Lecture 13 - Root Space Decomposition II

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1 Review

First let us recall the situation. Let $\mathfrak{g}$ be a simple algebra, with maximal toral subalgebra $\mathfrak{h}$ (which we are calling a CSA, or Cartan Subalgebra). We have that $\mathfrak{h}$ acts on $\mathfrak{g}$ via the adjoint action, and since $\mathfrak{h}$ has only mutually commuting, abstractly semisimple elements, we have that the action of $\mathfrak{h}$ is simultaneously diagonalizable. Thus $\mathfrak{g}$ decomposes into weight spaces, called in this special case *root spaces*:

$$\begin{aligned} \mathfrak{g} &= \bigoplus_{\alpha\in\Phi\cup\{0\}} \mathfrak{g}_\alpha \\ \mathfrak{g}_\alpha &= \{\, x\in\mathfrak{g} \mid h.x = \alpha(h)\,x \text{ for all } h\in\mathfrak{h}\,\}. \end{aligned} \tag{1}$$

We defined Φ to be the set of roots of $\mathfrak{g}$ relative to the choice of $\mathfrak{h}$, or in other words, the non-zero weights for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. We proved:

i) $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{g}_0$

ii) Φ spans $\mathfrak{h}^*$

iii) $\alpha\in\Phi$ implies $-\alpha\in\Phi$

iv) $[\mathfrak{g}_\alpha,\, \mathfrak{g}_\beta] \in \mathfrak{g}_{\alpha+\beta}$

v) $x_\alpha\in\mathfrak{g}_\alpha$, $y_\alpha\in\mathfrak{g}_{-\alpha}$ implies $[x_\alpha, y_\alpha] = \kappa_{\mathfrak{g}}(x_\alpha, y_\alpha)t_\alpha$ (where t_α is the $\kappa_{\mathfrak{g}}$-dual of α)

vi) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}\, t_\alpha \subseteq \mathfrak{h}$

vii) If $x_\alpha\in\mathfrak{g}_\alpha$ then $y_\alpha\in\mathfrak{g}_{-\alpha}$ exists so $\{x_\alpha, y_\alpha, h_\alpha\}$ is a standard basis for some $\mathfrak{sl}(2,\mathbb{C})\subseteq\mathfrak{g}$

viii) $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha,t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$

ix) $\kappa_{\mathfrak{g}}|_{\mathfrak{h}}$ is positive definite.

Since κ is nondegenerate, we can define an inner product on the dual space $\mathfrak{h}^*$ directly by

$$(\alpha,\,\beta) \;=\; \kappa(t_\alpha,\,t_\beta) \tag{2}$$

where, recall, t_α (resp. t_β) is the dual of α under κ.

Lemma 1.1 *Assume α and β are roots, and that $\alpha + c\beta$ are all roots where $c \in \mathbb{C}$. Then*

- *The number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer*
- *c is an integer*
- *The direct sum of root spaces of the form $\mathfrak{g}_{\beta+i\alpha}$ is an irreducible S_α module. In particular if $x_\alpha \in \mathfrak{g}$, we have that $(ad\, x_\alpha)^i : \mathfrak{g}_\alpha \to \mathfrak{g}_{\alpha+i\beta}$ is an isomorphism.*

Pf. The vector space

$$T \;=\; \bigoplus \mathfrak{g}_{\beta+c\alpha} \tag{3}$$

where c ranges over all numbers in $\mathbb{C}$ so $\beta + c\alpha \in \Phi$, is a finite dimensional S_α-module, and each $\mathfrak{g}_{\beta+i\alpha}$ is 1-dimensional. The weight (in the $\mathfrak{sl}_2$-sense) of the space $\mathfrak{g}_{\beta+c\alpha}$ is computed by selecting some $x \in \mathfrak{g}_{\alpha+c\beta}$ and computing $(ad\, h_\alpha)x$. We have

$$(ad\, h_\alpha)x \;=\; (\beta + c\alpha)(h_\alpha)\, x \;=\; \left(\frac{2\,(\alpha,\beta)}{(\alpha,\alpha)} \,+\, 2c\right) x \tag{4}$$

Since weights must be integers, we can take $c = 0$ to obtain the integrality of $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$. Then we also have that $2c \in \mathbb{Z}$.

Now we rule out c being half-integral. Assume $c = \frac{n}{2}$ (n odd). We can always choose n so $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ has weight 0 or else weight 1 (with respect to the weight operator h_α). We have

$$h_\alpha.x_{\beta+\frac{n}{2}\alpha} \;=\; \left(\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \,+\, n\right) x_{\beta+\frac{n}{2}\alpha} \tag{5}$$

so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is $-n$ or $1-n$.

Case I: Assume the S_α-weight of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ is 0, so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = -n$.
Consider the action of $S_{\beta+\frac{1}{2}\alpha} \approx \mathfrak{sl}_2$. We have

$$h_{\beta+\frac{n}{2}\alpha}.x_\beta \quad=\quad \beta(h_{\beta+\frac{n}{2}\alpha})\, x_\alpha \;=\; \frac{2(\beta,\beta) \,+\, n(\alpha,\beta)}{(\beta,\beta) + n(\alpha,\beta) + \frac{n^2}{4}(\alpha,\alpha)}\, x_\alpha \tag{6}$$

so with $(\alpha,\alpha) = -\frac{2}{n}(\alpha,\beta)$ we have

$$h_{\beta+\frac{1}{2}\alpha}.x_\beta \quad=\quad 2\, x_\alpha \tag{7}$$

Therefore $(ad\, y_{\beta+\frac{n}{2}\alpha})(x_\beta)$ is non-zero, and lies in $\mathfrak{g}_{-\frac{n}{2}\alpha}$. This is impossible because $-\frac{n}{2}\alpha$ is not a root.

Case II: Assume the S_α-weight of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$ is 1, so $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = 1-n$.
First note that since n is odd, the right side is even so that $\frac{\alpha\cdot\beta}{\alpha\cdot\alpha}$ is an integer. Applying a destruction operator, we get that $\mathfrak{g}_{\beta+\frac{n-2}{2}\alpha}$ is also a non-trivial weight space. Since β is a root, we have the algebra S_β, and we can find the S_β-weights of $\mathfrak{g}_{\beta+\frac{n}{2}\alpha}$, $\mathfrak{g}_{\beta+\frac{n-2}{2}\alpha}$ as follows:

$$\begin{aligned} h_\beta.x_{\beta+\frac{n}{2}\alpha} &= \left(\beta+\frac{n}{2}\alpha\right)\cdot\frac{2\beta}{\beta\cdot\beta}\,x_{\beta+\frac{n}{2}\alpha} \\ &= 2\left(1+\frac{n}{2}\frac{\alpha\cdot\beta}{\beta\cdot\beta}\right)x_{\beta+\frac{n}{2}\alpha}. \end{aligned} \tag{8}$$

Since $2+n\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer, n is odd, and $2\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer, we have that $\frac{\alpha\cdot\beta}{\beta\cdot\beta}$ is an integer. Now we cheat a bit and use the theorem below, which states that κ is positive definite (note that this is not cyclical: that result does not use this one). By Cauchy-Schwarz we have

$$\frac{(\alpha\cdot\beta)^2}{(\alpha\cdot\alpha)(\beta\cdot\beta)} \le 1 \tag{9}$$

so either $\alpha\cdot\beta = 0$, or α and β are parallel. We assumed they are not parallel, so $\alpha\cdot\beta=0$, meaning the calculation above gives

$$h_\beta.x_{\beta+\frac{n}{2}\alpha} \;=\; 2\,x_{\beta+\frac{n}{2}\alpha}. \tag{10}$$

Therefore we can apply a destruction operator (namely y_β) to obtain a non-trivial weight space, namely $g_{\frac{n}{2}\alpha}$. Yet this is impossible because $\frac{n}{2} \neq \pm 1$.

The lemma's final assertion is a direct consequence of the fact that $\beta + i\alpha$ is a root only when $i \in \mathbb{Z}$ and that each root space is 1-dimensional. $\square$

The numbers $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ are called the *Cartan integers*. The set of roots of the form $\beta+i\alpha$ is called the α-string through β.

Proposition 1.2 *Assume $\alpha,\beta\in\Phi$ and $\beta\neq\pm\alpha$ Then*

a) The S_α-weight of $\mathfrak{g}_\beta$ is $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

b) If β is a maximal weight for S_α, then the S_α module generated by $\mathfrak{g}_\beta$ is

$$\mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta-\alpha} \oplus \cdots \oplus \mathfrak{g}_{\beta-\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha} \tag{11}$$

c) If α,β are any roots, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \mathfrak{h}^*$ is a root.

Pf. For (a), we compute

$$(ad\, h_\alpha)(x_\beta) \;=\; \beta(h_\alpha)\, x_\beta \;=\; \left(t_\beta,\, \frac{2\, t_\alpha}{(\alpha,\alpha)}\right) x_\beta \;=\; \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\, x_\beta. \tag{12}$$

For (b), (c), let β be a root. The weight of the (possibly trivial) root space $g_{\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha}$ is

$$\left(\beta \,-\, \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha\right)(h_\alpha) \;=\; \beta(h_\alpha) \,-\, \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha(h_\alpha) \;=\; -\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \tag{13}$$

which is the negative of the weight of the root space of $\mathfrak{g}_\beta$. Since the negative of a weight is a weight, we have that $\mathfrak{g}_{\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha}$ must be non-trivial. □

2 The Euclidean space E and its inner product κ

Lemma 2.1 *If $\beta, \gamma \in \mathfrak{h}^*$ then $(\beta,\gamma) = \sum_{\alpha\in\Phi}(\alpha,\beta)(\alpha,\gamma)$.*

Pf. If $\Phi = \{\alpha_1, \ldots, \alpha_m\}$ then the decomposition

$$\mathfrak{g} \;=\; \mathfrak{h} \,\oplus\, \bigoplus_{\alpha_i\in\Phi} \mathfrak{g}_{\alpha_i} \tag{14}$$

diagonalizes the adjoint action of all elements of $\mathfrak{h}$. In fact if $h \in \mathfrak{h}$ then

$$ad\, h \;=\; \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \alpha_1(h) & & \\ & & & & \ddots & \\ & & & & & \alpha_m(h) \end{pmatrix} \tag{15}$$

Thus

$$\begin{aligned} (\beta,\, \gamma) \;&=\; \kappa(t_\beta,\, t_\gamma) \\ &=\; Tr\, ad\, t_\beta\, ad\, t_\gamma \\ &=\; \sum_{i=1}^m \alpha_i(t_\beta)\alpha_i(t_\gamma) \\ &=\; \sum_{i=1}^m (\beta,\, \alpha_i)\,(\gamma,\, \alpha_i)\,. \end{aligned} \tag{16}$$

□

Lemma 2.2 *Let $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi$ be a $\mathbb{C}$-basis of $\mathfrak{h}^*$. Then $\Phi \subset span_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_n\}$.*

Pf. We have $\beta = \sum_{i=1}^n c_i \alpha_i$ for some constants $c_i \in \mathbb{C}$. Then $(\beta, \alpha_j) = \sum_i c_i A_{ij}$ where $A_{ij} = (\alpha_i, \alpha_j)$. Note that A_{ij} is invertible. Then

$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_{i=1}^n c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \tag{17}$$

Since $M_{ij} = A_{ij}(\alpha_j, \alpha_j)^{-1}$ is invertible and all numbers besides the c_i are integers, the c_i are rationals. □

Lemma 2.3 *If $\beta, \gamma \in \Phi$, then (β, γ) is rational, and $(\beta, \beta) > 0$.*

Pf. We have $(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2$ so that

$$\frac{4}{(\beta, \beta)} = \sum_{i=1}^m \left(\frac{2(\alpha, \beta)}{(\beta, \beta)} \right)^2. \tag{18}$$

The numbers on the right are all integers, so (β, β) is rational. Now letting $\gamma \in \Phi$ we have

$$\begin{aligned} (\beta, \gamma) &= \sum_{\alpha \in \Phi} (\alpha, \beta)(\alpha, \gamma) \\ \frac{4(\beta, \gamma)}{(\beta, \beta)(\gamma, \gamma)} &= \sum_{\alpha \in \Phi} \frac{2(\alpha, \beta)}{(\beta, \beta)} \frac{2(\alpha, \gamma)}{(\gamma, \gamma)} \end{aligned} \tag{19}$$

where the right-hand side is integral, and (β, β) and (γ, γ) are rational. Therefore (β, γ) is rational. Finally with $(\beta, \beta) = \sum_{\alpha \in \Phi} (\alpha, \beta)^2$ again, we see that (β, β) is the sum of non-negative rationals. Thus κ is positive semi-definite. Since it is non-degenerate, it is therefore positive definite and $(\beta, \beta) > 0$. □

Theorem 2.4 *Setting $E = span_{\mathbb{R}} \Phi$ and restricting κ to E, we have that $dim_{\mathbb{R}} E = dim_{\mathbb{C}} \mathfrak{h}$ and κ is positive definite inner product.*

Pf. Trivlal. □

3 Root Space Axioms

It is useful to put some of our conclusions into one place; the theorem that follows verifies what we will call the *root space axioms*.

Theorem 3.1 *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h}$ any maximal toral subalgebra, Φ the set of roots associated to $\mathfrak{h}$, and $E = span_{\mathbb{R}}\Phi$ with positive definite inner product κ. Then*

a) Φ *spans* E*, and* $0 \notin \Phi$

b) If $\alpha \in \Phi$ *then* $-\alpha \in \Phi$*, but* $c\alpha \notin \Phi$ *for* $c \neq \pm 1$

c) If $\alpha, \beta \in \Phi$*, then* $\beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha \in \Phi$

d) If $\alpha, \beta \in \Phi$ *then* $\frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$

□