n_2, x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6$ ordered basis. The roots are therefore

$$\begin{aligned}
\alpha_1 &= \left(\frac{\sqrt{3}}{6}, 0\right) \\
\alpha_2 &= \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right) \\
\alpha_3 &= \left(\frac{\sqrt{3}}{12}, \frac{1}{4}\right) \\
\alpha_4 &= \left(0, \frac{1}{2}\right) \\
\alpha_5 &= \left(-\frac{\sqrt{3}}{12}, \frac{1}{4}\right) \\
\alpha_6 &= \left(-\frac{\sqrt{3}}{4}, \frac{1}{4}\right)
\end{aligned} \tag{27}$$

and their negatives. Root lengths are therefore of length $\frac{1}{2}$ and $\frac{\sqrt{3}}{6}$.