Chapter III

Root Systems

9. Axiomatics

9.1. Reflections in a euclidean space

Throughout this chapter we are concerned with a fixed euclidean space E, i.e., a finite dimensional vector space over **R** endowed with a positive definite symmetric bilinear form (α, β). Geometrically, a **reflection** in E is an invertible linear transformation leaving pointwise fixed some **hyperplane** (subspace of codimension one) and sending any vector orthogonal to that hyperplane into its negative. Evidently a reflection is *orthogonal*, i.e., preserves the inner product on E. Any nonzero vector α determines a reflection σ_α, with **reflecting hyperplane** $P_\alpha = \{\beta \in \mathsf{E} | (\beta, \alpha) = 0\}$. Of course, nonzero vectors proportional to α yield the same reflection. It is easy to write down an explicit formula: $\sigma_\alpha(\beta) = \beta - \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha$. (This works, because it sends α to $-\alpha$ and fixes all points in P_α.) Since the number $2(\beta, \alpha)/(\alpha, \alpha)$ occurs frequently, we abbreviate it by $\langle \beta, \alpha \rangle$. Notice that $\langle \beta, \alpha \rangle$ is linear only in the first variable.

For later use we record the following fact.

Lemma. *Let Φ be a finite set which spans* E. *Suppose all reflections $\sigma_\alpha (\alpha \in \Phi)$ leave Φ invariant. If $\sigma \in GL(\mathsf{E})$ leaves Φ invariant, fixes pointwise a hyperplane P of* E, *and sends some nonzero $\alpha \in \Phi$ to its negative, then $\sigma = \sigma_\alpha$ (and $P = P_\alpha$).*

Proof. Let $\tau = \sigma\sigma_\alpha$ $(=\sigma\sigma_\alpha^{-1})$. Then $\tau(\Phi) = \Phi$, $\tau(\alpha) = \alpha$, and τ acts as the identity on the subspace $\mathbf{R}\alpha$ as well as on the quotient $\mathsf{E}/\mathbf{R}\alpha$. So all eigenvalues of τ are 1, and the minimal polynomial of τ divides $(T-1)^\ell$ ($\ell = \dim \mathsf{E}$). On the other hand, since Φ is finite, not all vectors $\beta, \tau(\beta), \ldots, \tau^k(\beta)$ ($\beta \in \Phi$, $k \geq \text{Card } \Phi$) can be distinct, so some power of τ fixes β. Choose k large enough so that τ^k fixes all $\beta \in \Phi$. Because Φ spans E, this forces $\tau^k = 1$; so the minimal polynomial of τ divides $T^k - 1$. Combined with the previous step, this shows that τ has minimal polynomial $T - 1 = g.c.d.\,(T^k - 1, (T-1)^\ell)$, i.e., $\tau = 1$. ☐

9.2. Root systems

A subset Φ of the euclidean space E is called a **root system** in E if the following axioms are satisfied:

(R1) Φ is finite, spans E, and does not contain 0.
(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.
(R3) If $\alpha \in \Phi$, the reflection σ_α leaves Φ invariant.
(R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbf{Z}$.

There is some redundancy in the axioms; in particular, both (R2) and (R3) imply that $\Phi = -\Phi$. In the literature (R2) is sometimes omitted, and what we have called a "root system" is then referred to as a "reduced root system" (cf. Exercise 9). Notice that replacement of the given inner product on E by a positive multiple would not affect the axioms, since only *ratios* of inner products occur.

Let Φ be a root system in E. Denote by $\mathscr{W}$ the subgroup of $GL(\mathsf{E})$ generated by the reflections $\sigma_\alpha (\alpha \in \Phi)$. By (R3), $\mathscr{W}$ permutes the set Φ, which by (R1) is finite and spans E. This allows us to identify $\mathscr{W}$ with a subgroup of the symmetric group on Φ; in particular, $\mathscr{W}$ is finite. $\mathscr{W}$ is called the **Weyl group** of Φ, and plays an extremely important role in the sequel. The following lemma shows how certain automorphisms of E act on $\mathscr{W}$ by conjugation.

Lemma. *Let Φ be a root system in* E, *with Weyl group $\mathscr{W}$. If $\sigma \in GL(\mathsf{E})$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$, and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

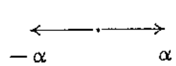
Proof. $\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma\sigma_\alpha(\beta) \in \Phi$, since $\sigma_\alpha(\beta) \in \Phi$. But this equals $\sigma(\beta - \langle \beta, \alpha \rangle \alpha) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$. Since $\sigma(\beta)$ runs over Φ as β runs over Φ, we conclude that $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant, while fixing pointwise the hyperplane $\sigma(P_\alpha)$ and sending $\sigma(\alpha)$ to $-\sigma(\alpha)$. By Lemma 9.1, $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$. But then, comparing the equation above with the equation $\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha)$, we also get the second assertion of the lemma. ☐

There is a natural notion of isomorphism between root systems Φ, Φ' in respective euclidean spaces E, E′: Call (Φ, E) and (Φ', E′) **isomorphic** if there exists a vector space isomorphism (not necessarily an isometry) ϕ: E → E′ sending Φ onto Φ' such that $\langle \phi(\beta), \phi(\alpha) \rangle = \langle \beta, \alpha \rangle$ for each pair of roots $\alpha, \beta \in \Phi$. It follows at once that $\sigma_{\phi(\alpha)}(\phi(\beta)) = \phi(\sigma_\alpha(\beta))$. Therefore an isomorphism of root systems induces a natural isomorphism $\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$ of Weyl groups. In view of the lemma above, an automorphism of Φ is the same thing as an automorphism of E leaving Φ invariant. In particular, we can regard $\mathscr{W}$ as a subgroup of Aut Φ (cf. Exercise 6).

It is useful to work not only with α but also with $\alpha^\vee = \dfrac{2\alpha}{(\alpha, \alpha)}$. Call $\Phi^\vee = \{\alpha^\vee | \alpha \in \Phi\}$ the **dual** (or **inverse**) of Φ. It is in fact a root system in E, whose Weyl group is canonically isomorphic to $\mathscr{W}$ (Exercise 2). (In the Lie algebra situation of §8, α corresponds to t_α, while $\alpha^\vee$ corresponds to h_α, under the Killing form identification of H^* with H.)

9.3. Examples

Call $\ell = \dim \mathsf{E}$ the **rank** of the root system Φ. When $\ell \leq 2$, we can describe Φ by simply drawing a picture. In view of (R2), there is only one possibility in case $\ell = 1$, labelled (A_1):



Of course, this actually is a root system (with Weyl group of order 2); in Lie algebra theory it belongs to $\mathfrak{sl}(2, \mathsf{F})$.

Rank 2 offers more possibilities, four of which are depicted in Figure 1 (these turn out to be the only possibilities). In each case the reader should check the axioms directly and determine $\mathscr{W}$.

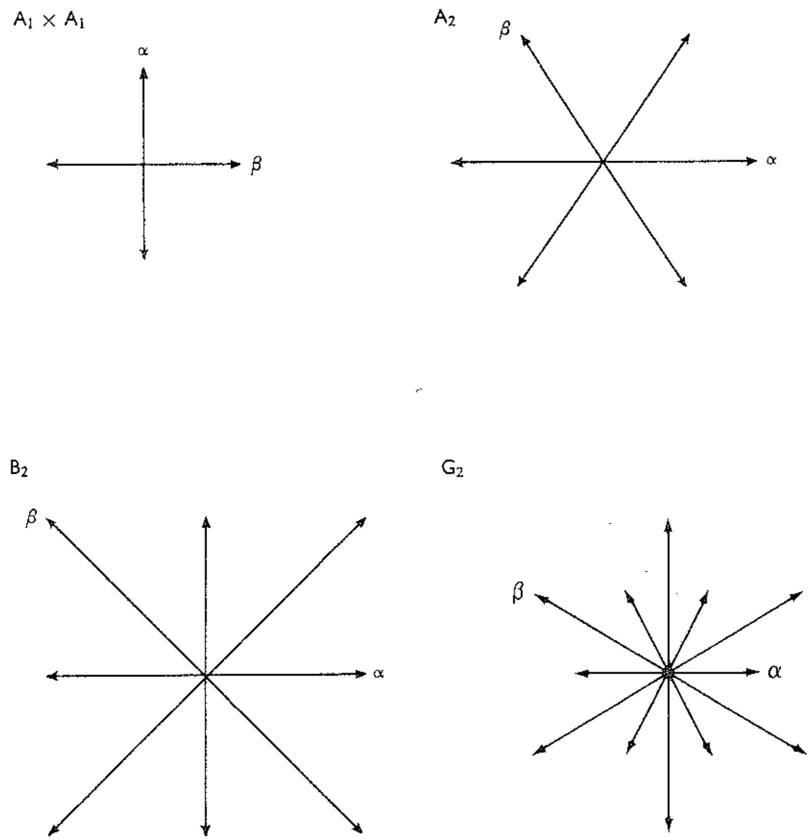


Figure 1

9.4. Pairs of roots

Axiom (R4) limits severely the possible angles occurring between pairs of roots. Recall that the cosine of the angle θ between vectors $\alpha, \beta \in \mathsf{E}$ is given by the formula $\|\alpha\|\;\|\beta\| \cos\theta = (\alpha, \beta)$. Therefore, $\langle \beta, \alpha\rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)} = 2\dfrac{\|\beta\|}{\|\alpha\|}\cos\theta$ and $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle = 4\cos^2\theta$. This last number is a nonnegative integer; but $0 \le \cos^2\theta \le 1$, and $\langle \alpha, \beta\rangle, \langle \beta, \alpha\rangle$ have like sign, so the following possibilities are the only ones when $\alpha \ne \pm\beta$ and $\|\beta\| \ge \|\alpha\|$ (Table 1).

Table 1.

$\langle \alpha, \beta\rangle$	$\langle \beta, \alpha\rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
−1	−1	$2\pi/3$	1
1	2	$\pi/4$	2
−1	−2	$3\pi/4$	2
1	3	$\pi/6$	3
−1	−3	$5\pi/6$	3

The reader will observe that these angles and relative lengths are just the ones portrayed in Figure 1 (9.3). (For $\mathsf{A}_1 \times \mathsf{A}_1$ it is harmless to change scale in one direction so as to insure that $\|\alpha\| = \|\beta\|$.) The following simple but very useful criterion can be read off from Table 1.

Lemma. *Let α, β be nonproportional roots. If $(\alpha, \beta) > 0$ (i.e., if the angle between α and β is strictly acute), then $\alpha-\beta$ is a root. If $(\alpha, \beta) < 0$, then $\alpha+\beta$ is a root.*

Proof. The second assertion follows from the first (applied to $-\beta$ in place of β). Since (α, β) is positive if and only if $\langle \alpha, \beta\rangle$ is, Table 1 shows that one or the other of $\langle \alpha, \beta\rangle, \langle \beta, \alpha\rangle$ equals 1. If $\langle \alpha, \beta\rangle = 1$, then $\sigma_\beta(\alpha) = \alpha-\beta \in \Phi$ (R3); similarly, if $\langle \beta, \alpha\rangle = 1$, then $\beta-\alpha \in \Phi$, hence $\sigma_{\beta-\alpha}(\beta-\alpha) = \alpha-\beta \in \Phi$. ▯

As an application, consider a pair of nonproportional roots α, β. Look at all roots of the form $\beta+i\alpha$ ($i \in \mathbf{Z}$), the **α-string through β**. Let $r, q \in \mathbf{Z}^+$ be the largest integers for which $\beta-r\alpha \in \Phi$, $\beta+q\alpha \in \Phi$ (respectively). If some $\beta+i\alpha \notin \Phi$ $(-r < i < q)$, we can find $p < s$ in this interval such that $\beta+p\alpha \in \Phi$, $\beta+(p+1)\alpha \notin \Phi$, $\beta+(s-1)\alpha \notin \Phi$, $\beta+s\alpha \in \Phi$. But then the lemma implies both $(\alpha, \beta+p\alpha) \ge 0$, $(\alpha, \beta+s\alpha) \le 0$. Since $p < s$ and $(\alpha, \alpha) > 0$, this is absurd. We conclude that *the α-string through β is unbroken, from $\beta-r\alpha$ to $\beta+q\alpha$.* Now σ_α just adds or subtracts a multiple of α to any root, so this string is invariant under σ_α. Geometrically, it is obvious that σ_α just reverses the string (the reader can easily supply an algebraic proof). In particular, $\sigma_\alpha(\beta+q\alpha) = \beta-r\alpha$. The left side is $\beta-\langle \beta, \alpha\rangle\alpha-q\alpha$, so finally we obtain: $r-q = \langle \beta, \alpha\rangle$ (cf. Proposition 8.4(e)). It follows at once that *root strings are of length at most* 4.

Exercises

(*Unless otherwise specified, Φ denotes a root system in* E, *with Weyl group* $\mathscr{W}$.)

1. Let E' be a subspace of E. If a reflection σ_α leaves E' invariant, prove that either $\alpha \in \mathsf{E}'$ or else $\mathsf{E}' \subset P_\alpha$.

2. Prove that $\Phi^\vee$ is a root system in E, whose Weyl group is naturally isomorphic to $\mathscr{W}$; show also that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$, and draw a picture of $\Phi^\vee$ in the cases A_1, A_2, B_2, G_2.
3. In Table 1, show that the order of $\sigma_\alpha \sigma_\beta$ in $\mathscr{W}$ is (respectively) 2, 3, 4, 6 when $\theta = \pi/2, \pi/3$ (or $2\pi/3$), $\pi/4$ (or $3\pi/4$), $\pi/6$ (or $5\pi/6$). [Note that $\sigma_\alpha \sigma_\beta =$ rotation through 2θ.]
4. Prove that the respective Weyl groups of $A_1 \times A_1$, A_2, B_2, G_2 are dihedral of order 4, 6, 8, 12. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.
5. Show by example that $\alpha - \beta$ may be a root even when $(\alpha, \beta) \leq 0$ (cf. Lemma 9.4).
6. Prove that $\mathscr{W}$ is a normal subgroup of Aut Φ (=group of all isomorphisms of Φ onto itself).
7. Let $\alpha, \beta \in \Phi$ span a subspace E′ of E. Prove that $E' \cap \Phi$ is a root system in E′. Prove similarly that $\Phi \cap (\mathbf{Z}\alpha + \mathbf{Z}\beta)$ is a root system in E′ (must this coincide with $E' \cap \Phi$?). More generally, let Φ' be a nonempty subset of Φ such that $\Phi' = -\Phi'$, and such that $\alpha, \beta \in \Phi'$, $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi'$. Prove that Φ' is a root system in the subspace of E it spans. [Use Table 1].
8. Compute root strings in G_2 to verify the relation $r - q = \langle \beta, \alpha \rangle$.
9. Let Φ be a set of vectors in a euclidean space E, satisfying only (R1), (R3), (R4). Prove that the only possible multiples of $\alpha \in \Phi$ which can be in Φ are $\pm 1/2\ \alpha$, $\pm \alpha$, $\pm 2\alpha$. Verify that $\{\alpha \in \Phi | 2\alpha \notin \Phi\}$ is a root system. *Example*: See Figure 2.

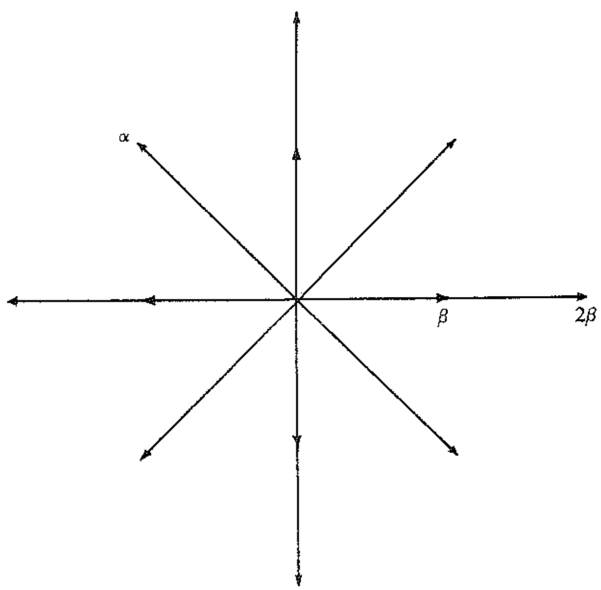


Figure 2

10. Let $\alpha, \beta \in \Phi$. Let the α-string through β be $\beta - r\alpha, \ldots, \beta + q\alpha$, and let the β-string through α be $\alpha - r'\beta, \ldots, \alpha + q'\beta$. Prove that $\dfrac{q(r+1)}{(\beta, \beta)} = \dfrac{q'(r'+1)}{(\alpha, \alpha)}$.
11. Let c be a positive real number. If Φ possesses any roots of squared length c, prove that the set of all such roots is a root system in the subspace of E it spans. Describe the possibilities occurring in Figure 1.

Notes

The axiomatic approach to root systems (as in Serre [2], Bourbaki [2]) has the advantage of providing results which apply simultaneously to Lie algebras, Lie groups, and linear algebraic groups. For historical remarks, consult Bourbaki [2].

10. Simple roots and Weyl group

Throughout this section Φ denotes a root system of rank ℓ in a euclidean space E, *with Weyl group* $\mathscr{W}$.

10.1. Bases and Weyl chambers

A subset Δ of Φ is called a **base** if:
(B1) Δ is a basis of E,
(B2) each root β can be written as $\beta = \Sigma k_\alpha \alpha$ ($\alpha \in \Delta$) with integral coefficients k_α all nonnegative or all nonpositive.

The roots in Δ are then called **simple.** In view of (B1), Card $\Delta = \ell$, and the expression for β in (B2) is unique. This allows us to define the **height** of a root (relative to Δ) by ht $\beta = \sum_{\alpha \in \Delta} k_\alpha$. If all $k_\alpha \geq 0$ (resp. all $k_\alpha \leq 0$), we call β **positive** (resp. **negative**) and write $\beta \succ 0$ (resp. $\beta \prec 0$). The collections of positive and negative roots (relative to Δ) will usually just be denoted Φ^+ and Φ^- (clearly, $\Phi^- = -\Phi^+$). If α and β are positive roots, and $\alpha + \beta$ is a root, then evidently $\alpha + \beta$ is also positive. Actually, Δ defines a *partial order* on E, compatible with the notation $\alpha \succ 0$: define $\mu \prec \lambda$ iff $\lambda - \mu$ is a sum of positive roots (equivalently, of simple roots) or $\mu = \lambda$.

The only problem with the definition of base is that it fails to guarantee *existence.* In the examples shown in (9.3), the roots labelled α, β do form a base in each case (verify!). Notice there that the angle between α and β is *obtuse,* i.e., $(\alpha, \beta) \leq 0$. This is no accident.

Lemma. *If Δ is a base of Φ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ in Δ, and $\alpha - \beta$ is not a root.*

Proof. Otherwise $(\alpha, \beta) > 0$. Since $\alpha \neq \beta$, by assumption, and since obviously $\alpha \neq -\beta$, Lemma 9.4 then says that $\alpha - \beta$ is a root. But this violates (B2). ☐

Our goal is the proof of the following theorem.

Theorem. *Φ has a base.*

The proof will in fact yield a concrete method for constructing all possible bases. For each vector $\gamma \in \mathsf{E}$, define $\Phi^+(\gamma) = \{\alpha \in \Phi | (\gamma, \alpha) > 0\}$ = the set of roots lying on the "positive" side of the hyperplane orthogonal to γ. It is an elementary fact in euclidean geometry that the union of the finitely many hyperplanes P_α $(\alpha \in \Phi)$ cannot exhaust E (we leave to the reader the task of formulating a rigorous proof). Call $\gamma \in \mathsf{E}$ **regular** if $\gamma \in \mathsf{E} - \bigcup_{\alpha \in \Phi} P_\alpha$, **singular** otherwise. When γ is regular, it is clear that $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. This is the case we shall now pursue. Call $\alpha \in \Phi^+(\gamma)$ **decomposable** if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \in \Phi^+(\gamma)$, **indecomposable** otherwise. Now it suffices to prove the following statement.

Theorem′. *Let $\gamma \in \mathsf{E}$ be regular. Then the set $\Delta(\gamma)$ of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ, and every base is obtainable in this manner.*

Proof. This will proceed in steps.

(1) *Each root in $\Phi^+(\gamma)$ is a nonnegative **Z**-linear combination of $\Delta(\gamma)$.* Otherwise some $\alpha \in \Phi^+(\gamma)$ cannot be so written; choose α so that (γ, α) is as small as possible. Obviously α itself cannot be in $\Delta(\gamma)$, so $\alpha = \beta_1 + \beta_2$ $(\beta_i \in \Phi^+(\gamma))$, whence $(\gamma, \alpha) = (\gamma, \beta_1) + (\gamma, \beta_2)$. But each of the (γ, β_i) is positive, so β_1 and β_2 must each be a nonnegative **Z**-linear combination of $\Delta(\gamma)$ (to avoid contradicting the minimality of (γ, α)), whence α also is. This contradiction proves the original assertion.

(2) *If $\alpha, \beta \in \Delta(\gamma)$, then $(\alpha, \beta) \leq 0$ unless $\alpha = \beta$.* Otherwise $\alpha - \beta$ is a root (Lemma 9.4), since β clearly cannot be $-\alpha$, so $\alpha - \beta$ or $\beta - \alpha$ is in $\Phi^+(\gamma)$. In the first case, $\alpha = \beta + (\alpha - \beta)$, which says that α is decomposable; in the second case, $\beta = \alpha + (\beta - \alpha)$ is decomposable. This contradicts the assumption.

(3) *$\Delta(\gamma)$ is a linearly independent set.* Suppose $\Sigma r_\alpha \alpha = 0$ $(\alpha \in \Delta(\gamma), r_\alpha \in \mathbf{R})$. Separating the indices α for which $r_\alpha > 0$ from those for which $r_\alpha < 0$, we can rewrite this as $\Sigma s_\alpha \alpha = \Sigma t_\beta \beta$ $(s_\alpha, t_\beta > 0$, the sets of α's and β's being disjoint). Call $\varepsilon = \Sigma s_\alpha \alpha$. Then $(\varepsilon, \varepsilon) = \sum_{\alpha, \beta} s_\alpha t_\beta (\alpha, \beta) \leq 0$ by step (2), forcing $\varepsilon = 0$. Then $0 = (\gamma, \varepsilon) = \Sigma s_\alpha (\gamma, \alpha)$, forcing all $s_\alpha = 0$. Similarly, all $t_\beta = 0$. (This argument actually shows that *any set of vectors lying strictly on one side of a hyperplane in E and forming pairwise obtuse angles must be linearly independent.*)

(4) *$\Delta(\gamma)$ is a base of Φ.* Since $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$, the requirement (B2) is satisfied thanks to step (1). It also follows that $\Delta(\gamma)$ spans E, which combined with step (3) yields (B1).

(5) *Each base Δ of Φ has the form $\Delta(\gamma)$ for some regular $\gamma \in \mathsf{E}$.* Given Δ, select $\gamma \in \mathsf{E}$ so that $(\gamma, \alpha) > 0$ for all $\alpha \in \Delta$. (This is possible, because the intersection of "positive" open half-spaces associated with any basis of E is nonvoid (Exercise 7).) In view of (B2), γ is regular and $\Phi^+ \subset \Phi^+(\gamma)$, $\Phi^- \subset -\Phi^+(\gamma)$ (so equality must hold in each instance). Since $\Phi^+ = \Phi^+(\gamma)$,

Δ clearly consists of indecomposable elements, i.e., $\Delta \subset \Delta(\gamma)$. But Card Δ = Card $\Delta(\gamma) = \ell$, so $\Delta = \Delta(\gamma)$. ☐

It is useful to introduce a bit of terminology. The hyperplanes P_α $(\alpha \in \Phi)$ partition E into finitely many regions; the connected components of $\mathsf{E} - \bigcup_\alpha P_\alpha$ are called the (open) **Weyl chambers** of E. Each regular $\gamma \in \mathsf{E}$ therefore belongs to precisely one Weyl chamber, denoted $\mathfrak{C}(\gamma)$. To say that $\mathfrak{C}(\gamma) = \mathfrak{C}(\gamma')$ is just to say that γ, γ' lie on the same side of each hyperplane P_α $(\alpha \in \Phi)$, i.e., that $\Phi^+(\gamma) = \Phi^+(\gamma')$, or $\Delta(\gamma) = \Delta(\gamma')$. This shows that *Weyl chambers are in natural 1–1 correspondence with bases.* Write $\mathfrak{C}(\Delta) = \mathfrak{C}(\gamma)$ if $\Delta = \Delta(\gamma)$, and call this the **fundamental Weyl chamber relative to Δ**. $\mathfrak{C}(\Delta)$ is the open convex set (intersection of open half-spaces) consisting of all $\gamma \in \mathsf{E}$ which satisfy the inequalities $(\gamma, \alpha) > 0$ $(\alpha \in \Delta)$. In rank 2, it is easy to draw the appropriate picture; this is done in Figure 1 for type A_2. Here there are six chambers, the shaded one being fundamental relative to the base $\{\alpha, \beta\}$.

The Weyl group obviously sends one Weyl chamber onto another: explicitly, $\sigma(\mathfrak{C}(\gamma)) = \mathfrak{C}(\sigma\gamma)$, if $\sigma \in \mathscr{W}$ and γ is regular. On the other hand, $\mathscr{W}$ permutes bases: σ sends Δ to $\sigma(\Delta)$, which is again a base (why?). These two actions of $\mathscr{W}$ are in fact compatible with the above correspondence

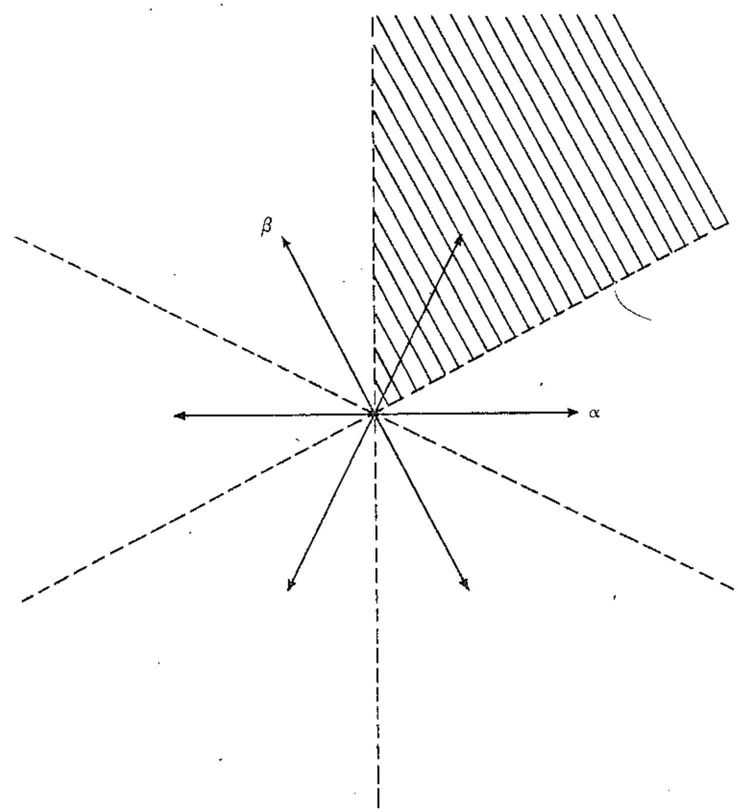


Figure 1

between Weyl chambers and bases; we have $\sigma(\Delta(\gamma)) = \Delta(\sigma\gamma)$, because $(\sigma\gamma, \sigma\alpha) = (\gamma, \alpha)$.

10.2. Lemmas on simple roots

Let Δ be a fixed base of Φ. We prove here several very useful lemmas about the behavior of simple roots.

Lemma A. *If α is positive but not simple, then $\alpha-\beta$ is a root (necessarily positive) for some $\beta \in \Delta$.*

Proof. If $(\alpha, \beta) \leq 0$ for all $\beta \in \Delta$, the parenthetic remark in step (3) in (10.1) would apply, showing that $\Delta \cup \{\alpha\}$ is a linearly independent set. This is absurd, since Δ is already a basis of E. So $(\alpha, \beta) > 0$ for some $\beta \in \Delta$ and then $\alpha-\beta \in \Phi$ (Lemma 9.4, which applies since β cannot be proportional to α). Write $\alpha = \sum_{\gamma\in\Delta} k_\gamma \gamma$ (all $k_\gamma \geq 0$, some $k_\gamma > 0$ for $\gamma \neq \beta$). Subtracting β from α yields a **Z**-linear combination of simple roots with at least one positive coefficient. This forces all coefficients to be nonnegative, thanks to the uniqueness of expression in (B2). ▯

Corollary. *Each $\beta \in \Phi^+$ can be written in the form $\alpha_1+\ldots+\alpha_k$ ($\alpha_i \in \Delta$, not necessarily distinct) in such a way that each partial sum $\alpha_1+\ldots+\alpha_i$ is a root.*

Proof. Use the lemma and induction on ht β. ▯

Lemma B. *Let α be simple. Then σ_α permutes the positive roots other than α.*

Proof. Let $\beta \in \Phi^+ - \{\alpha\}$, $\beta = \sum_{\gamma\in\Delta} k_\gamma \gamma$ ($k_\gamma \in \mathbf{Z}^+$). It is clear that β is not proportional to α. Therefore, $k_\gamma \neq 0$ for some $\gamma \neq \alpha$. But the coefficient of γ in $\sigma_\alpha(\beta) = \beta - \langle\beta, \alpha\rangle \alpha$ is still k_γ. In other words, $\sigma_\alpha(\beta)$ has at least one positive coefficient (relative to Δ), forcing it to be positive. Moreover, $\sigma_\alpha(\beta) \neq \alpha$, since α is the image of $-\alpha$. ▯

Corollary. *Set $\delta = \frac{1}{2}\sum_{\beta\succ 0}\beta$. Then $\sigma_\alpha(\delta) = \delta-\alpha$ for all $\alpha \in \Delta$.*

Proof. Obvious from the lemma. ▯

Lemma C. *Let $\alpha_1, \ldots, \alpha_t \in \Delta$ (not necessarily distinct). Write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \ldots \sigma_{t-1}(\alpha_t)$ is negative, then for some index $1 \leq s < t$, $\sigma_1 \ldots \sigma_t = \sigma_1 \ldots \sigma_{s-1}\sigma_{s+1} \ldots \sigma_{t-1}$.*

Proof. Write $\beta_i = \sigma_{i+1} \ldots \sigma_{t-1}(\alpha_t)$, $0 \leq i \leq t-2$, $\beta_{t-1} = \alpha_t$. Since $\beta_0 \prec 0$ and $\beta_{t-1} \succ 0$, we can find a smallest index s for which $\beta_s \succ 0$. Then $\sigma_s(\beta_s) = \beta_{s-1} \prec 0$, and Lemma B forces $\beta_s = \alpha_s$. In general (Lemma 9.2), $\sigma \in \mathscr{W}$ implies $\sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$; so in particular, $\sigma_s = (\sigma_{s+1} \ldots \sigma_{t-1})\sigma_t(\sigma_{t-1} \ldots \sigma_{s+1})$ which yields the lemma. ▯

Corollary. *If $\sigma = \sigma_1 \ldots \sigma_t$ is an expression for $\sigma \in \mathscr{W}$ in terms of reflections corresponding to simple roots, with t as small as possible, then $\sigma(\alpha_t) \prec 0$.* ▯

10.3. The Weyl group

Now we are in a position to prove that $\mathscr{W}$ permutes the bases of Φ (or, equivalently, the Weyl chambers) in a simply transitive fashion and that $\mathscr{W}$ is generated by the "simple reflections" relative to any base Δ (i.e., by the σ_α for $\alpha \in \Delta$).

Theorem. *Let Δ be a base of Φ.*

(a) If $\gamma \in$ E, γ regular, there exists $\sigma \in \mathscr{W}$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$ (so $\mathscr{W}$ acts transitively on Weyl chambers).

(b) If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathscr{W}$ (so $\mathscr{W}$ acts transitively on bases).

(c) If α is any root, there exists $\sigma \in \mathscr{W}$ such that $\sigma(\alpha) \in \Delta$.

(d) $\mathscr{W}$ is generated by the σ_α ($\alpha \in \Delta$).

(e) If $\sigma(\Delta) = \Delta$, $\sigma \in \mathscr{W}$, then $\sigma = 1$ (so $\mathscr{W}$ acts simply transitively on bases).

Proof. Let $\mathscr{W}'$ be the subgroup of $\mathscr{W}$ generated by all σ_α ($\alpha \in \Delta$). We shall prove (a)–(c) for $\mathscr{W}'$, then deduce that $\mathscr{W}' = \mathscr{W}$.

(a) Write $\delta = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$, and choose $\sigma \in \mathscr{W}'$ for which $(\sigma(\gamma), \delta)$ is as big as possible. If α is simple, then of course $\sigma_\alpha\sigma$ is also in $\mathscr{W}'$, so the choice of σ implies that $(\sigma(\gamma), \delta) \geq (\sigma_\alpha\sigma(\gamma), \delta) = (\sigma(\gamma), \sigma_\alpha(\delta)) = (\sigma(\gamma), \delta-\alpha) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha)$ (Corollary to Lemma 10.2B). This forces $(\sigma(\gamma), \alpha) \geq 0$ for all $\alpha \in \Delta$. Since γ is regular, we cannot have $(\sigma(\gamma), \alpha) = 0$ for any α, because then γ would be orthogonal to $\sigma^{-1}\alpha$. So all the inequalities are strict. Therefore $\sigma(\gamma)$ lies in the fundamental Weyl chamber $\mathfrak{C}(\Delta)$, and σ sends $\mathfrak{C}(\gamma)$ to $\mathfrak{C}(\Delta)$ as desired.

(b) Since $\mathscr{W}'$ permutes the Weyl chambers, by (a), it also permutes the bases of Φ (transitively).

(c) In view of (b), it suffices to prove that each root belongs to at least one base. Since the only roots proportional to α are $\pm\alpha$, the hyperplanes P_β ($\beta \neq \pm\alpha$) are distinct from P_α, so there exists $\gamma \in P_\alpha$, $\gamma \notin P_\beta$ (all $\beta \neq \pm\alpha$) (why?). Choose γ' close enough to γ so that $(\gamma', \alpha) = \varepsilon > 0$ while $|(\gamma', \beta)| > \varepsilon$ for all $\beta \neq \pm\alpha$. Evidently α then belongs to the base $\Delta(\gamma')$.

(d) To prove $\mathscr{W}' = \mathscr{W}$, it is enough to show that each reflection σ_α ($\alpha \in \Phi$) is in $\mathscr{W}'$. Using (c), find $\sigma \in \mathscr{W}'$ such that $\beta = \sigma(\alpha) \in \Delta$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$, so $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in \mathscr{W}'$.

(e) Let $\sigma(\Delta) = \Delta$, but $\sigma \neq 1$. If σ is written minimally as a product of one or more simple reflections (which is possible, thanks to (d)), then the Corollary to Lemma 10.2C is contradicted. ▯

We can use the lemmas of (10.2) to explore more precisely the significance of the generation of $\mathscr{W}$ by simple reflections.

When $\sigma \in \mathscr{W}$ is written as $\sigma_{\alpha_1} \ldots \sigma_{\alpha_t}$ ($\alpha_i \in \Delta$, t minimal), we call the expression **reduced**, and write $\ell(\sigma) = t$: this is the **length** of σ, relative to Δ. By definition, $\ell(1) = 0$. We can characterize length in another way, as follows. Define $n(\sigma)$ = number of positive roots α for which $\sigma(\alpha) \prec 0$.

Lemma A. *For all $\sigma \in \mathscr{W}$, $\ell(\sigma) = n(\sigma)$.*

Proof. Proceed by induction on $\ell(\sigma)$. The case $\ell(\sigma) = 0$ is clear: $\ell(\sigma) = 0$ implies $\sigma = 1$, so $n(\sigma) = 0$. Assume the lemma true for all $\tau \in \mathscr{W}$ with $\ell(\tau) < \ell(\sigma)$. Write σ in reduced form as $\sigma = \sigma_{\alpha_1} \ldots \sigma_{\alpha_t}$, and set $\alpha = \alpha_t$. By the Corollary of Lemma 10.2C, $\sigma(\alpha) \prec 0$. Then Lemma 10.2B implies that $n(\sigma\sigma_\alpha) = n(\sigma) - 1$. On the other hand, $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1 < \ell(\sigma)$, so by induction $\ell(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$. Combining these statements, we get $\ell(\sigma) = n(\sigma)$. ▯

Next we look more closely at the simply transitive action of $\mathscr{W}$ on Weyl chambers (parts (a) and (e) of the theorem). The next lemma shows that the closure $\overline{\mathfrak{C}(\Delta)}$ of the fundamental Weyl chamber relative to Δ is a **fundamental domain** for the action of $\mathscr{W}$ on E, i.e., each vector in E is $\mathscr{W}$-conjugate to precisely one point of this set (cf. Exercise 14).

Lemma B. *Let $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$. If $\sigma\lambda = \mu$ for some $\sigma \in \mathscr{W}$, then σ is a product of simple reflections which fix λ; in particular, $\lambda = \mu$.*

Proof. Use induction on $\ell(\sigma)$, the case $\ell(\sigma) = 0$ being clear. Let $\ell(\sigma) > 0$. By Lemma A, σ must send some positive root to a negative root; so σ cannot send all simple roots to positive roots. Say $\sigma\alpha \prec 0 (\alpha \in \Delta)$. Now $0 \geq (\mu, \sigma\alpha) = (\sigma^{-1}\mu, \alpha) = (\lambda, \alpha) \geq 0$, because $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$. This forces $(\lambda, \alpha) = 0, \sigma_\alpha\lambda = \lambda, (\sigma\sigma_\alpha)\lambda = \mu$. Thanks to Lemma 10.2B (and Lemma A), $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1$, so induction may be applied. ▯

10.4. Irreducible root systems

Φ is called **irreducible** if it cannot be partitioned into the union of two proper subsets such that each root in one set is orthogonal to each root in the other. (In (9.3), A_1, A_2, B_2, G_2 are irreducible, while $A_1 \times A_1$ is not.) Suppose Δ is a base of Φ. We claim that Φ *is irreducible if and only if Δ cannot be partitioned in the way just stated.* In one direction, let $\Phi = \Phi_1 \cup \Phi_2$, with $(\Phi_1, \Phi_2) = 0$. Unless Δ is wholly contained in Φ_1 or Φ_2, this induces a similar partition of Δ; but $\Delta \subset \Phi_1$ implies $(\Delta, \Phi_2) = 0$, or $(E, \Phi_2) = 0$, since Δ spans E. This shows that the "if" holds. Conversely, let Φ be irreducible, but $\Delta = \Delta_1 \cup \Delta_2$ with $(\Delta_1, \Delta_2) = 0$. Each root is conjugate to a simple root (Theorem 10.3(c)), so $\Phi = \Phi_1 \cup \Phi_2$, Φ_i the set of roots having a conjugate in Δ_i. Recall that $(\alpha, \beta) = 0$ implies $\sigma_\alpha\sigma_\beta = \sigma_\beta\sigma_\alpha$. Since $\mathscr{W}$ is generated by the σ_α $(\alpha \in \Delta)$, the formula for a reflection makes it clear that each root in Φ_i is gotten from one in Δ_i by adding or subtracting elements of Δ_i. Therefore, Φ_i lies in the subspace E_i of E spanned by Δ_i, and we see that $(\Phi_1, \Phi_2) = 0$. This forces $\Phi_1 = \varnothing$ or $\Phi_2 = \varnothing$, whence $\Delta_1 = \varnothing$ or $\Delta_2 = \varnothing$.

Lemma A. *Let Φ be irreducible. Relative to the partial ordering $\prec$, there is a unique maximal root β (in particular, $\alpha \neq \beta$ implies ht $\alpha <$ ht β for all $\alpha \in \Phi$, and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Delta$). If $\beta = \Sigma k_\alpha \alpha$ $(\alpha \in \Delta)$ then all $k_\alpha > 0$.*

Proof. Let $\beta = \Sigma k_\alpha \alpha$ $(\alpha \in \Delta)$ be maximal in the ordering; evidently $\beta \succ 0$. If $\Delta_1 = \{\alpha \in \Delta | k_\alpha > 0\}$ and $\Delta_2 = \{\alpha \in \Delta | k_\alpha = 0\}$, then $\Delta = \Delta_1 \cup \Delta_2$ is a partition. Suppose Δ_2 is nonvoid. Then $(\alpha, \beta) \leq 0$ for any $\alpha \in \Delta_2$ (Lemma 10.1); since Φ is irreducible, at least one $\alpha \in \Delta_2$ must be nonorthogonal to Δ_1, forcing $(\alpha, \alpha') < 0$ for some $\alpha' \in \Delta_1$, whence $(\alpha, \beta) < 0$. This implies that $\beta + \alpha$ is a root (Lemma 9.4), contradicting the maximality of β. Therefore Δ_2 is empty and all $k_\alpha > 0$. This argument shows also that $(\alpha, \beta) \geq 0$ for all $\alpha \in \Delta$ (with $(\alpha, \beta) > 0$ for at least one α, since Δ spans E). Now let β' be another maximal root. The preceding argument applies to β', so β' involves (with positive coefficient) at least one $\alpha \in \Delta$ for which $(\alpha, \beta) > 0$. It follows that $(\beta', \beta) > 0$, and $\beta - \beta'$ is a root (Lemma 9.4) unless $\beta = \beta'$. But if $\beta - \beta'$ is a root, then either $\beta \prec \beta'$ or else $\beta' \prec \beta$, which is absurd. So β is unique. ▯

Lemma B. *Let Φ be irreducible. Then $\mathscr{W}$ acts irreducibly on* E. *In particular, the $\mathscr{W}$-orbit of a root α spans* E.

Proof. The span of the $\mathscr{W}$-orbit of a root is a (nonzero) $\mathscr{W}$-invariant subspace of E, so the second statement follows from the first. As to the first, let E′ be a nonzero subspace of E invariant under $\mathscr{W}$. The orthogonal complement E″ of E′ is also $\mathscr{W}$-invariant, and $E = E' \oplus E''$. It is trivial to verify that for $\alpha \in \Phi$, either $\alpha \in E'$ or else $E' \subset P_\alpha$, since $\sigma_\alpha(E') = E'$ (Exercise 9.1). Thus, $\alpha \notin E'$ implies $\alpha \in E''$, so each root lies in one subspace or the other. This partitions Φ into orthogonal subsets, forcing one or the other to be empty. Since Φ spans E, we conclude that $E' = E$. ▯

Lemma C. *Let Φ be irreducible. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under $\mathscr{W}$.*

Proof. If α, β are arbitrary roots, then not all $\sigma(\alpha)$ $(\sigma \in \mathscr{W})$ can be orthogonal to β, since the $\sigma(\alpha)$ span E (Lemma B). If $(\alpha, \beta) \neq 0$, we know (cf. (9.4)) that the possible ratios of squared root lengths of α, β are 1, 2, 3, 1/2, 1/3. These two remarks easily imply the first assertion of the lemma, since the presence of three root lengths would yield also a ratio 3/2. Now let α, β have equal length. After replacing one of these by a $\mathscr{W}$-conjugate (as above), we may assume them to be nonorthogonal (and distinct: otherwise we're done!). According to (9.4), this in turn forces $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \pm 1$. Replacing β (if need be) by $-\beta = \sigma_\beta(\beta)$, we may assume that $\langle \alpha, \beta \rangle = 1$. Therefore, $(\sigma_\alpha\sigma_\beta\sigma_\alpha)(\beta) = \sigma_\alpha\sigma_\beta(\beta - \alpha) = \sigma_\alpha(-\beta - \alpha + \beta) = \alpha$. ▯

In case Φ is irreducible, with two distinct root lengths, we speak of **long** and **short roots**. (If all roots are of equal length, it is conventional to call all of them long.)

Lemma D. *Let Φ be irreducible, with two distinct root lengths. Then the maximal root β of Lemma A is long.*

Proof. Let $\alpha \in \Phi$ be arbitrary. It will suffice to show that $(\beta, \beta) \geq (\alpha, \alpha)$. For this we may replace α by a $\mathscr{W}$-conjugate lying in the closure of the fundamental Weyl chamber (relative to Δ). Since $\beta - \alpha \succ 0$ (Lemma A), we have $(\gamma, \beta-\alpha) \geq 0$ for any $\gamma \in \overline{\mathfrak{C}(\Delta)}$. This fact, applied to the cases $\gamma = \beta$ (cf. Lemma A) and $\gamma = \alpha$, yields $(\beta, \beta) \geq (\beta, \alpha) \geq (\alpha, \alpha)$. □

Exercises

1. Let $\Phi^\vee$ be the dual system of $\Phi, \Delta^\vee = \{\alpha^\vee | \alpha \in \Delta\}$. Prove that $\Delta^\vee$ is a base of $\Phi^\vee$. [Compare Weyl chambers of Φ and $\Phi^\vee$.]
2. If Δ is a base of Φ, prove that the set $(\mathbf{Z}\alpha + \mathbf{Z}\beta) \cap \Phi$ ($\alpha \neq \beta$ in Δ) is a root system of rank 2 in the subspace of E spanned by α, β (cf. Exercise 9.7). Generalize to an arbitrary subset of Δ.
3. Prove that each root system of rank 2 is isomorphic to one of those listed in (9.3).
4. Verify the Corollary of Lemma 10.2A directly for G_2.
5. If $\sigma \in \mathscr{W}$ can be written as a product of t simple reflections, prove that t has the same parity as $\ell(\sigma)$.
6. Define a function $sn: \mathscr{W} \to \{\pm 1\}$ by $sn(\sigma) = (-1)^{\ell(\sigma)}$. Prove that sn is a homomorphism (cf. the case A_2, where $\mathscr{W}$ is isomorphic to the symmetric group $\mathscr{S}_3$).
7. Prove that the intersection of "positive" open half-spaces associated with any basis $\gamma_1, \ldots, \gamma_\ell$ of E is nonvoid. [If δ_i is the projection of γ_i on the orthogonal complement of the subspace spanned by all basis vectors except γ_i, consider $\gamma = \Sigma r_i \delta_i$ when all $r_i > 0$.]
8. Let Δ be a base of Φ, $\alpha \neq \beta$ simple roots, $\Phi_{\alpha\beta}$ the rank 2 root system in $\mathsf{E}_{\alpha\beta} = \mathbf{R}\alpha + \mathbf{R}\beta$ (see Exercise 2 above). The Weyl group $\mathscr{W}_{\alpha\beta}$ of $\Phi_{\alpha\beta}$ is generated by the restrictions τ_α, τ_β to $\mathsf{E}_{\alpha\beta}$ of $\sigma_\alpha, \sigma_\beta$, and $\mathscr{W}_{\alpha\beta}$ may be viewed as a subgroup of $\mathscr{W}$. Prove that the "length" of an element of $\mathscr{W}_{\alpha\beta}$ (relative to τ_α, τ_β) coincides with the length of the corresponding element of $\mathscr{W}$.
9. Prove that there is a unique element σ in $\mathscr{W}$ sending Φ^+ to Φ^- (relative to Δ). Prove that any reduced expression for σ must involve all σ_α ($\alpha \in \Delta$). Discuss $\ell(\sigma)$.
10. Given $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ in Φ, let $\lambda = \sum_{i=1}^{\ell} k_i\alpha_i$ ($k_i \in \mathbf{Z}$, all $k_i \geq 0$ or all $k_i \leq 0$). Prove that either λ is a multiple (possibly 0) of a root, or else there exists $\sigma \in \mathscr{W}$ such that $\sigma\lambda = \sum_{i=1}^{\ell} k_i'\alpha_i$, with some $k_i' > 0$ and some $k_i' < 0$. [Sketch of proof: If λ is not a multiple of any root, then the hyperplane P_λ orthogonal to λ is not included in $\bigcup_{\alpha\in\Phi} P_\alpha$. Take $\mu \in P_\lambda - \bigcup_{\alpha\in\Phi} P_\alpha$. Then find $\sigma \in \mathscr{W}$ for which all $(\alpha_i, \sigma\mu) > 0$. It follows that $0 = (\lambda, \mu) = (\sigma\lambda, \sigma\mu) = \Sigma k(\alpha_i, \sigma\mu)$.]

11. Let Φ be irreducible. Prove that $\Phi^\vee$ is also irreducible. If Φ has all roots of equal length, so does $\Phi^\vee$ (and then $\Phi^\vee$ is isomorphic to Φ). On the other hand, if Φ has two root lengths, then so does $\Phi^\vee$; but if α is long, then $\alpha^\vee$ is short (and vice versa). Use this fact to prove that Φ has a unique maximal short root (relative to the partial order $\prec$ defined by Δ).
12. Let $\lambda \in \overline{\mathfrak{C}(\Delta)}$. If $\sigma\lambda = \lambda$ for some $\sigma \in \mathscr{W}$, then $\sigma = 1$.
13. The only reflections in $\mathscr{W}$ are those of the form σ_α ($\alpha \in \Phi$). [A vector in the reflecting hyperplane would, if orthogonal to no root, be fixed only by the identity in $\mathscr{W}$.]
14. Prove that each point of E is $\mathscr{W}$-conjugate to a point in the closure of the fundamental Weyl chamber relative to a base Δ. [Enlarge the partial order on E by defining $\mu \prec \lambda$ iff $\lambda - \mu$ is a nonnegative $\mathbf{R}$-linear combination of simple roots. If $\mu \in \mathsf{E}$, choose $\sigma \in \mathscr{W}$ for which $\lambda = \sigma\mu$ is maximal in this partial order.]

Notes

The exposition here is an expanded version of that in Serre [2].

11. Classification

In this section Φ denotes a root system of rank ℓ, $\mathscr{W}$ its Weyl group, Δ a base of Φ.

11.1. Cartan matrix of Φ

Fix an ordering $(\alpha_1, \ldots, \alpha_\ell)$ of the simple roots. The matrix $(\langle\alpha_i, \alpha_j\rangle)$ is then called the **Cartan matrix** of Φ. Its entries are called **Cartan integers**. *Examples*: For the systems of rank 2, the matrices are:

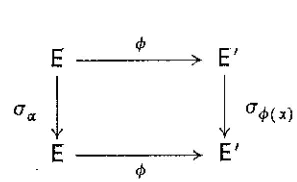
$$\mathsf{A}_1 \times \mathsf{A}_1 \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \quad \mathsf{A}_2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}; \quad \mathsf{B}_2 \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}; \quad \mathsf{G}_2 \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

The matrix of course depends on the chosen ordering, but this is not very serious. The important point is that the Cartan matrix is independent of the choice of Δ, thanks to the fact (Theorem 10.3(b)) that $\mathscr{W}$ acts transitively on the collection of bases. The Cartan matrix is nonsingular, as in (8.5), since Δ is a basis of E. It turns out to characterize Φ completely.

Proposition. *Let $\Phi' \subset \mathsf{E}'$ be another root system, with base $\Delta' = \{\alpha_1', \ldots, \alpha_\ell'\}$. If $\langle\alpha_i, \alpha_j\rangle = \langle\alpha_i', \alpha_j'\rangle$ for $1 \leq i, j \leq \ell$, then the bijection $\alpha_i \mapsto \alpha_i'$ extends (uniquely) to an isomorphism $\phi: \mathsf{E} \to \mathsf{E}'$ mapping Φ onto Φ' and satisfying $\langle\phi(\alpha), \phi(\beta)\rangle = \langle\alpha, \beta\rangle$ for all $\alpha, \beta \in \Phi$. Therefore, the Cartan matrix of Φ determines Φ up to isomorphism.*

Proof. Since Δ (resp. Δ') is a basis of E (resp. E'), there is a unique vector space isomorphism $\phi: \mathsf{E} \to \mathsf{E}'$ sending α_i to α_i' ($1 \leq i \leq \ell$). If $\alpha, \beta \in \Delta$, the hypothesis insures that $\sigma_{\phi(\alpha)}(\phi(\beta)) = \sigma_{\alpha'}(\beta') = \beta' - \langle\beta', \alpha'\rangle\alpha' = \phi(\beta) -$

$\langle\beta,\alpha\rangle\ \phi(\alpha) = \phi(\beta - \langle\beta,\alpha\rangle\ \alpha) = \phi(\sigma_\alpha(\beta))$. In other words, the following diagram commutes for each $\alpha \in \Delta$:



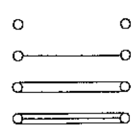
The respective Weyl groups $\mathscr{W}$, $\mathscr{W}'$ are generated by simple reflections (Theorem 10.3(d)), so it follows that the map $\sigma \mapsto \phi \circ \sigma \circ \phi^{-1}$ is an isomorphism of $\mathscr{W}$ onto $\mathscr{W}'$, sending σ_α to $\sigma_{\phi(\alpha)}$ ($\alpha \in \Delta$). But each $\beta \in \Phi$ is conjugate under $\mathscr{W}$ to a simple root (Theorem 10.3(c)), say $\beta = \sigma(\alpha)$ ($\alpha \in \Delta$). This in turn forces $\phi(\beta) = (\phi \circ \sigma \circ \phi^{-1})(\phi(\alpha)) \in \Phi'$. It follows that ϕ maps Φ onto Φ'; moreover, the formula for a reflection shows that ϕ preserves all Cartan integers. ▯

The proposition shows that it is possible in principle to recover Φ from a knowledge of the Cartan integers. In fact, it is not too hard to devise a practical algorithm for writing down all roots (or just all positive roots). Probably the best approach is to consider *root strings* (9.4). Start with the roots of height one, i.e., the simple roots. For any pair $\alpha_i \neq \alpha_j$, the integer r for the α_j-string through α_i is 0 (i.e., $\alpha_i - \alpha_j$ is not a root, thanks to Lemma 10.1), so the integer q equals $-\langle\alpha_i, \alpha_j\rangle$. This enables us in particular to write down all roots α of height 2, hence all integers $\langle\alpha, \alpha_j\rangle$. For each root α of height 2, the integer r for the α_j-string through α can be determined easily, since α_j can be subtracted at most once (why?), and then q is found, because we know $r - q = \langle\alpha, \alpha_j\rangle$. The corollary of Lemma 10.2A assures us that all positive roots are eventually obtained if we repeat this process enough times.

11.2. Coxeter graphs and Dynkin diagrams

If α, β are distinct positive roots, then we know that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 0$, 1, 2, or 3 (9.4). Define the **Coxeter graph** of Φ to be a graph having ℓ vertices, the ith joined to the jth ($i \neq j$) by $\langle\alpha_i,\alpha_j\rangle\langle\alpha_j,\alpha_i\rangle$ edges. *Examples*:

$A_1 \times A_1$
A_2
B_2
G_2

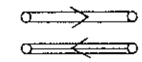


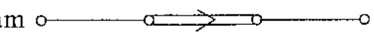
The Coxeter graph determines the numbers $\langle\alpha_i,\alpha_j\rangle$ in case all roots have equal length, since then $\langle\alpha_i,\alpha_j\rangle = \langle\alpha_j,\alpha_i\rangle$. In case more than one root length occurs (e.g., B_2 or G_2), the graph fails to tell us which of a pair of vertices should correspond to a short simple root, which to a long (in case these vertices are joined by two or three edges). (It can, however, be proved that the Coxeter graph determines the Weyl group completely, essentially

because it determines the orders of products of generators of $\mathscr{W}$, cf. Exercise 9.3.)

Whenever a double or triple edge occurs in the Coxeter graph of Φ, we can add an arrow pointing to the shorter of the two roots. This additional information allows us to recover the Cartan integers; we call the resulting figure the **Dynkin diagram** of Φ. (As before, this depends on the numbering of simple roots.) For example:

B_2
G_2



Another example: Given the diagram  (which turns out to be associated with the root system F_4), the reader can easily recover the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

11.3. Irreducible components

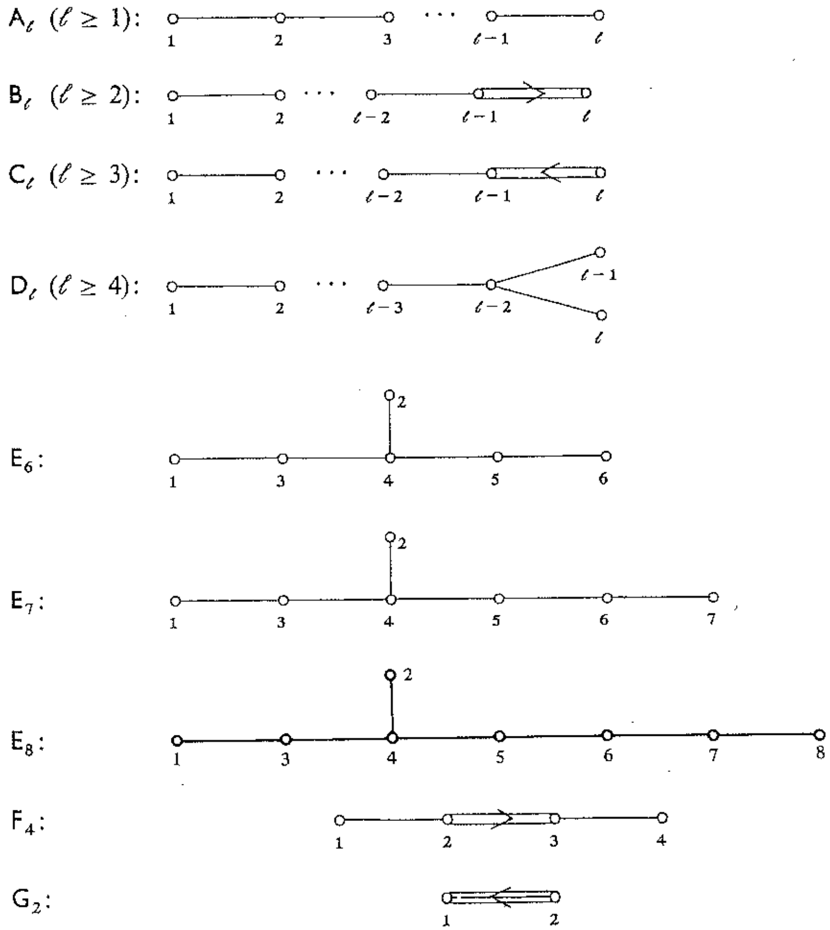
Recall (10.4) that Φ is irreducible if and only if Φ (or, equivalently, Δ) cannot be partitioned into two proper, orthogonal subsets. It is clear that Φ is *irreducible if and only if its Coxeter graph is connected* (in the usual sense). In general, there will be a number of connected components of the Coxeter graph; let $\Delta = \Delta_1 \cup \ldots \cup \Delta_t$ be the corresponding partition of Δ into mutually orthogonal subsets. If E_i is the span of Δ_i, it is clear that $E = E_1 \oplus \ldots \oplus E_t$ (orthogonal direct sum). Moreover, the $\mathbf{Z}$-linear combinations of Δ_i which are roots (call this set Φ_i) obviously form a root system in E_i, whose Weyl group is the restriction to E_i of the subgroup of $\mathscr{W}$ generated by all σ_α ($\alpha \in \Delta_i$). Finally, each E_i is $\mathscr{W}$-invariant (since $\alpha \notin \Delta_i$ implies that σ_α acts trivially on E_i), so the (easy) argument required for Exercise 9.1 shows immediately that each root lies in one of the E_i, i.e., $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$.

Proposition. *Φ decomposes (uniquely) as the union of irreducible root systems Φ_i (in subspaces E_i of E) such that $E = E_1 \oplus \ldots \oplus E_t$ (orthogonal direct sum).* ▯

11.4. Classification theorem

The discussion in (11.3) shows that it is sufficient to classify the irreducible root systems, or equivalently, the connected Dynkin diagrams (cf. Proposition 11.1).

Theorem. *If Φ is an irreducible root system of rank ℓ, its Dynkin diagram is one of the following (ℓ vertices in each case):*



The restrictions on ℓ for types $A_\ell - D_\ell$ are imposed in order to avoid duplication. Relative to the indicated numbering of simple roots, the corresponding Cartan matrices are given in Table 1. Inspection of the diagrams listed above reveals that in all cases except B_ℓ, C_ℓ, the Dynkin diagram can be deduced from the Coxeter graph. However, B_ℓ and C_ℓ both come from a single Coxeter graph, and differ in the relative numbers of short and long simple roots. (These root systems are actually dual to each other, cf. Exercise 5.)

Proof of Theorem. The idea of the proof is to classify first the possible Coxeter graphs (ignoring relative lengths of roots), then see what Dynkin diagrams result. Therefore, we shall merely apply some elementary euclidean geometry to finite sets of vectors whose pairwise angles are those prescribed by the Coxeter graph. Since we are ignoring lengths, it is easier to work for the time being with sets of unit vectors. For maximum flexibility, we make

Table 1. *Cartan matrices*

$$A_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ -1 & 2 & -1 & 0 & & \cdot & \cdot & \cdot & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \cdot & \cdot & -1 & 2 \end{pmatrix}$$

$$B_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}$$

$$C_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2 \end{pmatrix}$$

$$D_\ell: \begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 0 & 2 \end{pmatrix}$$

$$E_6: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_7: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$E_8: \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$F_4: \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \qquad G_2: \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

only the following assumptions: E is a euclidean space (of arbitrary dimension), $\mathfrak{A} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is a set of n linearly independent unit vectors which satisfy $(\varepsilon_i, \varepsilon_j) \le 0$ $(i \ne j)$ and $4(\varepsilon_i, \varepsilon_j)^2 = 0, 1, 2$, or 3 $(i \ne j)$. Such a set of vectors is called (for brevity) **admissible**. (*Example*: Elements of a base for a root system, each divided by its length.) We attach a graph Γ to the set $\mathfrak{A}$ just as we did above to the simple roots in a root system, with vertices i and j $(i \ne j)$ joined by $4(\varepsilon_i, \varepsilon_j)^2$ edges. Now our task is to determine all the connected graphs associated with admissible sets of vectors (these include all connected Coxeter graphs). This we do in steps, the first of which is obvious. (Γ is not assumed to be connected until later on.)

(1) *If some of the ε_i are discarded, the remaining ones still form an admissible set, whose graph is obtained from Γ by omitting the corresponding vertices and all incident edges.*

(2) *The number of pairs of vertices in Γ connected by at least one edge is strictly less than n.* Set $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. Since the ε_i are linearly independent, $\varepsilon \ne 0$. So $0 < (\varepsilon, \varepsilon) = n + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j)$. Let i, j be a pair of (distinct) indices for which $(\varepsilon_i, \varepsilon_j) \ne 0$ (i.e., let vertices i and j be joined). Then $4(\varepsilon_i, \varepsilon_j)^2 = 1, 2$, or 3, so in particular $2(\varepsilon_i, \varepsilon_j) \le -1$. In view of the above inequality, the number of such pairs cannot exceed $n-1$.

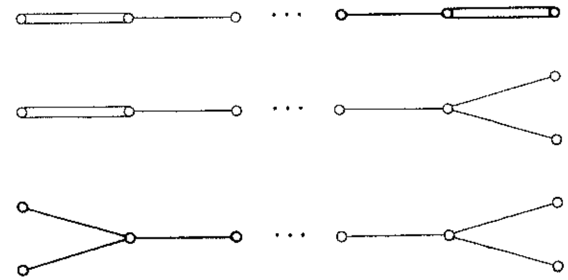
(3) *Γ contains no cycles.* A cycle would be the graph Γ' of an admissible subset $\mathfrak{A}'$ of $\mathfrak{A}$ (cf. (1)), and then Γ' would violate (2), with n replaced by Card $\mathfrak{A}'$.

(4) *No more than three edges can originate at a given vertex of Γ.* Say $\varepsilon \in \mathfrak{A}$, and $\eta_1, \ldots, \eta_k$ are the vectors in $\mathfrak{A}$ connected to ε (by 1, 2, or 3 edges each), i.e., $(\varepsilon, \eta_i) < 0$ with $\varepsilon, \eta_1, \ldots, \eta_k$ all distinct. In view of (3), no two η's can be connected, so $(\eta_i, \eta_j) = 0$ for $i \ne j$. Because $\mathfrak{A}$ is linearly independent, some unit vector η_0 in the span of $\varepsilon, \eta_1, \ldots, \eta_k$ is orthogonal to $\eta_1, \ldots, \eta_k$; clearly $(\varepsilon, \eta_0) \ne 0$ for such η_0. Now $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\eta_i$, so $1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2$. This forces $\sum_{i=1}^{k} (\varepsilon, \eta_i)^2 < 1$, or $\sum_{i=1}^{k} 4(\varepsilon, \eta_i)^2 < 4$. But $4(\varepsilon, \eta_i)^2$ is the number of edges joining ε to η_i in Γ.

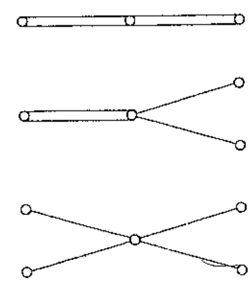
(5) *The only connected graph Γ of an admissible set $\mathfrak{A}$ which can contain a triple edge is*  *(the Coxeter graph* G_2*).* This follows at once from (4).

(6) *Let $\{\varepsilon_1, \ldots, \varepsilon_k\} \subset \mathfrak{A}$ have subgraph*  *(a simple chain in Γ. If $\mathfrak{A}' = (\mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}) \cup \{\varepsilon\}$, $\varepsilon = \sum_{i=1}^{k} \varepsilon_i$, then $\mathfrak{A}'$ is admissible.* (The graph of $\mathfrak{A}'$ is obtained from Γ by shrinking the simple chain to a point.) Linear independence of $\mathfrak{A}'$ is obvious. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1$ $(1 \le i \le k-1)$, so $(\varepsilon, \varepsilon) = k + 2\sum_{i<j} (\varepsilon_i, \varepsilon_j) = k - (k-1) = 1$. So ε is a unit vector. Any $\eta \in \mathfrak{A} - \{\varepsilon_1, \ldots, \varepsilon_k\}$ can be connected to at most one of $\varepsilon_1, \ldots, \varepsilon_k$ (by (3)), so $(\eta, \varepsilon) = 0$ or else $(\eta, \varepsilon) = (\eta, \varepsilon_i)$ for $1 \le i \le k$. In either case, $4(\eta, \varepsilon)^2 = 0, 1, 2$, or 3.

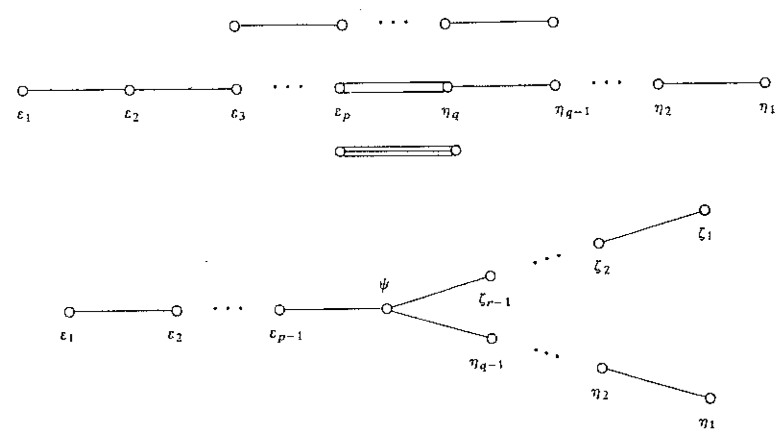
(7) *Γ contains no subgraph of the form:*

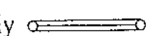


Suppose one of these graphs occurred in Γ; by (1) it would be the graph of an admissible set. But (6) allows us to replace the simple chain in each case by a single vertex, yielding (respectively) the following graphs which violate (4):

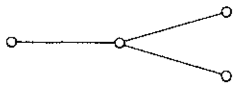


(8) *Any connected graph Γ of an admissible set has one of the following forms:*

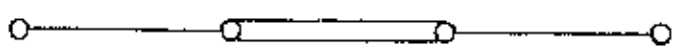
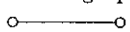
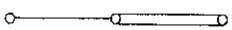


Indeed, only  contains a triple edge, by (5). A connected graph containing more than one double edge would contain a subgraph

which (7) forbids, so at most one double edge occurs. Moreover, if Γ has a double edge, it cannot also have a "node" (branch point)

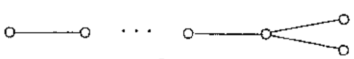


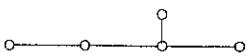
(again by (7)), so the second graph pictured is the only possibility (cycles being forbidden by (3)). Finally, let Γ have only single edges; if Γ has no node, it must be a simple chain (again because no cycles are allowed). It cannot contain more than one node (7), so the fourth graph is the only remaining possibility.

(9) *The only connected Γ of the second type in* (8) *is the Coxeter graph* F_4  *or the Coxeter graph* $\mathsf{B}_n(=\mathsf{C}_n)$  ⋯ .

Set $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\eta_i$. By hypothesis, $2(\varepsilon_i, \varepsilon_{i+1}) = -1 = 2(\eta_i, \eta_{i+1})$, and other pairs are orthogonal, so $(\varepsilon, \varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p(p+1)/2$, $(\eta, \eta) = q(q+1)/2$. Since $4(\varepsilon_p, \eta_q)^2 = 2$, we also have $(\varepsilon, \eta)^2 = p^2q^2(\varepsilon_p, \eta_q)^2 = p^2q^2/2$. The Schwartz inequality implies (since ε, η are obviously independent) that $(\varepsilon, \eta)^2 < (\varepsilon, \varepsilon)(\eta, \eta)$, or $p^2q^2/2 < p(p+1)q(q+1)/4$, whence $(p-1)(q-1) < 2$. The possibilities are: $p = q = 2$ (whence F_4) or $p = 1$ (q arbitrary), $q = 1$ (p arbitrary).

(10) *The only connected Γ of the fourth type in* (8) *is the Coxeter graph* D_n

 *or the Coxeter graph* E_n ($n = 6$, 7 *or* 8)

 ⋯ o. Set $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum i\eta_i$, $\zeta = \sum i\zeta_i$. It is clear that ε, η, ζ are mutually orthogonal, linearly independent vectors, and that ψ is not in their span. As in the proof of (4) we therefore obtain $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1$, where $\theta_1, \theta_2, \theta_3$ are the respective angles between ψ and ε, η, ζ. The same calculation as in (9), with $p-1$ in place of p, shows that $(\varepsilon, \varepsilon) = p(p-1)/2$, and similarly for η, ζ. Therefore $\cos^2\theta_1 = (\varepsilon, \psi)^2/(\varepsilon, \varepsilon)(\psi, \psi) = (p-1)^2(\varepsilon_{p-1}, \psi)^2/(\varepsilon, \varepsilon) = \frac{1}{4}(2(p-1)^2/p(p-1)) = (p-1)/2p = \frac{1}{2}(1-1/p)$. Similarly for θ_2, θ_3. Adding, we get the inequality $\frac{1}{2}(1-1/p+1-1/q+1-1/r) < 1$, or (*) $1/p+1/q+1/r > 1$. (This inequality, by the way, has a long mathematical history.) By changing labels we may assume that $1/p \le 1/q \le 1/r$ ($\le 1/2$; if p, q, or r equals 1, we are back in type A_n). In particular, the inequality (*) implies $3/2 \ge 3/r > 1$, so $r = 2$. Then $1/p+1/q > 1/2$, $2/q > 1/2$, and $2 \le q < 4$. If $q = 3$, then $1/p > 1/6$ and necessarily $p < 6$. So the possible triples (p, q, r) turn out to be: $(p, 2, 2) = \mathsf{D}_n$; $(3, 3, 2) = \mathsf{E}_6$; $(4, 3, 2) = \mathsf{E}_7$; $(5, 3, 2) = \mathsf{E}_8$.

The preceding argument shows that the connected graphs of admissible sets of vectors in euclidean space are all to be found among the Coxeter graphs of types A–G. In particular, the Coxeter graph of a root system must be of one of these types. But in all cases except B_ℓ, C_ℓ, the Coxeter graph

uniquely determines the Dynkin diagram, as remarked at the outset. So the theorem follows. □

Exercises

1. Verify the Cartan matrices (Table 1).
2. Calculate the determinants of the Cartan matrices (using induction on ℓ for types A_ℓ–D_ℓ), which are as follows:

$$\mathsf{A}_\ell: \ell+1;\ \mathsf{B}_\ell: 2;\ \mathsf{C}_\ell: 2;\ \mathsf{D}_\ell: 4;\ \mathsf{E}_6: 3;\ \mathsf{E}_7: 2;\ \mathsf{E}_8,\ \mathsf{F}_4 \text{ and } \mathsf{G}_2: 1.$$

3. Use the algorithm of (11.1) to write down all roots for G_2. Do the same for C_3: $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$.
4. Prove that the Weyl group of a root system Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.
5. Prove that each irreducible root system is isomorphic to its dual, except that B_ℓ, C_ℓ are dual to each other.
6. Prove that an inclusion of one Dynkin diagram in another (e.g., E_6 in E_7 or E_7 in E_8) induces an inclusion of the corresponding root systems.

Notes

Our proof of the classification theorem follows Jacobson [1]. For a somewhat different approach, see Carter [1]. Bourbaki [2] emphasizes the classification of Coxeter groups, of which the Weyl groups of root systems are important examples.

12. Construction of root systems and automorphisms

In §11 the possible (connected) Dynkin diagrams of (irreducible) root systems were all determined. It remains to be shown that each diagram of type A–G does in fact belong to a root system Φ. Afterwards we shall briefly discuss Aut Φ. The existence of root systems of type $\mathsf{A}_\ell - \mathsf{D}_\ell$ could actually be shown by verifying for each classical linear Lie algebra (1.2) that its root system is of the indicated type, which of course requires that we first prove the semisimplicity of these algebras (cf. §19). But it is easy enough to give a direct construction of the root system, which moreover makes plain the structure of its Weyl group.

12.1. Construction of types A–G

We shall work in various spaces $\mathbf{R}^n$, where the inner product is the usual one and where $\varepsilon_1, \ldots, \varepsilon_n$ denote the usual orthonormal unit vectors which form a basis of $\mathbf{R}^n$. The $\mathbf{Z}$-span of this basis is (by definition) a **lattice**, denoted I. In each case we shall take E to be $\mathbf{R}^n$ (or a suitable subspace thereof, with the inherited inner product). Then Φ will be defined to be the set of all vectors in I (or a closely related subgroup J of E) having specified length or lengths.

Since the group I (or J) is discrete in the usual topology of $\mathbf{R}^n$, while the set of vectors in $\mathbf{R}^n$ having one or two given lengths is compact (closed and bounded), Φ is then obviously finite, and will exclude 0 by definition. In each case it will be evident that Φ spans E (indeed, a base of Φ will be exhibited explicitly). Therefore (R1) is satisfied. The choice of lengths will also make it obvious that (R2) holds. For (R3) it is enough to check that the reflection σ_α ($\alpha \in \Phi$) maps Φ back into J, since then $\sigma_\alpha(\Phi)$ automatically consists of vectors of the required lengths. But then (R3) follows from (R4). As to (R4), it usually suffices to choose squared lengths dividing 2, since it is automatic that all inner products $(\alpha,\beta) \in \mathbf{Z}$ $(\alpha,\beta \in I)$.

Having made these preliminary remarks, we now treat the separate cases $\mathsf{A}-\mathsf{G}$. After verifying (R1) to (R4) in the way just sketched, the reader should observe that the resulting Cartan matrix matches that in Table 1 (11.4).

A_ℓ ($\ell \geq 1$): Let E be the ℓ-dimensional subspace of $\mathbf{R}^{\ell+1}$ orthogonal to the vector $\varepsilon_1+\ldots+\varepsilon_{\ell+1}$. Let $I' = I \cap \mathsf{E}$, and take Φ to be the set of all vectors $\alpha \in I'$ for which $(\alpha,\alpha) = 2$. It is obvious that $\Phi = \{\varepsilon_i - \varepsilon_j,\ i \neq j\}$. The vectors $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq \ell)$ are independent, and $\varepsilon_i - \varepsilon_j = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \ldots + (\varepsilon_{j-1} - \varepsilon_j)$ if $i < j$, which shows that they form a base of Φ. It is clear that the Cartan matrix A_ℓ results. Finally, notice that the reflection with respect to α_i permutes the subscripts i, $i+1$ and leaves all other subscripts fixed. Thus σ_{α_i} corresponds to the transposition $(i, i+1)$ in the symmetric group $\mathscr{S}_{\ell+1}$; these transpositions generate $\mathscr{S}_{\ell+1}$, so we obtain a natural isomorphism of $\mathscr{W}$ onto $\mathscr{S}_{\ell+1}$.

B_ℓ ($\ell \geq 2$): Let $\mathsf{E} = \mathbf{R}^\ell$, $\Phi = \{\alpha \in I | (\alpha, \alpha) = 1 \text{ or } 2\}$. It is easy to check that Φ consists of the vectors $\pm\varepsilon_i$ (of squared length 1) and the vectors $\pm(\varepsilon_i \pm \varepsilon_j)$, $i \neq j$ (of squared length 2). The ℓ vectors $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_\ell$ are independent; a short root $\varepsilon_i = (\varepsilon_i - \varepsilon_{i+1}) + (\varepsilon_{i+1} - \varepsilon_{i+2}) + \ldots + (\varepsilon_{\ell-1} - \varepsilon_\ell) + \varepsilon_\ell$, while a long root $\varepsilon_i - \varepsilon_j$ or $\varepsilon_i + \varepsilon_j$ is similarly expressible. The Cartan matrix for this (ordered) base is clearly B_ℓ. $\mathscr{W}$ acts as the group of all permutations and sign changes of the set $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$, so $\mathscr{W}$ is isomorphic to the semidirect product of $(\mathbf{Z}/2\mathbf{Z})^\ell$ and $\mathscr{S}_\ell$ (the latter acting on the former).

C_ℓ ($\ell \geq 3$): C_ℓ ($\ell \geq 2$) may be viewed most conveniently as the root system dual to B_ℓ (with $\mathsf{B}_2 = \mathsf{C}_2$), cf. Exercise 11.5. The reader can verify directly that in $\mathsf{E} = \mathbf{R}^\ell$, the set of all $\pm 2\varepsilon_i$ and all $\pm(\varepsilon_i \pm \varepsilon_j)$, $i \neq j$, forms a root system of type C_ℓ, with base $(\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{\ell-1} - \varepsilon_\ell, 2\varepsilon_\ell\}$. Of course the Weyl group is isomorphic to that of B_ℓ.

D_ℓ ($\ell \geq 4$): Let $\mathsf{E} = \mathbf{R}^\ell$, $\Phi = \{\alpha \in I | (\alpha, \alpha) = 2\} = \{\pm(\varepsilon_i \pm \varepsilon_j),\ i \neq j\}$. For a base take the ℓ independent vectors $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{\ell-1} - \varepsilon_\ell, \varepsilon_{\ell-1} + \varepsilon_\ell$ (so D_ℓ results). The Weyl group is the group of permutations and sign changes

involving only *even numbers of signs* of the set $\{\varepsilon_1, \ldots, \varepsilon_\ell\}$. So $\mathscr{W}$ is isomorphic to the semidirect product of $(\mathbf{Z}/2\mathbf{Z})^{\ell-1}$ and $\mathscr{S}_\ell$.

E_6, E_7, E_8: We know that E_6, E_7 can be identified canonically with subsystems of E_8 (Exercise 11.6), so it suffices to construct E_8. This is slightly complicated. Take $\mathsf{E} = \mathbf{R}^8$, $I' = I + \mathbf{Z}((\varepsilon_1+\ldots+\varepsilon_8)/2)$, I'' = subgroup of I' consisting of all elements $\Sigma c_i\varepsilon_i + \frac{c}{2}(\varepsilon_1 + \ldots + \varepsilon_8)$ for which Σc_i is an *even* integer. (Check that this is a subgroup!) Define $\Phi = \{\alpha \in I'' | (\alpha, \alpha) = 2\}$. It is easy to see that Φ consists of the obvious vectors $\pm(\varepsilon_i \pm \varepsilon_j)$, $i \neq j$, along with the less obvious ones $\frac{1}{2}\sum_{i=1}^{8}(-1)^{k(i)}\varepsilon_i$ (where the $k(i) = 0, 1$, add up to an even integer). By inspection, all inner products here are in $\mathbf{Z}$ (this has to be checked, because we are working in a larger lattice than I). As a base we take $\{\frac{1}{2}(\varepsilon_1 + \varepsilon_8 - (\varepsilon_2 + \ldots + \varepsilon_7)),\ \varepsilon_1 + \varepsilon_2,\ \varepsilon_2 - \varepsilon_1,\ \varepsilon_3 - \varepsilon_2,\ \varepsilon_4 - \varepsilon_3,\ \varepsilon_5 - \varepsilon_4,\ \varepsilon_6 - \varepsilon_5,\ \varepsilon_7 - \varepsilon_6\}$. (This has been ordered so as to correspond to the Cartan matrix for E_8 in Table 1 (11.4).) The reader is invited to contemplate for himself the action of the Weyl group, whose order can be shown to be $2^{14}3^5 5^2 7$.

F_4: Let $\mathsf{E} = \mathbf{R}^4$, $I' = I + \mathbf{Z}((\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)/2)$, $\Phi = \{\alpha \in I' | (\alpha, \alpha) = 1 \text{ or } 2\}$. Then Φ consists of all $\pm\varepsilon_i$, all $\pm(\varepsilon_i \pm \varepsilon_j)$, $i \neq j$, as well as all $\pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$, where the signs may be chosen independently. By inspection, all numbers $\langle\alpha,\beta\rangle$ are integral. As a base take $\{\varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4, \varepsilon_4, \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\}$. Here $\mathscr{W}$ has order 1152.

G_2: We already constructed G_2 explicitly in §9. Abstractly, we can take E to be the subspace of $\mathbf{R}^3$ orthogonal to $\varepsilon_1+\varepsilon_2+\varepsilon_3$, $I' = I \cap \mathsf{E}$, $\Phi = \{\alpha \in I' | (\alpha, \alpha) = 2 \text{ or } 6\}$. So $\Phi = \pm\{\varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_3,\ \varepsilon_1 - \varepsilon_3,\ 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3,\ 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3,\ 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$. As a base choose $\varepsilon_1 - \varepsilon_2$, $-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. (How does $\mathscr{W}$ act?)

Theorem. *For each Dynkin diagram (or Cartan matrix) of type* A–G, *there exists an irreducible root system having the given diagram.* ☐

12.2. Automorphisms of Φ

We are going to give a complete description of Aut Φ, for each root system Φ. Recall that Lemma 9.2 implies that $\mathscr{W}$ is a normal subgroup of Aut Φ (Exercise 9.6). Let $\Gamma = \{\sigma \in \text{Aut } \Phi | \sigma(\Delta) = \Delta\}$, Δ a fixed base of Φ. Evidently, Γ is a subgroup of Aut Φ. If $\tau \in \Gamma \cap \mathscr{W}$, then $\tau = 1$ by virtue of the simple transitivity of $\mathscr{W}$ (Theorem 10.3(e)). Moreover, if $\tau \in \text{Aut } \Phi$ is arbitrary, then $\tau(\Delta)$ is evidently another base of Φ, so there exists $\sigma \in \mathscr{W}$ such that $\sigma\tau(\Delta) = \Delta$ (Theorem 10.3(b)), whence $\tau \in \Gamma\mathscr{W}$. It follows that *Aut* Φ *is the semidirect product of* Γ *and* $\mathscr{W}$.

For all $\tau \in \text{Aut } \Phi$, all $\alpha, \beta \in \Phi$, we have $\langle\alpha, \beta\rangle = \langle\tau(\alpha), \tau(\beta)\rangle$. Therefore, each $\tau \in \Gamma$ determines an automorphism (in the obvious sense) of the Dynkin diagram of Φ. If τ acts trivially on the diagram, then $\tau = 1$ (because Δ spans E). On the other hand, each automorphism of the Dynkin diagram obviously determines an automorphism of Φ (cf. Proposition 11.1). So Γ *may be*

Table 1.

Type	Number of Positive Roots	Order of $\mathscr{W}$	Structure of $\mathscr{W}$	Γ
A_ℓ	$\binom{\ell+1}{2}$	$(\ell+1)!$	$\mathscr{S}_{\ell+1}$	$\mathbf{Z}/2\mathbf{Z}$ $(\ell \geq 2)$
B_ℓ, C_ℓ	ℓ^2	$2^\ell \ell!$	$(\mathbf{Z}/2\mathbf{Z})^\ell \rtimes \mathscr{S}_\ell$	1
D_ℓ	$\ell^2-\ell$	$2^{\ell-1}\ \ell!$	$(\mathbf{Z}/2\mathbf{Z})^{\ell-1} \rtimes \mathscr{S}_\ell$	$\mathscr{S}_3$ $(\ell=4)$; $\mathbf{Z}/2\mathbf{Z}$ $(\ell>4)$
E_6	36	$2^7\ 3^4\ 5$		$\mathbf{Z}/2\mathbf{Z}$
E_7	63	$2^{10}\ 3^4\ 5\ 7$		1
E_8	120	$2^{14}\ 3^5\ 5^2\ 7$		1
F_4	24	$2^7\ 3^2$		1
G_2	6	$2^2\ 3$	$\mathscr{D}_6$	1

identified with the group of **diagram automorphisms**. A glance at the list in (11.4) yields a description of Γ, summarized in Table 1 along with other useful data, for Φ irreducible. (Since diagram automorphisms other than the identity exist only in cases of single root length, when the Dynkin diagram and Coxeter graph coincide, the term **graph automorphism** may also be used.)

Exercises

1. Verify the details of the constructions in (12.1).
2. Verify Table 2.
3. Let $\Phi \subset E$ satisfy (R1), (R3), (R4), but not (R2), cf. Exercise 9.9. Suppose moreover that Φ is irreducible, in the sense of §11. Prove that Φ is the union of root systems of type B_n, C_n in E (if dim $E = n > 1$), where the long roots of B_n are also the short roots of C_n. (This is called the *non-reduced root system* of type BC_n in the literature.)

Table 2. *Highest long and short roots*

Type	Long	Short
A_ℓ	$\alpha_1+\alpha_2+\ldots+\alpha_\ell$	
B_ℓ	$\alpha_1+2\alpha_2+2\alpha_3+\ldots+2\alpha_\ell$	$\alpha_1+\alpha_2+\ldots+\alpha_\ell$
C_ℓ	$2\alpha_1+2\alpha_2+\ldots+2\alpha_{\ell-1}+\alpha_\ell$	$\alpha_1+2\alpha_2+\ldots+2\alpha_{\ell-1}+\alpha_\ell$
D_ℓ	$\alpha_1+2\alpha_2+\ldots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell$	
E_6	$\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$	
E_7	$2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$	
E_8	$2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$	
F_4	$2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$	$\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$
G_2	$3\alpha_1+2\alpha_2$	$2\alpha_1+\alpha_2$

4. Prove that the long roots in G_2 form a root system in E of type A_2.
5. In constructing C_ℓ, would it be correct to characterize Φ as the set of all vectors in I of squared length 2 or 4? Explain.
6. Prove that the map $\alpha \mapsto -\alpha$ is an automorphism of Φ. Try to decide for which irreducible Φ this belongs to the Weyl group.
7. Describe Aut Φ when Φ is not irreducible.

Notes

The treatment here follows Serre [2]. More information about the individual root systems may be found in Bourbaki [2].

13. Abstract theory of weights

In this section we describe that part of the representation theory of semisimple Lie algebras which depends only on the root system. (None of this is needed until Chapter VI.) Let Φ be a root system in a euclidean space E, with Weyl group $\mathscr{W}$.

13.1. Weights

Let Λ be the set of all $\lambda \in E$ for which $\langle\lambda, \alpha\rangle \in \mathbf{Z}$ $(\alpha \in \Phi)$, and call its elements **weights**. Since $\langle\lambda, \alpha\rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$ depends linearly on λ, Λ is a subgroup of E including Φ. Thanks to Exercise 10.1, $\lambda \in \Lambda$ iff $\langle\lambda,\alpha\rangle \in \mathbf{Z}$ for all $\alpha \in \Delta$. Denote by Λ_r the **root lattice** (=subgroup of Λ generated by Φ). Λ_r is a lattice in E in the technical sense: it is the **Z**-span of an **R**-basis of E (namely, any set of simple roots). Fix a base $\Delta \subset \Phi$, and define $\lambda \in \Lambda$ to be **dominant** if all the integers $\langle\lambda,\alpha\rangle(\alpha \in \Delta)$ are nonnegative, **strongly dominant** if these integers are positive. Let Λ^+ be the set of all dominant weights. In the language of (10.1), Λ^+ is the set of all weights lying in the closure of the fundamental Weyl chamber $\mathfrak{C}(\Delta)$, while $\Lambda \cap \mathfrak{C}(\Delta)$ is the set of strongly dominant weights.

If $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$, then the vectors $2\alpha_i/(\alpha_i, \alpha_i)$ again form a basis of E. Let $\lambda_1, \ldots, \lambda_\ell$ be the dual basis (relative to the inner product on E): $\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$. Since all $\langle\lambda_i, \alpha\rangle$ $(\alpha \in \Delta)$ are nonnegative integers, the λ_i are dominant weights. We call them the **fundamental dominant weights (relative to Δ)**. Notice that $\sigma_i\lambda_j = \lambda_j - \delta_{ij}\alpha_i$ where $\sigma_i = \sigma_{\alpha_i}$. If $\lambda \in E$ is arbitrary, let $m_i = \langle\lambda, \alpha_i\rangle$. Then $0 = \langle\lambda - \Sigma m_i\lambda_i, \alpha\rangle$ for each simple root α, which implies that $(\lambda - \Sigma m_i\lambda_i, \alpha) = 0$ as well, or that $\lambda = \Sigma m_i\lambda_i$. Therefore, Λ *is a lattice with basis* $(\lambda_i, 1 \leq i \leq \ell)$, *and* $\lambda \in \Lambda^+$ *if and only if all* $m_i \geq 0$. (Cf. Figure 1, for type A_2.)

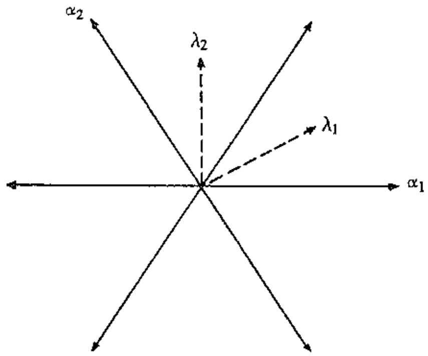


Figure 1

It is an elementary fact about lattices that Λ/Λ_r must be a finite group (called the **fundamental group** of Φ). We can see this directly as follows. Write $\alpha_i = \sum_j m_{ij}\lambda_j$ ($m_{ij} \in \mathbf{Z}$). Then $\langle \alpha_i, \alpha_k\rangle = \sum_j m_{ij} \langle \lambda_j, \alpha_k\rangle = m_{ik}$. In other words, the Cartan matrix expresses the change of basis. To write the λ_j in terms of the α_i, we have only to invert the Cartan matrix; its determinant (cf. Exercise 11.2) is the sole denominator involved, so this measures the index of Λ_r in Λ. For example, in type A_1, $\alpha_1 = 2\lambda_1$. (This is the only case in which a simple root is dominant, for reasons which will later become apparent.) In type A_2, the Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2\end{pmatrix}$, so $\alpha_1 = 2\lambda_1 - \lambda_2$ and $\alpha_2 = -\lambda_1 + 2\lambda_2$. Inverting, we get $(1/3)\begin{pmatrix} 2 & 1 \\ 1 & 2\end{pmatrix}$, so that $\lambda_1 = (1/3)$ $(2\alpha_1+\alpha_2)$ and $\lambda_2 = (1/3)$ $(\alpha_1+2\alpha_2)$. By computing determinants of Cartan matrices one verifies the following list of orders for the fundamental groups Λ/Λ_r in the irreducible cases:

$$A_\ell,\ \ell+1;\ B_\ell,\ C_\ell,\ E_7,\ 2;\ D_\ell,\ 4;\ E_6,\ 3;\ E_8,\ F_4,\ G_2,\ 1.$$

With somewhat more labor one can calculate explicitly the λ_i in terms of the α_j. This information is listed in Table 1, for the reader's convenience, although strictly speaking we shall not need it in what follows. The exact structure of the fundamental group can be found by computing elementary divisors, or can be deduced from Table 1 once the latter is known (Exercise 4).

13.2. Dominant weights

The Weyl group $\mathscr{W}$ of Φ preserves the inner product on E, hence leaves Λ invariant. (In fact, we already made the more precise observation that $\sigma_i\lambda_j = \lambda_j - \delta_{ij}\alpha_i$.) Orbits of weights under $\mathscr{W}$ occur frequently in the study of representations. In view of Lemma 10.3B and Exercise 10.14, we can state:

Lemma A. *Each weight is conjugate under $\mathscr{W}$ to one and only one dominant weight. If λ is dominant, then $\sigma\lambda \prec \lambda$ for all $\sigma \in \mathscr{W}$, and if λ is strongly dominant, then $\sigma\lambda = \lambda$ only when $\sigma = 1$.* ☐

As a subset of E, Λ is partially ordered by the relation: $\lambda \succ \mu$ if and only if $\lambda - \mu$ is a sum of positive roots (10.1). Unfortunately, this ordering does not have too close a connection with the property of being dominant; for example, it is easy to have μ dominant, $\mu \prec \lambda$, but λ not dominant (Exercise 2). Our next lemma shows, however, that dominant weights are not *too* badly behaved relative to $\prec$.

Table 1.

A_ℓ: $\lambda_i = \frac{1}{\ell+1}[(\ell-i+1)\alpha_1+2(\ell-i+1)\alpha_2+\ldots+(i-1)(\ell-i+1)\alpha_{i-1}$
$\quad +i(\ell-i+1)\alpha_i+i(\ell-i)\alpha_{i+1}+\ldots+i\alpha_\ell]$

B_ℓ: $\lambda_i = \alpha_1+2\alpha_2+\ldots+(i-1)\alpha_{i-1}+i(\alpha_i+\alpha_{i+1}+\ldots+\alpha_\ell)$ $\quad (i<\ell)$
$\lambda_\ell = \frac{1}{2}(\alpha_1+2\alpha_2+\ldots+\ell\alpha_\ell)$

C_ℓ: $\lambda_i = \alpha_1+2\alpha_2+\ldots+(i-1)\alpha_{i-1}+i(\alpha_i+\ldots+\alpha_{\ell-1}+\frac{1}{2}\alpha_\ell)$

D_ℓ: $\lambda_i = \alpha_1+2\alpha_2+\ldots+(i-1)\alpha_{i-1}+i(\alpha_i+\ldots+\alpha_{\ell-2})+\frac{1}{2}i(\alpha_{\ell-1}+\alpha_\ell)$ $\quad (i<\ell-1)$
$\lambda_{\ell-1} = \frac{1}{2}(\alpha_1+2\alpha_2+\ldots+(\ell-2)\alpha_{\ell-2}+\frac{1}{2}\ell\alpha_{\ell-1}+\frac{1}{2}(\ell-2)\alpha_\ell)$
$\lambda_\ell = \frac{1}{2}(\alpha_1+2\alpha_2+\ldots+(\ell-2)\alpha_{\ell-2}+\frac{1}{2}(\ell-2)\alpha_{\ell-1}+\frac{1}{2}\ell\alpha_\ell)$

($\Sigma q_i\alpha_i$ is abbreviated $(q_1,\ldots q_\ell)$ in the following lists.)

E_6: $\lambda_1 = \frac{1}{3}(4, 3, 5, 6, 4, 2)$
$\lambda_2 = (1, 2, 2, 3, 2, 1)$
$\lambda_3 = \frac{1}{3}(5, 6, 10, 12, 8, 4)$
$\lambda_4 = (2, 3, 4, 6, 4, 2)$
$\lambda_5 = \frac{1}{3}(4, 6, 8, 12, 10, 5)$
$\lambda_6 = \frac{1}{3}(2, 3, 4, 6, 5, 4)$

E_7: $\lambda_1 = (2, 2, 3, 4, 3, 2, 1)$
$\lambda_2 = \frac{1}{2}(4, 7, 8, 12, 9, 6, 3)$
$\lambda_3 = (3, 4, 6, 8, 6, 4, 2)$
$\lambda_4 = (4, 6, 8, 12, 9, 6, 3)$
$\lambda_5 = \frac{1}{2}(6, 9, 12, 18, 15, 10, 5)$
$\lambda_6 = (2, 3, 4, 6, 5, 4, 2)$
$\lambda_7 = \frac{1}{2}(2, 3, 4, 6, 5, 4, 3)$

E_8: $\lambda_1 = (4, 5, 7, 10, 8, 6, 4, 2)$
$\lambda_2 = (5, 8, 10, 15, 12, 9, 6, 3)$
$\lambda_3 = (7, 10, 14, 20, 16, 12, 8, 4)$
$\lambda_4 = (10, 15, 20, 30, 24, 18, 12, 6)$
$\lambda_5 = (8, 12, 16, 24, 20, 15, 10, 5)$
$\lambda_6 = (6, 9, 12, 18, 15, 12, 8, 4)$
$\lambda_7 = (4, 6, 8, 12, 10, 8, 6, 3)$
$\lambda_8 = (2, 3, 4, 6, 5, 4, 3, 2)$

F_4: $\lambda_1 = (2, 3, 4, 2)$
$\lambda_2 = (3, 6, 8, 4)$
$\lambda_3 = (2, 4, 6, 3)$
$\lambda_4 = (1, 2, 3, 2)$

G_2: $\lambda_1 = (2, 1)$
$\lambda_2 = (3, 2)$

Lemma B. *Let $\lambda \in \Lambda^+$. Then the number of dominant weights $\mu \prec \lambda$ is finite.*

Proof. Since $\lambda+\mu \in \Lambda^+$ and $\lambda-\mu$ is a sum of positive roots, $0 \leq (\lambda+\mu, \lambda-\mu) = (\lambda,\lambda) - (\mu,\mu)$. Thus μ lies in the *compact* set $\{x \in \mathsf{E} | (x,x) \leq (\lambda,\lambda)\}$, whose intersection with the *discrete* set Λ^+ is finite. ▯

13.3. The weight δ

Recall (Corollary to Lemma 10.2B) that $\delta = \frac{1}{2}\sum_{\alpha \succ 0} \alpha$, and that $\sigma_i \delta = \delta - \alpha_i$ ($1 \leq i \leq \ell$). Of course, δ may or may not lie in the root lattice Λ_r (cf. type A_1); but δ does lie in Λ. More precisely:

Lemma A. $\delta = \sum_{j=1}^{\ell} \lambda_j$, *so δ is a (strongly) dominant weight.*

Proof. Since $\sigma_i\delta = \delta - \alpha_i = \delta - \langle \delta, \alpha_i \rangle \alpha_i$, $\langle \delta, \alpha_i \rangle = 1$ $(1 \leq i \leq \ell)$. But $\delta = \sum_i \langle \delta, \alpha_i \rangle \lambda_i$ (cf. (13.1)), so the lemma follows. ▯

The next lemma is merely an auxiliary result, needed in (13.4).

Lemma B. *Let $\mu \in \Lambda^+$, $\nu = \sigma^{-1}\mu$ ($\sigma \in \mathscr{W}$). Then $(\nu+\delta, \nu+\delta) \leq (\mu+\delta, \mu+\delta)$, with equality only if $\nu = \mu$.*

Proof. $(\nu+\delta, \nu+\delta) = (\sigma(\nu+\delta), \sigma(\nu+\delta)) = (\mu+\sigma\delta, \mu+\sigma\delta) = (\mu+\delta, \mu+\delta) - 2(\mu, \delta-\sigma\delta)$. Since $\mu \in \Lambda^+$, and $\delta - \sigma\delta$ is a sum of positive roots (13.2A, 13.3A), the right side is $\leq (\mu+\delta, \mu+\delta)$, with equality only if $(\mu, \delta-\sigma\delta) = 0$, i.e., $(\mu,\delta) = (\mu,\sigma\delta) = (\nu,\delta)$, or $(\mu-\nu, \delta) = 0$. But $\mu-\nu$ is a sum of positive roots (13.2A) and δ is strongly dominant, so $\mu = \nu$. ▯

13.4. Saturated sets of weights

Certain finite sets of weights, stable under $\mathscr{W}$, play a prominent role in representation theory. We call a subset Π of Λ **saturated** if for all $\lambda \in \Pi$, $\alpha \in \Phi$, and i between 0 and $\langle \lambda, \alpha \rangle$, the weight $\lambda - i\alpha$ also lies in Π. Notice first that any saturated set is automatically stable under $\mathscr{W}$, since $\sigma_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha$ and $\mathscr{W}$ is generated by reflections. We say that a saturated set Π has **highest weight** λ ($\lambda \in \Lambda^+$) if $\lambda \in \Pi$ and $\mu \prec \lambda$ for all $\mu \in \Pi$. *Examples*: (1) The set consisting of 0 alone is saturated, with highest weight 0. (2) The set Φ of all roots of a semisimple Lie algebra, along with 0, is saturated. In case Φ is irreducible, there is a unique highest root (relative to a fixed base Δ of Φ) (Lemma 10.4A), so Π has this root as its highest weight (why?).

Lemma A. *A saturated set of weights having highest weight λ must be finite.*

Proof. Use Lemma 13.2B. ▯

Lemma B. *Let Π be saturated, with highest weight λ. If $\mu \in \Lambda^+$ and $\mu \prec \lambda$, then $\mu \in \Pi$.*

Proof. Suppose $\mu' = \mu + \sum_{\alpha \in \Delta} k_\alpha \alpha \in \Pi$ ($k_\alpha \in \mathbf{Z}^+$). (*Important*: We do not

assume that μ' is dominant.) We shall show how to reduce one of the k_α by one while still remaining in Π, thus eventually arriving at the conclusion that $\mu \in \Pi$. Of course, our starting point is the fact that λ itself is such a μ'. Now suppose $\mu' \neq \mu$, so some k_α is positive. From $(\sum_\alpha k_\alpha \alpha, \sum_\alpha k_\alpha \alpha) > 0$, we deduce that $(\sum_\alpha k_\alpha \alpha, \beta) > 0$ for some $\beta \in \Delta$, with $k_\beta > 0$. In particular, $\langle \sum_\alpha k_\alpha \alpha, \beta \rangle$ is positive. Since μ is dominant, $\langle \mu, \beta \rangle$ is nonnegative. Therefore, $\langle \mu', \beta \rangle$ is positive. By definition of saturated set, it is now possible to subtract β once from μ' without leaving Π, thus reducing k_β by one. ▯

From Lemma B emerges a very clear picture of a saturated set Π having highest weight λ: Π consists of all dominant weights lower than or equal to λ in the partial ordering, along with their conjugates under $\mathscr{W}$. In particular, for given $\lambda \in \Lambda^+$, at most one such set Π can exist. Conversely, given $\lambda \in \Lambda^+$, we may simply define Π to be the set consisting of all dominant weights below λ, along with their $\mathscr{W}$-conjugates. Since Π is stable under $\mathscr{W}$, it can be seen to be saturated (Exercise 10), and thanks to Lemma 13.2A, Π has λ as highest weight.

To conclude this section, we prove an inequality which is essential to the application of Freudenthal's formula (§22).

Lemma C. *Let Π be saturated, with highest weight λ. If $\mu \in \Pi$, then $(\mu+\delta, \mu+\delta) \leq (\lambda+\delta, \lambda+\delta)$, with equality only if $\mu = \lambda$.*

Proof. In view of Lemma 13.3B, it is enough to prove this when μ is dominant. Write $\mu = \lambda - \pi$, where π is a sum of positive roots. Then $(\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) = (\lambda+\delta, \lambda+\delta) - (\lambda+\delta-\pi, \lambda+\delta-\pi) = (\lambda+\delta, \pi) + (\pi, \mu+\delta) \geq (\lambda+\delta, \pi) \geq 0$, the inequalities holding because $\mu+\delta$ and $\lambda+\delta$ are dominant. Equality holds only if $\pi = 0$, since $\lambda+\delta$ is strongly dominant. ▯

Exercises

1. Let $\Phi = \Phi_1 \cup \ldots \cup \Phi_t$ be the decomposition of Φ into its irreducible components, with $\Delta = \Delta_1 \cup \ldots \cup \Delta_t$. Prove that Λ decomposes into a direct sum $\Lambda_1 \oplus \ldots \oplus \Lambda_t$; what about Λ^+?
2. Show by example (e.g., for A_2) that $\lambda \notin \Lambda^+$, $\alpha \in \Delta$, $\lambda - \alpha \in \Lambda^+$ is possible.
3. Verify some of the data in Table 1, e.g., for F_4.
4. Using Table 1, show that the fundamental group of A_ℓ is cyclic of order $\ell+1$, while that of D_ℓ is isomorphic to $\mathbf{Z}/4\mathbf{Z}$ (ℓ odd), or $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ (ℓ even). (It is easy to remember which is which, since $\mathsf{A}_3 = \mathsf{D}_3$.)
5. If Λ' is any subgroup of Λ which includes Λ_r, prove that Λ' is $\mathscr{W}$-invariant. Therefore, we obtain a homomorphism ϕ: Aut $\Phi/\mathscr{W} \to$ Aut (Λ/Λ_r). Prove that ϕ is injective, then deduce that $-1 \in \mathscr{W}$ if and only if $\Lambda_r \supset 2\Lambda$ (cf. Exercise 12.6). Show that $-1 \in \mathscr{W}$ for precisely the irreducible root systems A_1, B_ℓ, C_ℓ, D_ℓ (ℓ even), E_7, E_8, F_4, G_2.

6. Prove that the roots in Φ which are dominant weights are precisely the highest long root and (if two root lengths occur) the highest short root (cf. (10.4) and Exercise 10.11), when Φ is irreducible.
7. If $\varepsilon_1, \ldots, \varepsilon_\ell$ is an *obtuse* basis of the euclidean space E (i.e., all $(\varepsilon_i, \varepsilon_j) \leq 0$ for $i \neq j$), prove that the dual basis is *acute* (i.e., all $(\varepsilon_i^*, \varepsilon_j^*) \geq 0$ for $i \neq j$). [Reduce to the case $\ell = 2$.]
8. Let Φ be irreducible. Without using the data in Table 1, prove that each λ_i is of the form $\sum_j q_{ij}\alpha_j$, where all q_{ij} are *positive* rational numbers. [Deduce from Exercise 7 that all q_{ij} are nonnegative. From $(\lambda_i, \lambda_i) > 0$ obtain $q_{ii} > 0$. Then show that if $q_{ij} > 0$ and $(\alpha_j, \alpha_k) < 0$, then $q_{ik} > 0$.]
9. Let $\lambda \in \Lambda^+$. Prove that $\sigma(\lambda+\delta)-\delta$ is dominant only for $\sigma = 1$.
10. If $\lambda \in \Lambda^+$, prove that the set Π consisting of all dominant weights $\mu \prec \lambda$ and their $\mathscr{W}$-conjugates is saturated, as asserted in (13.4).
11. Prove that each subset of Λ is contained in a unique smallest saturated set, which is finite if the subset in question is finite.
12. For the root system of type A_2, write down the effect of each element of the Weyl group on each of λ_1, λ_2. Using this data, determine which weights belong to the saturated set having highest weight $\lambda_1+3\lambda_2$. Do the same for type G_2 and highest weight $\lambda_1+2\lambda_2$.
13. Call $\lambda \in \Lambda^+$ **minimal** if $\mu \in \Lambda^+$, $\mu \prec \lambda$ implies that $\mu = \lambda$. Show that each coset of Λ_r in Λ contains precisely one minimal λ. Prove that λ is minimal if and only if the $\mathscr{W}$-orbit of λ is saturated (with highest weight λ), if and only if $\lambda \in \Lambda^+$ and $\langle \lambda, \alpha \rangle = 0, 1, -1$ for *all* roots α. Determine (using Table 1) the nonzero minimal λ for each irreducible Φ, as follows:

A_ℓ: $\lambda_1, \ldots, \lambda_\ell$
B_ℓ: λ_ℓ
C_ℓ: λ_1
D_ℓ: $\lambda_1, \lambda_{\ell-1}, \lambda_\ell$
E_6: λ_1, λ_6
E_7: λ_7

Notes

Part of the material in this section is drawn from the text and exercises of Bourbaki [2], Chapter VI, §1, No. 9–10 (and Exercise 23). But we have gone somewhat beyond what is usually done outside representation theory in order to emphasize the role played by the root system.

Chapter IV

Isomorphism and Conjugacy Theorems

14. Isomorphism theorem

We return now to the situation of Chapter II: L is a semisimple Lie algebra over the algebraically closed field F of characteristic 0, H is a maximal toral subalgebra of L, $\Phi \subset H^*$ the set of roots of L relative to H. In (8.5) it was shown that the rational span of Φ in H^* is of dimension ℓ over $\mathbf{Q}$, where $\ell = \dim_F H^*$. By extending the base field from $\mathbf{Q}$ to $\mathbf{R}$ we therefore obtain an ℓ-dimensional real vector space E spanned by Φ. Moreover, the symmetric bilinear form dual to the Killing form is carried along to E, making E a euclidean space. Then Theorem 8.5 affirms that Φ is a root system in E.

Our aim in this section is to prove that two semisimple Lie algebras having the same root system are isomorphic. Actually, we can prove a more precise statement, which leads to the construction of certain automorphisms as well.

14.1. Reduction to the simple case

Proposition. *Let L be a simple Lie algebra, H and Φ as above. Then Φ is an irreducible root system in the sense of* (10.4).

Proof. Suppose not. Then Φ decomposes as $\Phi_1 \cup \Phi_2$, where the Φ_i are orthogonal. If $\alpha \in \Phi_1$, $\beta \in \Phi_2$, then $(\alpha+\beta, \alpha) \neq 0$, $(\alpha+\beta, \beta) \neq 0$, so $\alpha+\beta$ cannot be a root, and $[L_\alpha L_\beta] = 0$. This shows that the subalgebra K of L generated by all L_α $(\alpha \in \Phi_1)$ is centralized by all L_β $(\beta \in \Phi_2)$; in particular, K is a proper subalgebra of L, because $Z(L) = 0$. Furthermore, K is normalized by all L_α $(\alpha \in \Phi_1)$, hence by all L_α $(\alpha \in \Phi)$, hence by L (Proposition 8.4 (f)). Therefore K is a proper ideal of L, different from 0, contrary to the simplicity of L. ▯

Next let L be an arbitrary semisimple Lie algebra. Then L can be written uniquely as a direct sum $L_1 \oplus \ldots \oplus L_t$ of simple ideals (Theorem 5.2). If H is a maximal toral subalgebra of L, then $H = H_1 \oplus \ldots \oplus H_t$, where $H_i = L_i \cap H$ (cf. Exercise 5.8). Evidently each H_i is a toral algebra in L_i, in fact *maximal toral*: Any toral subalgebra of L_i larger than H_i would automatically be toral in L, centralize all H_j, $j \neq i$, and generate with them a toral subalgebra of L larger than H. Let Φ_i denote the root system of L_i relative to H_i, in the real vector space E_i. If $\alpha \in \Phi_i$, we can just as well view α as a linear function on H, by decreeing that $\alpha(H_j) = 0$ for $j \neq i$. Then α is clearly a root of L relative to H, with $L_\alpha \subset L_i$. Conversely, if $\alpha \in \Phi$, then