Preface

This book is designed to introduce the reader to the theory of semisimple Lie algebras over an algebraically closed field of characteristic 0, with emphasis on representations. A good knowledge of linear algebra (including eigenvalues, bilinear forms, euclidean spaces, and tensor products of vector spaces) is presupposed, as well as some acquaintance with the methods of abstract algebra. The first four chapters might well be read by a bright undergraduate; however, the remaining three chapters are admittedly a little more demanding.

Besides being useful in many parts of mathematics and physics, the theory of semisimple Lie algebras is inherently attractive, combining as it does a certain amount of depth and a satisfying degree of completeness in its basic results. Since Jacobson's book appeared a decade ago, improvements have been made even in the classical parts of the theory. I have tried to incorporate some of them here and to provide easier access to the subject for non-specialists. For the specialist, the following features should be noted:

(1) The Jordan-Chevalley decomposition of linear transformations is emphasized, with "toral" subalgebras replacing the more traditional Cartan subalgebras in the semisimple case.

(2) The conjugacy theorem for Cartan subalgebras is proved (following D. J. Winter and G. D. Mostow) by elementary Lie algebra methods, avoiding the use of algebraic geometry.

(3) The isomorphism theorem is proved first in an elementary way (Theorem 14.2), but later obtained again as a corollary of Serre's Theorem (18.3), which gives a presentation by generators and relations.

(4) From the outset, the simple algebras of types A, B, C, D are emphasized in the text and exercises.

(5) Root systems are treated axiomatically (Chapter III), along with some of the theory of weights.

(6) A conceptual approach to Weyl's character formula, based on Harish-Chandra's theory of "characters" and independent of Freudenthal's multiplicity formula (22.3), is presented in §23 and §24. This is inspired by D.-N. Verma's thesis, and recent work of I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand.

(7) The basic constructions in the theory of Chevalley groups are given in Chapter VII, following lecture notes of R. Steinberg.

I have had to omit many standard topics (most of which I feel are better suited to a second course), e.g., cohomology, theorems of Levi and Mal'cev, theorems of Ado and Iwasawa, classification over non-algebraically closed fields, Lie algebras in prime characteristic. I hope the reader will be stimulated to pursue these topics in the books and articles listed under References, especially Jacobson [1], Bourbaki [1], [2], Winter [1], Seligman [1].

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A few words about mechanics: Terminology is mostly traditional, and notation has been kept to a minimum, to facilitate skipping back and forth in the text. After Chapters I–III, the remaining chapters can be read in almost any order if the reader is willing to follow up a few references (except that VII depends on §20 and §21, while VI depends on §17). A reference to Theorem 14.2 indicates the (unique) theorem in subsection 14.2 (of §14).

Notes following some sections indicate nonstandard sources or further reading, but I have not tried to give a history of each theorem (for historical remarks, cf. Bourbaki [2] and Freudenthal-deVries [1]): The reference list consists largely of items mentioned explicitly; for more extensive bibliographies, consult Jacobson [1], Seligman [1]. Some 240 exercises, of all shades of difficulty, have been included; a few of the easier ones are needed in the text.

This text grew out of lectures which I gave at the N.S.F. Advanced Science Seminar on Algebraic Groups at Bowdoin College in 1968; my intention then was to enlarge on J.-P. Serre's excellent but incomplete lecture notes [2]. My other literary debts (to the books and lecture notes of N. Bourbaki, N. Jacobson, R. Steinberg, D. J. Winter, and others) will be obvious. Less obvious is my personal debt to my teachers, George Seligman and Nathan Jacobson, who first aroused my interest in Lie algebras. I am grateful to David J. Winter for giving me pre-publication access to his book, to Robert L. Wilson for making many helpful criticisms of an earlier version of the manuscript, to Connie Engle for her help in preparing the final manuscript, and to Michael J. DeRise for moral support. Financial assistance from the Courant Institute of Mathematical Sciences and the National Science Foundation is also gratefully acknowledged.

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J. E. Humphreys

Notation and Conventions

Z, Z_+, Q, R, C denote (respectively) the integers, nonnegative integers, rationals, reals, and complex numbers

$\amalg$ denotes direct sum of vector spaces

$A \ltimes B$ denotes the semidirect product of groups A and B, with B normal

Card = cardinality   Ker = kernel

char = characteristic   Im = image

det = determinant   Tr = trace

dim = dimension

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