Extra Credit I Math 116

Due Nov 20, 2012

Remember: No credit will be given without mathematical or logical justification.
This extra credit is worth one homework assignment.

Part 1: Cauchy and Cauchy-Schwarz

In the following problems, we will prove the Cauchy-Schwarz inequality from the Cauchy inequality.

1) The original Cauchy inequality is as follows: If $a, b \in \mathbb{R}$, then

$$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2. \tag{1}$$

   a) Prove the Cauchy inequality. (Hint: Start with the fact that $(a-b)^2 \ge 0$).

   b) Prove that equality holds in (1) if and only if $a = b$.

2) Prove the *weighted Cauchy inequality.* That is, prove that if $\epsilon > 0$ then

$$ab \le \frac{1}{2}\epsilon a^2 + \frac{1}{2}\epsilon^{-1}b^2 \tag{2}$$

3) Let $\vec{A} = (a_1, \ldots, a_n)$ and $\vec{B} = (b_1, \ldots, b_n)$ be vectors in $\mathbb{R}^n$. As usual the *dot product* is given by $\vec{A} \cdot \vec{B} = \sum_{i=1}^n a_i b_i$. Using (1), prove the Cauchy inequality for vectors:

$$\vec{A} \cdot \vec{B} \le \frac{1}{2}\|\vec{A}\|^2 + \frac{1}{2}\|\vec{B}\|^2. \tag{3}$$

   What is the condition for equality?

4) Prove the weighted Cauchy inequality for vectors: if $\epsilon > 0$ then

$$\vec{A} \cdot \vec{B} \le \frac{1}{2}\epsilon\|\vec{A}\|^2 + \frac{1}{2}\epsilon^{-1}\|\vec{B}\|^2. \tag{4}$$

5) In this problem we will use the Cauchy inequality to prove the Cauchy-Schwarz inequality. Let $\vec{v}, \vec{w} \in \mathbb{R}^n$ be arbitrary non-zero vectors.

   a) Set $\vec{A} = \|\vec{v}\|^{-1}\vec{v}$ and $\vec{B} = \|\vec{w}\|^{-1}\vec{w}$. Prove that $\vec{A}$ and $\vec{B}$ are unit vectors.

   b) Using only (3), prove that $\vec{A} \cdot \vec{B} \le 1$.

   c) Using part (*b*), prove the Cauchy-Schwarz inequality: $\vec{v} \cdot \vec{w} \le \|\vec{v}\|\|\vec{w}\|$ for any $\vec{v}, \vec{w} \in \mathbb{R}^n$. What is the condition for equality?

Part 2: Convexity and Young's Inequality

In the following problems, we will use the convexity of the exponential function to prove Young's inequality.

Definition. A real-valued function f defined on $[A, B]$ (or (A, B) or $(A, B])$ or $[A, B)$) is called *convex* if, whenever $A \leq a < b \leq B$, the line segment between the points $(a, f(a))$ and $(b, f(b))$ lies above or on the graph of f on $[a, b]$.

6) If $A \leq a < b \leq B$ show that the equation of the segment of the secant line between $(a, f(a))$ and $(b, f(b))$ is

$$\begin{aligned} x(t) &= (1-t)\,a + tb \\ y(t) &= (1-t)\,f(a) + t f(b) \end{aligned} \tag{5}$$

where t varies between 0 and 1. Show that a function $f(x)$ defined on an interval $[A, B]$ is convex if and only if

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b) \tag{6}$$

whenever $A \leq a < b \leq B$, and $t \in [0, 1]$.

7) Below are listed several functions along their domains of definition. Which are convex? (No formal proofs—you can justify with a graph or some intuitive reasoning.)

   a) $f(x) = x$ on $[-1, 1]$
   b) $f(x) = sgn(x)$ on $[-1, 1]$
   c) $f(x) = |x|$ on $[-1, 1]$
   d) $f(x) = \sin(x)$ on $[0, \pi]$
   e) $f(x) = \sin(x)$ on $[\pi, 2\pi]$
   f) $f(x) = x^2$ on $(-\infty, \infty)$
   g) $f(x) = x^3 - x$ on $(-\infty, \infty)$
   h) $f(x) = x^3 - x$ on $[-1, \infty)$
   i) $f(x) = x^3 - x$ on $[0, \infty)$

8) Assume $f(x)$ has a second derivative everywhere on the interval (A, B). Prove that if $f''(x) \geq 0$ on (A, B) then f is convex on (A, B) (you do not have to prove the converse, which, incidentally, is also true: if $f''(x)$ exists and f is convex, then $f''(x) \geq 0$).

9) Prove that the function $f(x) = e^x$ is convex.

10) Prove that, whenever $t \in [0, 1]$, we have

$$e^{(1-t)\,a + t\,b} \leq (1-t)\,e^a + t\,e^b \tag{7}$$

11) Here we finally prove *Young's inequality*, a generalization of Cauchy's inequality. Assume p, q are any positive real numbers that satisfy

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{8}$$

If a and b are also positive (but otherwise have no special relationship) then $ln(a)$ and $ln(b)$ are well-defined real numbers. Prove the following:

a) $e^{\ln a + \ln b} \; = \; ab$

b) Using the convexity of the function $f(x) = e^x$, prove that

$$e^{\frac{1}{p}\cdot p\cdot \ln a \,+\, \frac{1}{q}\cdot q\cdot \ln b} \;\leq\; \frac{1}{p}e^{p\ln a} \;+\; \frac{1}{q}e^{q\ln b} \tag{9}$$

c) Prove Young's inequality, namely that whenever $a, b, p, q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \;\leq\; \frac{1}{p}\,a^p \;+\; \frac{1}{q}\,b^q. \tag{10}$$

d) Prove that Cauchy's inequality is a special case of Young's inequality.