29. Let $g(x) = xe^{x^2}$ and let $f(x) = \int_1^x g(t)(t + 1/t)\,dt$. Compute the limit of $f''(x)/g''(x)$ as $x \to +\infty$.
30. Let $g(x) = x^c e^{2x}$ and let $f(x) = \int_0^x e^{2t}(3t^2 + 1)^{1/2}\,dt$. For a certain value of c, the limit of $f'(x)/g'(x)$ as $x \to +\infty$ is finite and nonzero. Determine c and compute the value of the limit.
31. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, and let $f(0) = 0$.
(a) Prove that for every $m > 0$, $f(x)/x^m \to 0$ as $x \to 0$.
(b) Prove that for $x \neq 0$ the nth derivative of f has the form $f^{(n)}(x) = f(x)P(1/x)$, where $P(t)$ is a polynomial in t.
(c) Prove that $f^{(n)}(0) = 0$ for all $n \geq 1$. This shows that every Taylor polynomial generated by f at 0 is the zero polynomial.
32. An amount of P dollars is deposited in a bank which pays interest at a rate r per year, compounded m times a year. (For example, $r = 0.06$ when the annual rate is 6%.) (a) Prove that the total amount of principal plus interest at the end of n years is $P(1 + r/m)^{mn}$. If r and n are kept fixed, this amount approaches the limit Pe^{rn} as $m \to +\infty$. This motivates the following definition: We say that money grows at an annual rate r when compounded continuously if the amount $f(t)$ after t years is $f(0)e^{rt}$, where t is any nonnegative real number. Approximately how long does it take for a bank account to double in value if it receives interest at an annual rate of 6% compounded (b) continuously? (c) four times a year?

8

INTRODUCTION TO DIFFERENTIAL EQUATIONS

8.1 Introduction

A large variety of scientific problems arise in which one tries to determine something from its rate of change. For example, we could try to compute the position of a moving particle from a knowledge of its velocity or acceleration. Or a radioactive substance may be disintegrating at a known rate and we may be required to determine the amount of material present after a given time. In examples like these, we are trying to determine an *unknown function* from prescribed information expressed in the form of an equation involving at least one of the derivatives of the unknown function. These equations are called *differential equations*, and their study forms one of the most challenging branches of mathematics.

Differential equations are classified under two main headings: *ordinary* and *partial*, depending on whether the unknown is a function of just *one* variable or of *two or more* variables. A simple example of an ordinary differential equation is the relation

$$f'(x) = f(x) \tag{8.1}$$

which is satisfied, in particular, by the exponential function, $f(x) = e^x$. We shall see presently that every solution of (8.1) must be of the form $f(x) = Ce^x$, where C may be any constant.

On the other hand, an equation like

$$\frac{\partial^2 f(x, y)}{\partial x^2} + \frac{\partial^2 f(x, y)}{\partial y^2} = 0$$

is an example of a partial differential equation. This particular one, called *Laplace's equation*, appears in the theory of electricity and magnetism, fluid mechanics, and elsewhere. It has many different kinds of solutions, among which are $f(x, y) = x + 2y$, $f(x, y) = e^x \cos y$, and $f(x, y) = \log(x^2 + y^2)$.

The study of differential equations is one part of mathematics that, perhaps more than any other, has been directly inspired by mechanics, astronomy, and mathematical physics. Its history began in the 17th century when Newton, Leibniz, and the Bernoullis solved some simple differential equations arising from problems in geometry and mechanics.

These early discoveries, beginning about 1690, gradually led to the development of a now-classic "bag of tricks" for solving certain special kinds of differential equations. Although these special tricks are applicable in relatively few cases, they do enable us to solve many differential equations that arise in mechanics and geometry, so their study is of practical importance. Some of these special methods and some of the problems which they help us solve are discussed near the end of this chapter.

Experience has shown that it is difficult to obtain mathematical theories of much generality about solutions of differential equations, except for a few types. Among these are the so-called *linear* differential equations which occur in a great variety of scientific problems. The simplest types of linear differential equations and some of their applications are also discussed in this introductory chapter. A more thorough study of linear equations is carried out in Volume II.

8.2 Terminology and notation

When we work with a differential equation such as (8.1), it is customary to write y in place of $f(x)$ and y' in place of $f'(x)$, the higher derivatives being denoted by y'', y''', etc. Of course, other letters such as u, v, z, etc. are also used instead of y. By the *order* of an equation is meant the order of the highest derivative which appears. For example, (8.1) is a first-order equation which may be written as $y' = y$. The differential equation $y' = x^3y + \sin(xy'')$ is one of second order.

In this chapter we shall begin our study with first-order equations which can be solved for y' and written as follows:

$$y' = f(x, y), \tag{8.2}$$

where the expression $f(x, y)$ on the right has various special forms. A differentiable function $y = Y(x)$ will be called a *solution* of (8.2) on an interval I if the function Y and its derivative Y' satisfy the relation

$$Y'(x) = f[x, Y(x)]$$

for every x in I. The simplest case occurs when $f(x, y)$ is independent of y. In this case, (8.2) becomes

$$y' = Q(x), \tag{8.3}$$

say, where Q is assumed to be a given function defined on some interval I. To solve the differential equation (8.3) means to find a primitive of Q. The second fundamental theorem of calculus tells us how to do it when Q is continuous on an open interval I. We simply integrate Q and add any constant. Thus, every solution of (8.3) is included in the formula

$$y = \int Q(x)\,dx + C, \tag{8.4}$$

where C is any constant (usually called an arbitrary constant of integration). The differential equation (8.3) has infinitely many solutions, one for each value of C.

If it is not possible to evaluate the integral in (8.4) in terms of familiar functions, such as polynomials, rational functions, trigonometric and inverse trigonometric functions, logarithms, and exponentials, still we consider the differential equation as having been solved if the solution can be expressed in terms of integrals of known functions. In actual practice, there are various methods for obtaining approximate evaluations of integrals which lead to useful information about the solution. Automatic high-speed computing machines are often designed with this kind of problem in mind.

EXAMPLE. *Linear motion determined from the velocity.* Suppose a particle moves along a straight line in such a way that its velocity at time t is $2 \sin t$. Determine its position at time t.

Solution. If $Y(t)$ denotes the position at time t measured from some starting point, then the derivative $Y'(t)$ represents the velocity at time t. We are given that

$$Y'(t) = 2 \sin t.$$

Integrating, we find that

$$Y(t) = 2\int \sin t\,dt + C = -2\cos t + C.$$

This is all we can deduce about $Y(t)$ from a knowledge of the velocity alone; some other piece of information is needed to fix the position function. We can determine C if we know the value of Y at some particular instant. For example, if $Y(0) = 0$, then $C = 2$ and the position function is $Y(t) = 2 - 2\cos t$. But if $Y(0) = 2$, then $C = 4$ and the position function is $Y(t) = 4 - 2\cos t$.

In some respects the example just solved is typical of what happens in general. Somewhere in the process of solving a first-order differential equation, an integration is required to remove the derivative y' and in this step an arbitrary constant C appears. The way in which the arbitrary constant C enters into the solution will depend on the nature of the given differential equation. It may appear as an additive constant, as in Equation (8.4), but it is more likely to appear in some other way. For example, when we solve the equation $y' = y$ in Section 8.3, we shall find that every solution has the form $y = Ce^x$.

In many problems it is necessary to select from the collection of all solutions one having a prescribed value at some point. The prescribed value is called an *initial condition*, and the problem of determining such a solution is called an *initial-value problem*. This terminology originated in mechanics where, as in the above example, the prescribed value represents the displacement at some initial time.

We shall begin our study of differential equations with an important special case.

8.3 A first-order differential equation for the exponential function

The exponential function is equal to its own derivative, and the same is true of any constant multiple of the exponential. It is easy to show that these are the only functions that satisfy this property on the whole real axis.

THEOREM 8.1. *If C is a given real number, there is one and only one function f which satisfies the differential equation*

$$f'(x) = f(x)$$

for all real x and which also satisfies the initial condition $f(0) = C$. This function is given by the formula

$$f(x) = Ce^x .$$

Proof. It is easy to verify that the function $f(x) = Ce^x$ satisfies both the given differential equation and the given initial condition. Now we must show that this is the *only* solution. Let $y = g(x)$ be any solution of this initial-value problem:

$$g'(x) = g(x) \qquad \text{for all } x, \qquad g(0) = C .$$

We wish to show that $g(x) = Ce^x$ or that $g(x)e^{-x} = C$. We consider the function $h(x) = g(x)e^{-x}$ and show that its derivative is always zero. The derivative of h is given by

$$h'(x) = g'(x)e^{-x} - g(x)e^{-x} = e^{-x}[g'(x) - g(x)] = 0 .$$

Hence, by the zero-derivative theorem, h is constant. But $g(0) = C$ so $h(0) = g(0)e^0 = C$. Hence, we have $h(x) = C$ for all x which means that $g(x) = Ce^x$, as required.

Theorem 8.1 is an example of an existence-uniqueness theorem. It tells us that the given initial-value problem *has* a solution (existence) and that it has *only one* solution (uniqueness). The object of much of the research in the theory of differential equations is to discover existence and uniqueness theorems for wide classes of equations.

We discuss next an important type which includes both the differential equation $y' = Q(x)$ and the equation $y' = y$ as special cases.

8.4 First-order linear differential equations

A differential equation of the form

$$y' + P(x)y = Q(x), \tag{8.5}$$

where P and Q are given functions, is called a *first-order linear* differential equation. The terms involving the unknown function y and its derivative y' appear as a linear combination of y and y'. The functions P and Q are assumed to be continuous on some open interval I. We seek all solutions y defined on I.

First we consider the special case in which the right member, $Q(x)$, is identically zero. The equation

$$y' + P(x)y = 0 \tag{8.6}$$

is called the *homogeneous* or *reduced* equation corresponding to (8.5). We will show how to solve the homogeneous equation and then use the result to help us solve the nonhomogeneous equation (8.5).

If y is nonzero on I, Equation (8.6) is equivalent to the equation

$$\frac{y'}{y} = -P(x) . \tag{8.7}$$

That is, every nonzero y which satisfies (8.6) also satisfies (8.7) and vice versa. Now suppose y is a positive function satisfying (8.7). Since the quotient y'/y is the derivative of $\log y$, Equation (8.7) becomes $D \log y = -P(x)$, from which we find $\log y = -\int P(x)\,dx + C$, so we have

$$y = e^{-A(x)}, \qquad \text{where} \quad A(x) = \int P(x)\,dx - C . \tag{8.8}$$

In other words, if there is a positive solution of (8.6), it must necessarily have the form (8.8) for some C. But now it is easy to verify that every function in (8.8) is a solution of the homogeneous equation (8.6). In fact, we have

$$y' = -e^{-A(x)}A'(x) = -P(x)e^{-A(x)} = -P(x)y .$$

Thus, we have found all positive solutions of (8.6). But now it is easy to describe all solutions. We state the result as an existence-uniqueness theorem.

THEOREM 8.2. *Assume P is continuous on an open interval I. Choose any point a in I and let b be any real number. Then there is one and only one function $y = f(x)$ which satisfies the initial-value problem*

$$y' + P(x)y = 0, \qquad \textit{with} \qquad f(a) = b , \tag{8.9}$$

on the interval I. This function is given by the formula

$$f(x) = be^{-A(x)}, \qquad \textit{where} \quad A(x) = \int_a^x P(t)\,dt . \tag{8.10}$$

Proof. Let f be defined by (8.10). Then $A(a) = 0$ so $f(a) = be^0 = b$. Differentiation shows that f satisfies the differential equation in (8.9), so f is a solution of the initial-value problem. Now we must show that it is the only solution.

Let g be an arbitrary solution. We wish to show that $g(x) = be^{-A(x)}$ or that $g(x)e^{A(x)} = b$. Therefore it is natural to introduce $h(x) = g(x)e^{A(x)}$. The derivative of h is given by

$$h'(x) = g'(x)e^{A(x)} + g(x)e^{A(x)}A'(x) = e^{A(x)}[g'(x) + P(x)g(x)] . \tag{8.11}$$

Now since g satisfies the differential equation in (8.9), we have $g'(x) + P(x)g(x) = 0$ everywhere on I, so $h'(x) = 0$ for all x in I. This means that h is constant on I. Hence, we have $h(x) = h(a) = g(a)e^{A(a)} = g(a) = b$. In other words, $g(x)e^{A(x)} = b$, so $g(x) = be^{-A(x)}$, which shows that $g = f$. This completes the proof.

The last part of the foregoing proof suggests a method for solving the nonhomogeneous differential equation in (8.5). Suppose that g is any function satisfying (8.5), and let $h(x) = g(x)e^{A(x)}$ where, as above, $A(x) = \int_a^x P(t)\,dt$. Then Equation (8.11) is again valid, but since g satisfies (8.5), the formula for $h'(x)$ gives us

$$h'(x) = e^{A(x)}Q(x) .$$

Now we may invoke the second fundamental theorem to write

$$h(x) = h(a) + \int_a^x e^{A(t)}Q(t)\,dt\,.$$

Hence, since $h(a) = g(a)$, every solution g of (8.5) has the form

$$g(x) = e^{-A(x)}h(x) = g(a)e^{-A(x)} + e^{-A(x)}\int_a^x Q(t)e^{A(t)}\,dt\,. \tag{8.12}$$

Conversely, by direct differentiation of (8.12), it is easy to verify that each such g is a solution of (8.5), so we have found *all* solutions. We state the result as follows.

THEOREM 8.3. *Assume P and Q are continuous on an open interval I. Choose any point a in I and let b be any real number. Then there is one and only one function $y = f(x)$ which satisfies the initial-value problem*

$$y' + P(x)y = Q(x), \qquad \text{with} \quad f(a) = b\,,$$

on the interval I. This function is given by the formula

$$f(x) = be^{-A(x)} + e^{-A(x)}\int_a^x Q(t)e^{A(t)}\,dt\,,$$

where $A(x) = \int_a^x P(t)\,dt$.

Up to now the word "interval" has meant a bounded interval of the form (a, b), $[a, b]$, $[a, b)$, or $(a, b]$, with $a < b$. It is convenient to consider also unbounded intervals. They are denoted by the symbols $(a, +\infty)$, $(-\infty, a)$, $[a, +\infty)$ and $(-\infty, a]$, and they are defined as follows:

$$(a, +\infty) = \{x \mid x > a\}\,, \qquad (-\infty, a) = \{x \mid x < a\}\,,$$

$$[a, +\infty) = \{x \mid x \geq a\}\,, \qquad (-\infty, a] = \{x \mid x \leq a\}\,.$$

In addition, it is convenient to refer to the collection of *all* real numbers as the interval $(-\infty, +\infty)$. Thus, when we discuss a differential equation or its solution over an interval I, it will be understood that I is one of the nine types just described.

EXAMPLE. Find all solutions of the first-order differential equation $xy' + (1 - x)y = e^{2x}$ on the interval $(0, +\infty)$.

Solution. First we transform the equation to the form $y' + P(x)y = Q(x)$ by dividing through by x. This gives us

$$y' + \left(\frac{1}{x} - 1\right)y = \frac{e^{2x}}{x}\,,$$

so $P(x) = 1/x - 1$ and $Q(x) = e^{2x}/x$. Since P and Q are continuous on the in $(0, +\infty)$, there is a unique solution $y = f(x)$ satisfying any given initial condition form $f(a) = b$. We shall express all solutions in terms of the initial value at the point In other words, given any real number b, we will determine all solutions for which $f(1$

First we compute

$$A(x) = \int_1^x P(t)\,dt = \int_1^x \left(\frac{1}{t} - 1\right)dt = \log x - (x - 1)\,.$$

Hence we have $e^{-A(x)} = e^{x-1-\log x} = e^{x-1}/x$, and $e^{A(t)} = te^{1-t}$, so Theorem 8.3 te that the solution is given by the formula

$$f(x) = b\,\frac{e^{x-1}}{x} + \frac{e^{x-1}}{x}\int_1^x \frac{e^{2t}}{t}\,te^{1-t}\,dt = b\,\frac{e^{x-1}}{x} + \frac{e^x}{x}\int_1^x e^t\,dt$$

$$= b\,\frac{e^{x-1}}{x} + \frac{e^x}{x}(e^x - e) = b\,\frac{e^{x-1}}{x} + \frac{e^{2x}}{x} - \frac{e^{x+1}}{x}\,.$$

We can also write this in the form

$$f(x) = \frac{e^{2x} + Ce^x}{x}\,,$$

where $C = be^{-1} - e$. This gives all solutions on the interval $(0, +\infty)$.

It may be of interest to study the behavior of the solutions as $x \to 0$. If we appro the exponential by its linear Taylor polynomial, we find that $e^{2x} = 1 + 2x + o($ $e^x = 1 + x + o(x)$ as $x \to 0$, so we have

$$f(x) = \frac{(1 + C) + (2 + C)x + o(x)}{x} = \frac{1 + C}{x} + (2 + C) + o(1)\,.$$

Therefore, only the solution with $C = -1$ tends to a finite limit as $x \to 0$, this limit b

8.5 Exercises

In each of Exercises 1 through 5, solve the initial-value problem on the specified interval.

1. $y' - 3y = e^{2x}$ on $(-\infty, +\infty)$, with $y = 0$ when $x = 0$.
2. $xy' - 2y = x^5$ on $(0, +\infty)$, with $y = 1$ when $x = 1$.
3. $y' + y\tan x = \sin 2x$ on $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, with $y = 2$ when $x = 0$.
4. $y' + xy = x^3$ on $(-\infty, +\infty)$, with $y = 0$ when $x = 0$.
5. $\dfrac{dx}{dt} + x = e^{2t}$ on $(-\infty, +\infty)$, with $x = 1$ when $t = 0$.
6. Find all solutions of $y'\sin x + y\cos x = 1$ on the interval $(0, \pi)$. Prove that exactly these solutions has a finite limit as $x \to 0$, and another has a finite limit as $x \to \pi$.
7. Find all solutions of $x(x + 1)y' + y = x(x + 1)^2e^{-x^2}$ on the interval $(-1, 0)$. Prove t solutions approach 0 as $x \to -1$, but that only one of them has a finite limit as $x \to 0$
8. Find all solutions of $y' + y\cot x = 2\cos x$ on the interval $(0, \pi)$. Prove that exactly these is also a solution on $(-\infty, +\infty)$.

9. Find all solutions of $(x - 2)(x - 3)y' + 2y = (x - 1)(x - 2)$ on each of the following intervals: (a) $(-\infty, 2)$; (b) $(2, 3)$; (c) $(3, +\infty)$. Prove that all solutions tend to a finite limit as $x \to 2$, but that none has a finite limit as $x \to 3$.
10. Let $s(x) = (\sin x)/x$ if $x \neq 0$, and let $s(0) = 1$. Define $T(x) = \int_0^x s(t)\,dt$. Prove that the function $f(x) = xT(x)$ satisfies the differential equation $xy' - y = x \sin x$ on the interval $(-\infty, +\infty)$ and find all solutions on this interval. Prove that the differential equation has no solution satisfying the initial condition $f(0) = 1$, and explain why this does not contradict Theorem 8.3.
11. Prove that there is exactly one function f, continuous on the positive real axis, such that

$$f(x) = 1 + \frac{1}{x}\int_1^x f(t)\,dt$$

for all $x > 0$ and find this function.
12. The function f defined by the equation

$$f(x) = xe^{(1-x^2)/2} - xe^{-x^2/2}\int_1^x t^{-2}e^{t^2/2}\,dt$$

for $x > 0$ has the properties that (i) it is continuous on the positive real axis, and (ii) it satisfies the equation

$$f(x) = 1 - x\int_1^x f(t)\,dt$$

for all $x > 0$. Find all functions with these two properties.

The Bernoulli equation. A differential equation of the form $y' + P(x)y = Q(x)y^n$, where n is not 0 or 1, is called a Bernoulli equation. This equation is nonlinear because of the presence of y^n. The next exercise shows that it can always be transformed into a linear first-order equation for a new unknown function v, where $v = y^k$, $k = 1 - n$.

13. Let k be a nonzero constant. Assume P and Q are continuous on an interval I. If $a \in I$ and if b is any real number, let $v = g(x)$ be the unique solution of the initital-value problem $v' + kP(x)v = kQ(x)$ on I, with $g(a) = b$. If $n \neq 1$ and $k = 1 - n$, prove that a function $y = f(x)$, which is never zero on I, is a solution of the initial-value problem

$$y' + P(x)y = Q(x)y^n \quad \text{on } I, \quad \text{with } f(a)^k = b$$

if and only if the kth power of f is equal to g on I.

In each of Exercises 14 through 17, solve the initial-value problem on the specified interval.
14. $y' - 4y = 2e^x y^{1/2}$ on $(-\infty, +\infty)$, with $y = 2$ when $x = 0$.
15. $y' - y = -y^2(x^2 + x + 1)$ on $(-\infty, +\infty)$, with $y = 1$ when $x = 0$.
16. $xy' - 2y = 4x^3y^{1/2}$ on $(-\infty, +\infty)$, with $y = 0$ when $x = 1$.
17. $xy' + y = y^2x^2 \log x$ on $(0, +\infty)$, with $y = \frac{1}{2}$ when $x = 1$.
18. $2xyy' + (1 + x)y^2 = e^x$ on $(0, +\infty)$, with (a) $y = \sqrt{e}$ when $x = 1$; (b) $y = -\sqrt{e}$ when $x = 1$; (c) a finite limit as $x \to 0$.
19. An equation of the form $y' + P(x)y + Q(x)y^2 = R(x)$ is called a *Riccati equation.* (There is no known method for solving the general Riccati equation.) Prove that if u is a known solution of this equation, then there are further solutions of the form $y = u + 1/v$, where v satisfies a first-order linear equation.

20. The Riccati equation $y' + y + y^2 = 2$ has two constant solutions. Start with each of these and use Exercise 19 to find further solutions as follows: (a) If $-2 \leq b < 1$, find a solution on $(-\infty, +\infty)$ for which $y = b$ when $x = 0$. (b) If $b \geq 1$ or $b < -2$, find a solution on the interval $(-\infty, +\infty)$ for which $y = b$ when $x = 0$.

8.6 Some physical problems leading to first-order linear differential equations

In this section we will discuss various physical problems that can be formulated mathematically as differential equations. In each case, the differential equation represents an idealized simplification of the physical problem and is called a *mathematical model* of the problem. The differential equation occurs as a translation of some physical law, such as Newton's second law of motion, a "conservation" law, etc. Our purpose here is not to justify the choice of the mathematical model but rather to deduce logical consequences from it. Each model is only an approximation to reality, and its justification properly belongs to the science from which the problem emanates. If intuition or experimental evidence agrees with the results deduced mathematically, then we feel that the model is a useful one. If not, we try to find a more suitable model.

EXAMPLE 1. *Radioactive decay.* Although various radioactive elements show marked differences in their rates of decay, they all seem to share a common property—the rate at which a given substance decomposes at any instant is proportional to the amount present at that instant. If we denote by $y = f(t)$ the amount present at time t, the derivative $y' = f'(t)$ represents the rate of change of y at time t, and the "law of decay" states that

$$y' = -ky\,,$$

where k is a positive constant (called the *decay constant*) whose actual value depends on the particular element that is decomposing. The minus sign comes in because y decreases as t increases, and hence y' is always negative. The differential equation $y' = -ky$ is the mathematical model used for problems concerning radioactive decay. Every solution $y = f(t)$ of this differential equation has the form

$$f(t) = f(0)e^{-kt}. \tag{8.13}$$

Therefore, to determine the amount present at time t, we need to know the initial amount $f(0)$ and the value of the decay constant k.

It is interesting to see what information can be deduced from (8.13), without knowing the exact value of $f(0)$ or of k. First we observe that there is no finite time t at which $f(t)$ will be zero because the exponential e^{-kt} never vanishes. Therefore, it is not useful to study the "total lifetime" of a radioactive substance. However, it is possible to determine the time required for any particular *fraction* of a sample to decay. The fraction $\frac{1}{2}$ is usually chosen for convenience and the time T at which $f(T)/f(0) = \frac{1}{2}$ is called the *half-life* of the substance. This can be determined by solving the equation $e^{-kT} = \frac{1}{2}$ for T. Taking logarithms, we get $-kT = -\log 2$ or $T = (\log 2)/k$. This equation relates the half-life to the decay constant. Since we have

$$\frac{f(t+T)}{f(t)} = \frac{f(0)e^{-k(t+T)}}{f(0)e^{-kt}} = e^{-kT} = \frac{1}{2}\,,$$

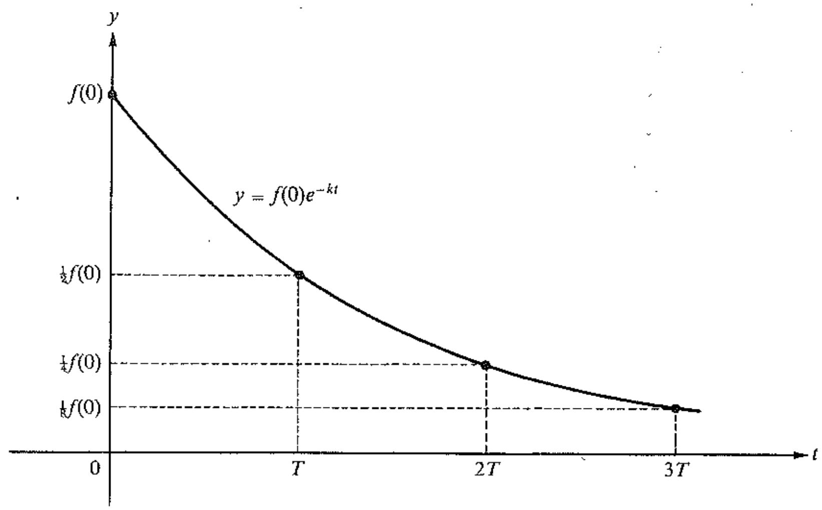


FIGURE 8.1 Radioactive decay with half-life T.

ve see that the half-life is the same for every sample of a given material. Figure 8.1 illustrates he general shape of a radioactive decay curve.

EXAMPLE 2. *Falling body in a resisting medium.* A body of mass m is dropped from est from a great height in the earth's atmosphere. Assume that it falls in a straight line nd that the only forces acting on it are the earth's gravitational attraction (mg, where g is he acceleration due to gravity, assumed to be constant) and a resisting force (due to air esistance) which is proportional to its velocity. It is required to discuss the resulting notion.

Let $s = f(t)$ denote the distance the body has fallen at time t and let $v = s' = f'(t)$ denote s velocity. The assumption that it falls from rest means that $f'(0) = 0$.

There are two forces acting on the body, a downward force mg (due to its weight) and n upward force $-kv$ (due to air resistance), where k is some positive constant. Newton's econd law states that the net sum of the forces acting on the body at any instant is equal o the product of its mass m and its acceleration. If we denote the acceleration at time t y a, then $a = v' = s''$ and Newton's law gives us the equation

$$ma = mg - kv .$$

his can be considered as a second-order differential equation for the displacement s or s a first-order equation for the velocity v. As a first-order equation for v, it is linear and an be written in the form

$$v' + \frac{k}{m} v = g .$$

his equation is the mathematical model of the problem. Since $v = 0$ when $t = 0$, the

unique solution of the differential equation is given by the formula

$$v = e^{-kt/m}\int_0^t g e^{ku/m}\,du = \frac{mg}{k}(1 - e^{-kt/m}) . \tag{8.14}$$

Note that $v \to mg/k$ as $t \to +\infty$. If we differentiate Equation (8.14), we find that the acceleration at every instant is $a = ge^{-kt/m}$. Note that $a \to 0$ as $t \to +\infty$. Interpreted physically, this means that the air resistance tends to balance out the force of gravity.

Since $v = s'$, Equation (8.14) is itself a differential equation for the displacement s, and it may be integrated directly to give

$$s = \frac{mg}{k} t + \frac{gm^2}{k^2} e^{-kt/m} + C .$$

Since $s = 0$ when $t = 0$, we find that $C = -gm^2/k^2$ and the equation of motion becomes

$$s = \frac{mg}{k} t + \frac{gm^2}{k^2}(e^{-kt/m} - 1) .$$

If the initial velocity is v_0 when $t = 0$, formula (8.14) for the velocity at time t must be replaced by

$$v = \frac{mg}{k}(1 - e^{-kt/m}) + v_0 e^{-kt/m} .$$

It is interesting to note that for *every* initial velocity (positive, negative, or zero), the limiting velocity, as t increases without bound, is mg/k, a number independent of v_0. The reader should convince himself, on physical grounds, that this seems reasonable.

EXAMPLE 3. *A cooling problem.* The rate at which a body changes temperature is proportional to the difference between its temperature and that of the surrounding medium. (This is called *Newton's law of cooling.*) If $y = f(t)$ is the (unknown) temperature of the body at time t and if $M(t)$ denotes the (known) temperature of the surrounding medium, Newton's law leads to the differential equation

$$y' = -k[y - M(t)] \quad \text{or} \quad y' + ky = kM(t) , \tag{8.15}$$

where k is a positive constant. This first-order linear equation is the mathematical model we use for cooling problems. The unique solution of the equation satisfying the initial condition $f(a) = b$ is given by the formula

$$f(t) = be^{-kt} + e^{-kt}\int_a^t kM(u)e^{ku}\,du . \tag{8.16}$$

Consider now a specific problem in which a body cools from 200° to 100° in 40 minutes while immersed in a medium whose temperature is kept constant, say $M(t) = 10°$. If we measure t in minutes and $f(t)$ in degrees, we have $f(0) = 200$ and Equation (8.16) gives us

$$(8.17)\qquad f(t) = 200e^{-kt} + 10ke^{-kt}\int_0^t e^{ku}\,du$$

$$= 200e^{-kt} + 10(1 - e^{-kt}) = 10 + 190e^{-kt}.$$

We can compute k from the information that $f(40) = 100$. Putting $t = 40$ in (8.17), we find $90 = 190e^{-40k}$, so $-40k = \log(90/190)$, $k = \frac{1}{40}(\log 19 - \log 9)$.

Next, let us compute the time required for this same material to cool from 200° to 100° if the temperature of the medium is kept at 5°. Then Equation (8.16) is valid with the same constant k but with $M(u) = 5$. Instead of (8.17), we get the formula

$$f(t) = 5 + 195e^{-kt}.$$

To find the time t for which $f(t) = 100$, we get $95 = 195e^{-kt}$, so $-kt = \log(95/195) = \log(19/39)$, and hence

$$t = \frac{1}{k}(\log 39 - \log 19) = 40\,\frac{\log 39 - \log 19}{\log 19 - \log 9}.$$

From a four-place table of natural logarithms, we find $\log 39 = 3.6636$, $\log 19 = 2.9444$, and $\log 9 = 2.1972$ so, with slide-rule accuracy, we get $t = 40(0.719)/(0.747) = 38.5$ minutes.

The differential equation in (8.15) tells us that the rate of cooling decreases considerably as the temperature of the body begins to approach the temperature of the medium. To illustrate, let us find the time required to cool the same substance from 100° to 10° with the medium kept at 5°. The calculation leads to $\log(5/95) = -kt$, or

$$t = \frac{1}{k}\log 19 = 40\,\frac{\log 19}{\log 19 - \log 9} = \frac{40(2.944)}{0.747} = 158 \text{ minutes}.$$

Note that the temperature drop from 100° to 10° takes more than four times as long as the change from 200° to 100°.

EXAMPLE 4. *A dilution problem.* A tank contains 100 gallons of brine whose concentration is 2.5 pounds of salt per gallon. Brine containing 2 pounds of salt per gallon runs into the tank at a rate of 5 gallons per minute and the mixture (kept uniform by stirring) runs out at the same rate. Find the amount of salt in the tank at every instant.

Let $y = f(t)$ denote the number of pounds of salt in the tank at time t minutes after mixing begins. There are two factors which cause y to change, the incoming brine which brings salt in at a rate of 10 pounds per minute and the outgoing mixture which removes salt at a rate of $5(y/100)$ pounds per minute. (The fraction $y/100$ represents the concentration at time t.) Hence the differential equation is

$$y' = 10 - \tfrac{1}{20}y \qquad \text{or} \qquad y' + \tfrac{1}{20}y = 10.$$

This linear equation is the mathematical model for our problem. Since $y = 250$ when $t = 0$, the unique solution is given by the formula

$$(8.18)\qquad y = 250e^{-t/20} + e^{-t/20}\int_0^t 10e^{u/20}\,du = 200 + 50e^{-t/20}.$$

This equation shows that $y > 200$ for all t and that $y \to 200$ as t increases without bound. Hence, the minimum salt content is 200 pounds. (This could also have been guessed from the statement of the problem.) Equation (8.18) can be solved for t in terms of y to yield

$$t = 20\log\left(\frac{50}{y - 200}\right).$$

This enables us to find the time at which the salt content will be a given amount y, provided that $200 < y < 250$.

EXAMPLE 5. *Electric circuits.* Figure 8.2(a), page 318, shows an electric circuit which has an electromotive force, a resistor, and an inductor connected in series. The electromotive force produces a voltage which causes an electric current to flow in the circuit. If the reader is not familiar with electric circuits, he should not be concerned. For our purposes, all we need to know about the circuit is that the voltage, denoted by $V(t)$, and the current, denoted by $I(t)$, are functions of time t related by a differential equation of the form

$$(8.19)\qquad LI'(t) + RI(t) = V(t).$$

Here L and R are assumed to be positive constants. They are called, respectively, the *inductance* and *resistance* of the circuit. The differential equation is a mathematical formulation of a conservation law known as *Kirchhoff's voltage law*, and it serves as a mathematical model for the circuit.

Those readers unfamiliar with circuits may find it helpful to think of the current as being analogous to water flowing in a pipe. The electromotive force (usually a battery or a generator) is analogous to a pump which causes the water to flow; the resistor is analogous to friction in the pipe, which tends to oppose the flow; and the inductance is a stabilizing influence which tends to oppose sudden changes in the current due to sudden changes in the voltage.

The usual type of question concerning such circuits is this: If a given voltage $V(t)$ is impressed on the circuit, what is the resulting current $I(t)$? Since we are dealing with a first-order linear differential equation, the solution is a routine matter. If $I(0)$ denotes the initial current at time $t = 0$, the equation has the solution

$$I(t) = I(0)e^{-Rt/L} + e^{-Rt/L}\int_0^t \frac{V(x)}{L}\,e^{Rx/L}\,dx.$$

An important special case occurs when the impressed voltage is constant, say $V(t) = E$ for all t. In this case, the integration is easy to perform and we are led to the formula

$$I(t) = \frac{E}{R} + \left(I(0) - \frac{E}{R}\right)e^{-Rt/L}.$$

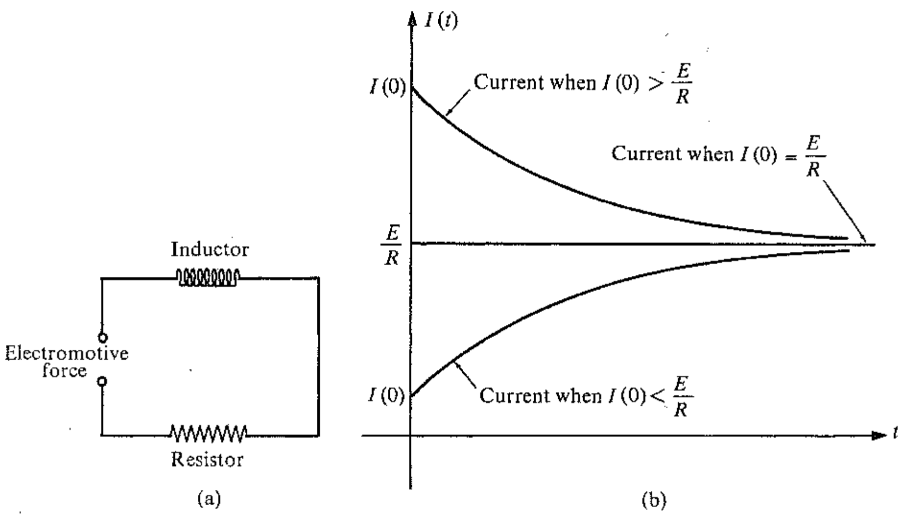


FIGURE 8.2 (a) Diagram for a simple series circuit. (b) The current resulting from a constant impressed voltage E.

This shows that the nature of the solution depends on the relation between the initial current $I(0)$ and the quotient E/R. If $I(0) = E/R$, the exponential term is not present and the current is constant, $I(t) = E/R$. If $I(0) > E/R$, the coefficient of the exponential term is positive and the current decreases to the limiting value E/R as $t \to +\infty$. If $I(0) < E/R$, the current increases to the limiting value E/R. The constant E/R is called the *steady-state current*, and the exponential term $[I(0) - E/R]e^{-Rt/L}$ is called the *transient current*. Examples are illustrated in Figure 8.2(b).

The foregoing examples illustrate the unifying power and practical utility of differential equations. They show how several different types of physical problems may lead to exactly the same type of differential equation.

The differential equation in (8.19) is of special interest because it suggests the possibility of attacking a wide variety of physical problems by electrical means. For example, suppose a physical problem leads to a differential equation of the form

$$y' + ay = Q,$$

where a is a positive constant and Q is a known function. We can try to construct an electric circuit with inductance L and resistance R in the ratio $R/L = a$ and then try to impress a voltage LQ on the circuit. We would then have an electric circuit with exactly the same mathematical model as the physical problem. Thus, we can hope to get numerical data about the solution of the physical problem by making measurements of current in the electric circuit. This idea has been used in practice and has led to the development of the *analog computer*.

8.7 Exercises

In the following exercises, use an appropriate first-order differential equation as a mathematical model of the problem.

1. The half-life for radium is approximately 1600 years. Find what percentage of a given quantity of radium disintegrates in 100 years.
2. If a strain of bacteria grows at a rate proportional to the amount present and if the population doubles in one hour, by how much will it increase at the end of two hours?
3. Denote by $y = f(t)$ the amount of a substance present at time t. Assume it disintegrates at a rate proportional to the amount present. If n is a positive integer, the number T for which $f(T) = f(0)/n$ is called the $1/n$th life of the substance.
   (a) Prove that the $1/n$th life is the same for every sample of a given material, and compute T in terms of n and the decay constant k.
   (b) If a and b are given, prove that f can be expressed in the form
   $$f(t) = f(a)^{w(t)}f(b)^{1-w(t)}$$
   and determine $w(t)$. This shows that the amount present at time t is a weighted geometric mean of the amounts present at two instants $t = a$ and $t = b$.
4. A man wearing a parachute jumps from a great height. The combined weight of man and parachute is 192 pounds. Let $v(t)$ denote his speed (in feet per second) at time t seconds after falling. During the first 10 seconds, before the parachute opens, assume the air resistance is $\frac{3}{4}v(t)$ pounds. Thereafter, while the parachute is open, assume the resistance is $12v(t)$ pounds. Assume the acceleration of gravity is 32 ft/sec^2 and find explicit formulas for the speed $v(t)$ at time t. (You may use the approximation $e^{-5/4} = 37/128$ in your calculations.)
5. Refer to Example 2 of Section 8.6. Use the chain rule to write
   $$\frac{dv}{dt} = \frac{ds}{dt}\frac{dv}{ds} = v\frac{dv}{ds}$$
   and thus show that the differential equation in the example can be expressed as follows:
   $$\frac{ds}{dv} = \frac{bv}{c - v},$$
   where $b = m/k$ and $c = gm/k$. Integrate this equation to express s in terms of v. Check your result with the formulas for v and s derived in the example.
6. Modify Example 2 of Section 8.6 by assuming the air resistance is proportional to v^2. Show that the differential equation can be put in each of the following forms:
   $$\frac{ds}{dv} = \frac{m}{k}\frac{v}{c^2 - v^2}; \qquad \frac{dt}{dv} = \frac{m}{k}\frac{1}{c^2 - v^2},$$
   where $c = \sqrt{mg/k}$. Integrate each of these and obtain the following formulas for v:
   $$v^2 = \frac{mg}{k}(1 - e^{-2ks/m}); \qquad v = c\frac{e^{bt} - e^{-bt}}{e^{bt} + e^{-bt}} = c \tanh bt,$$
   where $b = \sqrt{kg/m}$. Determine the limiting value of v as $t \to +\infty$.

7. A body in a room at 60° cools from 200° to 120° in half an hour.
(a) Show that its temperature after t minutes is $60 + 140e^{-kt}$, where $k = (\log 7 - \log 3)/30$.
(b) Show that the time t required to reach a temperature of T degrees is given by the formula $t = [\log 140 - \log (T - 60)]/k$, where $60 < T \le 200$.
(c) Find the time at which the temperature is 90°.
(d) Find a formula for the temperature of the body at time t if the room temperature is not kept constant but falls at a rate of 1° each ten minutes. Assume the room temperature is 60° when the body temperature is 200°.
8. A thermometer has been stored in a room whose temperature is 75°. Five minutes after being taken outdoors it reads 65°. After another five minutes, it reads 60°. Compute the outdoor temperature.
9. In a tank are 100 gallons of brine containing 50 pounds of dissolved salt. Water runs into the tank at the rate of 3 gallons per minute, and the concentration is kept uniform by stirring. How much salt is in the tank at the end of one hour if the mixture runs out at a rate of 2 gallons per minute?
10. Refer to Exercise 9. Suppose the bottom of the tank is covered with a mixture of salt and insoluble material. Assume that the salt dissolves at a rate proportional to the difference between the concentration of the solution and that of a saturated solution (3 pounds of salt per gallon), and that if the water were fresh 1 pound of salt would dissolve per minute. How much salt will be in solution at the end of one hour?
11. Consider an electric circuit like that in Example 5 of Section 8.6. Assume the electromotive force is an alternating current generator which produces a voltage $V(t) = E \sin \omega t$, where E and ω are positive constants (ω is the Greek letter *omega*). If $I(0) = 0$, prove that the current has the form

$$I(t) = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega t - \alpha) + \frac{E\omega L}{R^2 + \omega^2 L^2} e^{-Rt/L},$$

where α depends only on ω, L, and R. Show that $\alpha = 0$ when $L = 0$.
12. Refer to Example 5 of Section 8.6. Assume the impressed voltage is a step function defined as follows: $E(t) = E$ if $a \le t \le b$, where $a > 0$; $E(t) = 0$ for all other t. If $I(0) = 0$ prove that the current is given by the following formulas: $I(t) = 0$ if $t \le a$;

$$I(t) = \frac{E}{R}(1 - e^{-R(t-a)/L}) \quad \text{if} \quad a \le t \le b\,; \qquad I(t) = \frac{E}{R} e^{-Rt/L}(e^{Rb/L} - e^{Ra/L}) \quad \text{if} \quad t \ge b\,.$$

Make a sketch indicating the nature of the graph of I.

Population growth. In a study of the growth of a population (whether human, animal, or bacterial), the function which counts the number x of individuals present at time t is necessarily a *step function* taking on only integer values. Therefore the true *rate of growth* dx/dt is zero (when t lies in an open interval where x is constant), or else the derivative dx/dt does not exist (when x jumps from one integer to another). Nevertheless, useful information can often be obtained if we assume that the population x is a continuous function of t with a continuous derivative dx/dt at each instant. We then postulate various "laws of growth" for the population, depending on the factors in the environment which may stimulate or hinder growth.

For example, if environment has little or no effect, it seems reasonable to assume that the rate of growth is proportional to the amount present. The simplest kind of growth law takes the form

$$\frac{dx}{dt} = kx\,, \tag{8.20}$$

where k is a constant that depends on the particular kind of population. Conditions may develop which cause the factor k to change with time, and the growth law (8.20) can be generalized as follows:

$$\frac{dx}{dt} = k(t)x\,. \tag{8.21}$$

If, for some reason, the population cannot exceed a certain maximum M (for example, because the food supply may be exhausted), we may reasonably suppose that the rate of growth is jointly proportional to both x and $M - x$. Thus we have a second type of growth law:

$$\frac{dx}{dt} = kx(M - x)\,, \tag{8.22}$$

where, as in (8.21), k may be constant or, more generally, k may change with time. Technological improvements may tend to increase or decrease the value of M slowly, and hence we can generalize (8.22) even further by allowing M to change with time.

13. Express x as a function of t for each of the "growth laws" in (8.20) and (8.22) (with k and M both constant). Show that the result for (8.22) can be expressed as follows:

$$x = \frac{M}{1 + e^{-\alpha(t - t_1)}}\,, \tag{8.23}$$

where α is a constant and t_1 is the time at which $x = M/2$.
14. Assume the growth law in formula (8.23) of Exercise 13, and suppose a census is taken at three equally spaced times t_1, t_2, t_3, the resulting numbers being x_1, x_2, x_3. Show that this suffices to determine M and that, in fact, we have

$$M = x_2 \frac{x_3(x_2 - x_1) - x_1(x_3 - x_2)}{x_2^2 - x_1 x_3}\,. \tag{8.24}$$

15. Derive a formula that generalizes (8.23) of Exercise 13 for the growth law (8.22) when k is not necessarily constant. Express the result in terms of the time t_0 for which $x = M/2$.
16. The Census Bureau reported the following population figures (in millions) for the United States at ten-year intervals from 1790 to 1950: 3.9, 5.3, 7.2, 9.6, 12.9, 17, 23, 31, 39, 50, 63, 76, 92, 108, 122, 135, 150.
(a) Use Equation (8.24) to determine a value of M on the basis of the census figures for 1790, 1850, and 1910.
(b) Same as (a) for the years 1910, 1930, 1950.
(c) On the basis of your calculations in (a) and (b), would you be inclined to accept or reject the growth law (8.23) for the population of the United States?
17. (a) Plot a graph of $\log x$ as a function of t, where x denotes the population figures quoted in Exercise 16. Use this graph to show that the growth law (8.20) was very nearly satisfied from 1790 to 1910. Determine a reasonable average value of k for this period.
(b) Determine a reasonable average value of k for the period from 1920 to 1950, assume that the growth law (8.20) will hold for this k, and predict the United States population for the years 2000 and 2050.
18. The presence of toxins in a certain medium destroys a strain of bacteria at a rate jointly proportional to the number of bacteria present and to the amount of toxin. If there were no

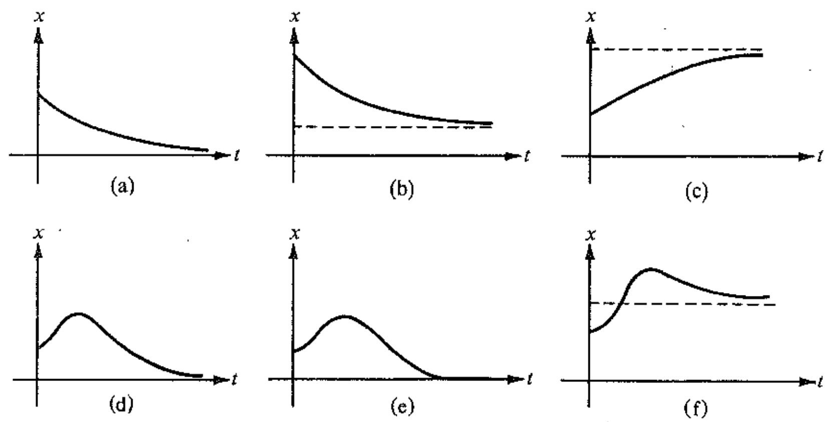


FIGURE 8.3 Exercise 18.

toxins present, the bacteria would grow at a rate proportional to the amount present. Let x denote the number of living bacteria present at time t. Assume that the amount of toxin is increasing at a constant rate and that the production of toxin begins at time $t = 0$. Set up a differential equation for x. Solve the differential equation. One of the curves shown in Figure 8.3 best represents the general behavior of x as a function of t. State your choice and explain your reasoning.

8.8 Linear equations of second order with constant coefficients

A differential equation of the form

$$y'' + P_1(x)y' + P_2(x)y = R(x)$$

is said to be a *linear equation of second order.* The functions P_1 and P_2 which multiply the unknown function y and its derivative y' are called the *coefficients* of the equation.

For first-order linear equations, we proved an existence-uniqueness theorem and determined all solutions by an explicit formula. Although there is a corresponding existence-uniqueness theorem for the general second-order linear equation, there is no explicit formula which gives all solutions, except in some special cases. A study of the general linear equation of second order is undertaken in Volume II. Here we treat only the case in which the coefficients P_1 and P_2 are constants. When the right-hand member $R(x)$ is identically zero, the equation is said to be *homogeneous.*

The homogeneous linear equation with constant coefficients was the first differential equation of a general type to be completely solved. A solution was first published by Euler in 1743. Apart from its historical interest, this equation arises in a great variety of applied problems, so its study is of practical importance. Moreover, we can give explicit formulas for all the solutions.

Consider a homogeneous linear equation with constant coefficients which we write as follows:

$$y'' + ay' + by = 0\,.$$

We seek solutions on the entire real axis $(-\infty, +\infty)$. One solution is the constant function $y = 0$. This is called the *trivial* solution. We are interested in finding nontrivial solutions, and we begin our study with some special cases for which nontrivial solutions can be found by inspection. In all these cases, the coefficient of y' is zero, and the equation has the form $y'' + by = 0$. We shall find that solving this special equation is tantamount to solving the general case.

8.9 Existence of solutions of the equation $y'' + by = 0$

EXAMPLE 1. *The equation* $y'' = 0$. Here both coefficients a and b are zero, and we can easily determine all solutions. Assume y is any function satisfying $y'' = 0$ on $(-\infty, +\infty)$. Then its derivative y' is constant, say $y' = c_1$. Integrating this relation, we find that y necessarily has the form

$$y = c_1x + c_2\,,$$

where c_1 and c_2 are constants. Conversely, for any choice of constants c_1 and c_2, the linear polynomial $y = c_1x + c_2$ satisfies $y'' = 0$, so we have found all solutions in this case.

Next we assume that $b \neq 0$ and treat separately the cases $b < 0$ and $b > 0$.

EXAMPLE 2. *The equation* $y'' + by = 0$, *where* $b < 0$. Since $b < 0$, we can write $b = -k^2$, where $k > 0$, and the differential equation takes the form

$$y'' = k^2y\,.$$

One obvious solution is $y = e^{kx}$, and another is $y = e^{-kx}$. From these we can obtain further solutions by constructing linear combinations of the form

$$y = c_1e^{kx} + c_2e^{-kx},$$

where c_1 and c_2 are arbitrary constants. It will be shown presently, in Theorem 8.6, that *all* solutions are included in this formula.

EXAMPLE 3. *The equation* $y'' + by = 0$, *where* $b > 0$. Here we can write $b = k^2$, where $k > 0$, and the differential equation takes the form

$$y'' = -k^2y\,.$$

Again we obtain some solutions by inspection. One solution is $y = \cos kx$, and another is $y = \sin kx$. From these we get further solutions by forming linear combinations,

$$y = c_1\cos kx + c_2\sin kx\,,$$

where c_1 and c_2 are arbitrary constants. Theorem 8.6 will show that this formula includes all solutions.

8.10 Reduction of the general equation to the special case $y'' + by = 0$

The problem of solving a second-order linear equation with constant coefficients can be reduced to that of solving the special cases just discussed. There is a method for doing this that also applies to more general equations. The idea is to consider three functions y, u, and v such that $y = uv$. Differentiation gives us $y' = uv' + u'v$, and $y'' = uv'' + 2u'v' + u''v$. Now we express the combination $y'' + ay' + by$ in terms of u and v. We have

$$(8.25)\qquad \begin{aligned} y'' + ay' + by &= uv'' + 2u'v' + u''v + a(uv' + u'v) + buv \\ &= (v'' + av' + bv)u + (2v' + av)u' + vu''. \end{aligned}$$

Next we choose v to make the coefficient of u' zero. This requires that $v' = -av/2$, so we may choose $v = e^{-ax/2}$. For this v we have $v'' = -av'/2 = a^2v/4$, and the coefficient of u in (8.25) becomes

$$v'' + av' + bv = \frac{a^2v}{4} - \frac{a^2v}{2} + bv = \frac{4b - a^2}{4}\, v .$$

Thus, Equation (8.25) reduces to

$$y'' + ay' + by = \left(u'' + \frac{4b - a^2}{4}\, u\right)v .$$

Since $v = e^{-ax/2}$, the function v is never zero, so y satisfies the differential equation $y'' + ay' + by = 0$ if and only if u satisfies $u'' + \frac{1}{4}(4b - a^2)u = 0$. Thus, we have proved the following theorem.

THEOREM 8.4. *Let y and u be two functions such that $y = ue^{-ax/2}$. Then, on the interval $(-\infty, +\infty)$, y satisfies the differential equation $y'' + ay' + by = 0$ if and only if u satisfies the differential equation*

$$u'' + \frac{4b - a^2}{4}\, u = 0 .$$

This theorem reduces the study of the equation $y'' + ay' + by = 0$ to the special case $y'' + by = 0$. We have exhibited nontrivial solutions of this equation but, except for the case $b = 0$, we have not yet shown that we have found *all* solutions.

8.11 Uniqueness theorem for the equation $y'' + by = 0$

The problem of determining all solutions of the equation $y'' + by = 0$ can be solved with the help of the following *uniqueness theorem.*

THEOREM 8.5. *Assume two functions f and g satisfy the differential equation $y'' + by = 0$ on $(-\infty, +\infty)$. Assume also that f and g satisfy the initial conditions*

$$f(0) = g(0) , \qquad f'(0) = g'(0) .$$

Then $f(x) = g(x)$ for all x.

Proof. Let $h(x) = f(x) - g(x)$. We wish to prove that $h(x) = 0$ for all x. We shall do this by expressing h in terms of its Taylor polynomial approximations.

First we note that h is also a solution of the differential equation $y'' + by = 0$ and satisfies the initial conditions $h(0) = 0$, $h'(0) = 0$. Now every function y satisfying the differential equation has derivatives of every order on $(-\infty, +\infty)$ and they can be computed by repeated differentiation of the differential equation. For example, since $y'' = -by$, we have $y''' = -by'$, and $y^{(4)} = -by'' = b^2y$. By induction we find that the derivatives of even order are given by

$$y^{(2n)} = (-1)^n b^n y ,$$

while those of odd order are $y^{(2n-1)} = (-1)^{n-1}b^{n-1}y'$. Since $h(0)$ and $h'(0)$ are both 0, it follows that all derivatives $h^{(n)}(0)$ are zero. Therefore, each Taylor polynomial generated by h at 0 has all its coefficients zero.

Now we apply Taylor's formula with remainder (Theorem 7.6), using a polynomial approximation of odd degree $2n - 1$, and we find that

$$h(x) = E_{2n-1}(x) ,$$

where $E_{2n-1}(x)$ is the error term in Taylor's formula. To complete the proof, we show that the error can be made arbitrarily small by taking n large enough.

We use Theorem 7.7 to estimate the size of the error term. For this we need estimates for the size of the derivative $h^{(2n)}$. Consider any finite closed interval $[-c, c]$, where $c > 0$. Since h is continuous on this interval, it is bounded there, say $|h(x)| \le M$ on $[-c, c]$. Since $h^{(2n)}(x) = (-1)^n b^n h(x)$, we have the estimate $|h^{(2n)}(x)| \le M\,|b|^n$ on $[-c, c]$. Theorem 7.7 gives us $|E_{2n-1}(x)| \le M\,|b|^n x^{2n}/(2n)!$ so, on the interval $[-c, c]$, we have the estimate

$$(8.26)\qquad 0 \le |h(x)| \le \frac{M\,|b|^n x^{2n}}{(2n)!} \le \frac{M\,|b|^n c^{2n}}{(2n)!} = \frac{MA^{2n}}{(2n)!} ,$$

where $A = |b|^{1/2} c$. Now we show that $A^m/m!$ tends to 0 as $m \to +\infty$. This is obvious if $0 \le A \le 1$. If $A > 1$, we may write

$$\frac{A^m}{m!} = \frac{A}{1}\cdot\frac{A}{2}\cdots\frac{A}{k}\cdot\frac{A}{k+1}\cdots\frac{A}{m} \le \frac{A^k}{k!}\left(\frac{A}{k+1}\right)^{m-k} ,$$

where $k < m$. If we choose k to be the greatest integer $\le A$, then $A < k + 1$ and the last factor tends to 0 as $m \to +\infty$. Hence $A^m/m!$ tends to 0 as $m \to \infty$, so inequality (8.26) shows that $h(x) = 0$ for every x in $[-c, c]$. But, since c is arbitrary, it follows that $h(x) = 0$ for all real x. This completes the proof.

Note: Theorem 8.5 tells us that two solutions of the differential equation $y'' + by = 0$ which have the same value and the same derivative at 0 must agree everywhere. The choice of the point 0 is not essential. The same argument shows that the theorem is also true if 0 is replaced by an arbitrary point c. In the foregoing proof, we simply use Taylor polynomial approximations at c instead of at 0.

8.12 Complete solution of the equation $y'' + by = 0$

The uniqueness theorem enables us to characterize all solutions of the differential equation $y'' + by = 0$.

THEOREM 8.6. *Given a real number b, define two functions u_1 and u_2 on $(-\infty, +\infty)$ as follows:*

(a) *If $b = 0$, let $u_1(x) = 1$, $u_2(x) = x$.*
(b) *If $b < 0$, write $b = -k^2$ and define $u_1(x) = e^{kx}$, $u_2(x) = e^{-kx}$.*
(c) *If $b > 0$, write $b = k^2$ and define $u_1(x) = \cos kx$, $u_2(x) = \sin kx$.*

Then every solution of the differential equation $y'' + by = 0$ on $(-\infty, +\infty)$ has the form

$$y = c_1u_1(x) + c_2u_2(x), \tag{8.27}$$

where c_1 and c_2 are constants.

Proof. We proved in Section 8.9 that for each choice of constants c_1 and c_2 the function y given in (8.27) is a solution of the equation $y'' + by = 0$. Now we show that all solutions have this form. The case $b = 0$ was settled in Section 8.9, so we may assume that $b \neq 0$.

The idea of the proof is this: Let $y = f(x)$ be any solution of $y'' + by = 0$. If we can show that constants c_1 and c_2 exist satisfying the pair of equations

$$c_1u_1(0) + c_2u_2(0) = f(0), \qquad c_1u_1'(0) + c_2u_2'(0) = f'(0), \tag{8.28}$$

then both f and $c_1u_1 + c_2u_2$ are solutions of the differential equation $y'' + by = 0$ having the same value and the same derivative at 0. By the uniqueness theorem, it follows that $f = c_1u_1 + c_2u_2$.

In case (b), we have $u_1(x) = e^{kx}$, $u_2(x) = e^{-kx}$, so $u_1(0) = u_2(0) = 1$ and $u_1'(0) = k$, $u_2'(0) = -k$. Thus the equations in (8.28) become $c_1 + c_2 = f(0)$, and $c_1 - c_2 = f'(0)/k$. They have the solution $c_1 = \frac{1}{2}f(0) + \frac{1}{2}f'(0)/k$, $c_2 = \frac{1}{2}f(0) - \frac{1}{2}f'(0)/k$.

In case (c), we have $u_1(x) = \cos kx$, $u_2(x) = \sin kx$, so $u_1(0) = 1$, $u_2(0) = 0$, $u_1'(0) = 0$, $u_2'(0) = k$, and the solutions are $c_1 = f(0)$, and $c_2 = f'(0)/k$. Since c_1 and c_2 always exist to satisfy (8.28), the proof is complete.

8.13 Complete solution of the equation $y'' + ay' + by = 0$

Theorem 8.4 tells us that y satisfies the differential equation $y'' + ay' + by = 0$ if and only if u satisfies $u'' + \frac{1}{4}(4b - a^2)u = 0$, where $y = e^{-ax/2}u$. From Theorem 8.6 we know that the nature of each solution u depends on the algebraic sign of the coefficient of u, that is, on the algebraic sign of $4b - a^2$ or, alternatively, of $a^2 - 4b$. We call the number $a^2 - 4b$ the *discriminant* of the differential equation $y'' + ay' + by = 0$ and denote it by d. When we combine the results of Theorem 8.4 and 8.6 we obtain the following.

THEOREM 8.7. *Let $d = a^2 - 4b$ be the discriminant of the linear differential equation $y'' + ay' + by = 0$. Then every solution of this equation on $(-\infty, +\infty)$ has the form*

$$y = e^{-ax/2}[c_1u_1(x) + c_2u_2(x)], \tag{8.29}$$

where c_1 and c_2 are constants, and the functions u_1 and u_2 are determined according to the algebraic sign of the discriminant as follows:

(a) *If $d = 0$, then $u_1(x) = 1$ and $u_2(x) = x$.*
(b) *If $d > 0$, then $u_1(x) = e^{kx}$ and $u_2(x) = e^{-kx}$, where $k = \frac{1}{2}\sqrt{d}$.*
(c) *If $d < 0$, then $u_1(x) = \cos kx$ and $u_2(x) = \sin kx$, where $k = \frac{1}{2}\sqrt{-d}$.*

Note: In case (b), where the discriminant d is positive, the solution y in (8.29) is a linear combination of two exponential functions,

$$y = e^{-ax/2}(c_1e^{kx} + c_2e^{-kx}) = c_1e^{r_1x} + c_2e^{r_2x},$$

where

$$r_1 = -\frac{a}{2} + k = \frac{-a + \sqrt{d}}{2}, \qquad r_2 = -\frac{a}{2} - k = \frac{-a - \sqrt{d}}{2}.$$

The two numbers r_1 and r_2 have sum $r_1 + r_2 = -a$ and product $r_1r_2 = \frac{1}{4}(a^2 - d) = b$. Therefore, they are the roots of the quadratic equation

$$r^2 + ar + b = 0.$$

This is called the *characteristic equation* associated with the differential equation

$$y'' + ay' + by = 0.$$

The number $d = a^2 - 4b$ is also called the discriminant of this quadratic equation; its algebraic sign determines the nature of the roots. If $d \geq 0$, the quadratic equation has real roots given by $(-a \pm \sqrt{d})/2$. If $d < 0$, the quadratic equation has no real roots but it does have *complex* roots r_1 and r_2. The definition of the exponential function can be extended so that e^{r_1x} and e^{r_2x} are meaningful when r_1 and r_2 are complex numbers. This extension, described in Chapter 9, is made in such a way that the linear combination in (8.29) can also be written as a linear combination of e^{r_1x} and e^{r_2x}, when r_1 and r_2 are complex.

We conclude this section with some miscellaneous remarks. Since all the solutions of the differential equation $y'' + ay' + by = 0$ are contained in formula (8.29), the linear combination on the right is often called the *general solution* of the differential equation. Any solution obtained by specializing the constants c_1 and c_2 is called a *particular solution*.

For example, taking $c_1 = 1$, $c_2 = 0$, and then $c_1 = 0$, $c_2 = 1$, we obtain the two particular solutions

$$v_1 = e^{-ax/2}u_1(x), \qquad v_2 = e^{-ax/2}u_2(x).$$

These two solutions are of special importance because linear combinations of them give us all solutions. Any pair of solutions with this property is called a *basis* for the set of all solutions.

A differential equation always has more than one basis. For example, the equation $y'' = 9y$ has the basis $v_1 = e^{3x}$, $v_2 = e^{-3x}$. But it also has the basis $w_1 = \cosh 3x$, $w_2 = \sinh 3x$. In fact, since $e^{3x} = w_1 + w_2$ and $e^{-3x} = w_1 - w_2$, every linear combination of e^{3x} and e^{-3x} is also a linear combination of w_1 and w_2. Hence, the pair w_1, w_2 is another basis.

It can be shown that any pair of solutions v_1 and v_2 of a differential equation $y'' + ay' + by = 0$ will be a basis if the ratio v_2/v_1 is not constant. Although we shall not need this fact, we mention it here because it is important in the theory of second-order linear equations with nonconstant coefficients. A proof is outlined in Exercise 23 of Section 8.14.

8.14 Exercises

Find all solutions of the following differential equations on $(-\infty, +\infty)$.

1. $y'' - 4y = 0$.
2. $y'' + 4y = 0$.
3. $y'' - 4y' = 0$.
4. $y'' + 4y' = 0$.
5. $y'' - 2y' + 3y = 0$.
6. $y'' + 2y' - 3y = 0$.
7. $y'' - 2y' + 2y = 0$.
8. $y'' - 2y' + 5y = 0$.
9. $y'' + 2y' + y = 0$.
10. $y'' - 2y' + y = 0$.

In Exercises 11 through 14, find the particular solution satisfying the given initial conditions.

11. $2y'' + 3y' = 0$, with $y = 1$ and $y' = 1$ when $x = 0$.
12. $y'' + 25y = 0$, with $y = -1$ and $y' = 0$ when $x = 3$.
13. $y'' - 4y' - y = 0$, with $y = 2$ and $y' = -1$ when $x = 1$.
14. $y'' + 4y' + 5y = 0$, with $y = 2$ and $y' = y''$ when $x = 0$.
15. The graph of a solution u of the differential equation $y'' - 4y' + 29y = 0$ intersects the graph of a solution v of the equation $y'' + 4y' + 13y = 0$ at the origin. The two curves have equal slopes at the origin. Determine u and v if $u'(\frac{1}{2}\pi) = 1$.
16. The graph of a solution u of the differential equation $y'' - 3y' - 4y = 0$ intersects the graph of a solution v of the equation $y'' + 4y' - 5y = 0$ at the origin. Determine u and v if the two curves have equal slopes at the origin and if

$$\lim_{x\to+\infty} \frac{v(x)^4}{u(x)} = \frac{5}{6}.$$

17. Find all values of the constant k such that the differential equation $y'' + ky = 0$ has a nontrivial solution $y = f_k(x)$ for which $f_k(0) = f_k(1) = 0$. For each permissible value of k, determine the corresponding solution $y = f_k(x)$. Consider both positive and negative values of k.
18. If (a, b) is a given point in the plane and if m is a given real number, prove that the differential equation $y'' + k^2y = 0$ has exactly one solution whose graph passes through (a, b) and has the slope m there. Discuss also the case $k = 0$.
19. (a) Let (a_1, b_1) and (a_2, b_2) be two points in the plane such that $a_1 - a_2 \neq n\pi$, where n is an integer. Prove that there is exactly one solution of the differential equation $y'' + y = 0$ whose graph passes through these two points.
(b) Is the statement in part (a) ever true if $a_1 - a_2$ is a multiple of π?
(c) Generalize the result in part (a) for the equation $y'' + k^2y = 0$. Discuss also the case $k = 0$.
20. In each case, find a linear differential equation of second order satisfied by u_1 and u_2.
(a) $u_1(x) = e^x$, $u_2(x) = e^{-x}$.
(b) $u_1(x) = e^{2x}$, $u_2(x) = xe^{2x}$.
(c) $u_1(x) = e^{-x/2}\cos x$, $u_2(x) = e^{-x/2}\sin x$.
(d) $u_1(x) = \sin(2x + 1)$, $u_2(x) = \sin(2x + 2)$.
(e) $u_1(x) = \cosh x$, $u_2(x) = \sinh x$.

The Wronskian. Given two functions u_1 and u_2, the function W defined by $W(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x)$ is called their *Wronskian,* after J. M. H. Wronski (1778–1853). The following exercises are concerned with properties of the Wronskian.

21. (a) If the Wronskian $W(x)$ of u_1 and u_2 is zero for all x in an open interval I, prove that the quotient u_2/u_1 is constant on I. In other words, if u_2/u_1 is not constant on I, then $W(c) \neq 0$ for at least one c in I.
(b) Prove that the derivative of the Wronskian is $W' = u_1u_2'' - u_2u_1''$.

22. Let W be the Wronskian of two solutions u_1, u_2 of the differential equation $y'' + ay' + by = 0$, where a and b are constants.
(a) Prove that W satisfies the first-order equation $W' + aW = 0$ and hence $W(x) = W(0)e^{-ax}$. This formula shows that if $W(0) \neq 0$, then $W(x) \neq 0$ for all x.
(b) Assume u_1 is not identically zero. Prove that $W(0) = 0$ if and only if u_2/u_1 is constant.
23. Let v_1 and v_2 be any two solutions of the differential equation $y'' + ay' + by = 0$ such that v_2/v_1 is not constant.
(a) Let $y = f(x)$ be any solution of the differential equation. Use properties of the Wronskian to prove that constants c_1 and c_2 exist such that

$$c_1v_1(0) + c_2v_2(0) = f(0), \qquad c_1v_1'(0) + c_2v_2'(0) = f'(0).$$

(b) Prove that every solution has the form $y = c_1v_1 + c_2v_2$. In other words, v_1 and v_2 form a basis for the set of all solutions.

8.15 Nonhomogeneous linear equations of second order with constant coefficients

We turn now to a discussion of nonhomogeneous equations of the form

$$y'' + ay' + by = R, \tag{8.30}$$

where the coefficients a and b are constants but the right-hand member R is any function continuous on $(-\infty, +\infty)$. The discussion may be simplified by the use of operator notation. For any function f with derivatives f' and f'', we may define an operator L which transforms f into another function $L(f)$ defined by the equation

$$L(f) = f'' + af' + bf.$$

In operator notation, the differential equation (8.30) is written in the simpler form

$$L(y) = R.$$

It is easy to verify that $L(y_1 + y_2) = L(y_1) + L(y_2)$, and that $L(cy) = cL(y)$ for every constant c. Therefore, for every pair of constants c_1 and c_2, we have

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2).$$

This is called the *linearity property* of the operator L.

Now suppose y_1 and y_2 are any two solutions of the equation $L(y) = R$. Since $L(y_1) = L(y_2) = R$, linearity gives us

$$L(y_2 - y_1) = L(y_2) - L(y_1) = R - R = 0,$$

so $y_2 - y_1$ is a solution of the homogeneous equation $L(y) = 0$. Therefore, we must have $y_2 - y_1 = c_1v_1 + c_2v_2$, where $c_1v_1 + c_2v_2$ is the general solution of the homogeneous equation, or

$$y_2 = c_1v_1 + c_2v_2 + y_1.$$

This equation must be satisfied by *every* pair of solutions y_1 and y_2 of the nonhomogeneous equation $L(y) = R$. Therefore, if we can determine *one particular solution* y_1 of the nonhomogeneous equation, *all* solutions are contained in the formula

$$y = c_1v_1 + c_2v_2 + y_1, \tag{8.31}$$

where c_1 and c_2 are arbitrary constants. Each such y is clearly a solution of $L(y) = R$ because $L(c_1v_1 + c_2v_2 + y_1) = L(c_1v_1 + c_2v_2) + L(y_1) = 0 + R = R$. Since all solutions of $L(y) = R$ are found in (8.31), the linear combination $c_1v_1 + c_2v_2 + y_1$ is called the *general solution* of (8.30). Thus, we have proved the following theorem.

THEOREM 8.8. *If y_1 is a particular solution of the nonhomogeneous equation $L(y) = R$, the general solution is obtained by adding to y_1 the general solution of the corresponding homogeneous equation $L(y) = 0$.*

Theorem 8.7 tells us how to find the general solution of the homogeneous equation $L(y) = 0$. It has the form $y = c_1v_1 + c_2v_2$, where

$$v_1(x) = e^{-ax/2}u_1(x), \qquad v_2(x) = e^{-ax/2}u_2(x), \tag{8.32}$$

the functions u_1 and u_2 being determined by the discriminant of the equation, as described in Theorem 8.7. Now we show that v_1 and v_2 can be used to construct a particular solution y_1 of the nonhomogeneous equation $L(y) = R$.

The construction involves a function W defined by the equation

$$W(x) = v_1(x)v_2'(x) - v_2(x)v_1'(x).$$

This is called the *Wronskian* of v_1 and v_2; some of its properties are described in Exercises 21 and 22 of Section 8.14. We shall need the property that $W(x)$ is never zero. This can be proved by the methods outlined in the exercises or it can be verified directly for the particular functions v_1 and v_2 given in (8.32).

THEOREM 8.9. *Let v_1 and v_2 be the solutions of the equation $L(y) = 0$ given by* (8.32), *where $L(y) = y'' + ay' + by$. Let W denote the Wronskian of v_1 and v_2. Then the nonhomogeneous equation $L(y) = R$ has a particular solution y_1 given by the formula*

$$y_1(x) = t_1(x)v_1(x) + t_2(x)v_2(x),$$

where

$$t_1(x) = -\int v_2(x)\frac{R(x)}{W(x)}\,dx, \qquad t_2(x) = \int v_1(x)\frac{R(x)}{W(x)}\,dx. \tag{8.33}$$

Proof. Let us try to find functions t_1 and t_2 such that the combination $y_1 = t_1v_1 + t_2v_2$ will satisfy the equation $L(y_1) = R$. We have

$$y' = t_1v_1' + t_2v_2' + (t_1'v_1 + t_2'v_2),$$

$$y_1'' = t_1v_1'' + t_2v_2'' + (t_1'v_1' + t_2'v_2') + (t_1'v_1 + t_2'v_2)'.$$

When we form the linear combination $L(y_1) = y_1'' + ay_1' + by_1$, the terms involving t_1 and t_2 drop out because of the relations $L(v_1) = L(v_2) = 0$. The remaining terms give us the relation

$$L(y_1) = (t_1'v_1' + t_2'v_2') + (t_1'v_1 + t_2'v_2)' + a(t_1'v_1 + t_2'v_2).$$

We want to choose t_1 and t_2 so that $L(y_1) = R$. We can satisfy this equation if we choose t_1 and t_2 so that

$$t_1'v_1 + t_2'v_2 = 0 \qquad \text{and} \qquad t_1'v_1' + t_2'v_2' = R.$$

This is a pair of algebraic equations for t_1' and t_2'. The determinant of the system is the Wronskian of v_1 and v_2. Since this is never zero, the system has a solution given by

$$t_1' = -v_2R/W \qquad \text{and} \qquad t_2' = v_1R/W.$$

Integrating these relations, we obtain Equation (8.33), thus completing the proof.

The method by which we obtained the solution y_1 is sometimes called *variation of parameters*. It was first used by Johann Bernoulli in 1697 to solve linear equations of first order, and then by Lagrange in 1774 to solve linear equations of second order.

Note: Since the functions t_1 and t_2 in Theorem 8.9 are expressed as indefinite integrals, each of them is determined only to within an additive constant. If we add a constant c_1 to t_1 and a constant c_2 to t_2 we change the function y_1 to a new function $y_2 = y_1 + c_1v_1 + c_2v_2$. By linearity, we have

$$L(y_2) = L(y_1) + L(c_1v_2 + c_2v_2) = L(y_1),$$

so the new function y_2 is also a particular solution of the nonhomogeneous equation.

EXAMPLE 1. Find the general solution of the equation $y'' + y = \tan x$ on $(-\pi/2, \pi/2)$.

Solution. The functions v_1 and v_2 of Equation (8.32) are given by

$$v_1(x) = \cos x, \qquad v_2(x) = \sin x.$$

Their Wronskian is $W(x) = v_1(x)v_2'(x) - v_2(x)v_1'(x) = \cos^2 x + \sin^2 x = 1$. Therefore Equation (8.33) gives us

$$t_1(x) = -\int \sin x \tan x\, dx = \sin x - \log|\sec x + \tan x|,$$

and

$$t_2(x) = \int \cos x \tan x\, dx = \int \sin x\, dx = -\cos x.$$

Thus, a particular solution of the nonhomogeneous equation is

$$y_1 = t_1(x)v_1(x) + t_2(x)v_2(x) = \sin x\cos x - \cos x \log|\sec x + \tan x| - \sin x \cos x$$

$$= -\cos x \log|\sec x + \tan x|.$$

By Theorem 8.8, its general solution is

$$y = c_1 \cos x + c_2 \sin x - \cos x \log |\sec x + \tan x| .$$

Although Theorem 8.9 provides a general method for determining a particular solution of $L(y) = R$, special methods are available that are often easier to apply when the function R has certain special forms. In the next section we describe a method that works when R is a polynomial or a polynomial times an exponential.

8.16 Special methods for determining a particular solution of the nonhomogeneous equation $y'' + ay' + by = R$

CASE 1. *The right-hand member R is a polynomial of degree n.* If $b \neq 0$, we can always find a polynomial of degree n that satisfies the equation. We try a polynomial of the form

$$y_1(x) = \sum_{k=0}^{n} a_k x^k$$

with undetermined coefficients. Substituting in the differential equation $L(y) = R$ and equating coefficients of like powers of x, we may determine $a_n, a_{n-1}, \ldots, a_1, a_0$ in succession. The method is illustrated by the following example.

EXAMPLE 1. Find the general solution of the equation $y'' + y = x^3$.

Solution. The general solution of the homogeneous equation $y'' + y = 0$ is given by $y = c_1 \cos x + c_2 \sin x$. To this we must add one particular solution of the nonhomogeneous equation. Since the right member is a cubic polynomial and since the coefficient of y is nonzero, we try to find a particular solution of the form $y_1(x) = Ax^3 + Bx^2 + Cx + D$. Differentiating twice, we find that $y''(x) = 6Ax + 2B$. The differential equation leads to the relation

$$(6Ax + 2B) + (Ax^3 + Bx^2 + Cx + D) = x^3 .$$

Equating coefficients of like powers of x, we obtain $A = 1$, $B = 0$, $C = -6$, and $D = 0$, so a particular solution is $y_1(x) = x^3 - 6x$. Thus, the general solution is

$$y = c_1 \cos x + c_2 \sin x + x^3 - 6x .$$

It may be of interest to compare this method with variation of parameters. Equation (8.33) gives us

$$t_1(x) = -\int x^3 \sin x \, dx = -(3x^2 - 6) \sin x + (x^3 - 6x) \cos x$$

and

$$t_2(x) = \int x^3 \cos x \, dx = (3x^2 - 6) \cos x + (x^3 - 6x) \sin x .$$

When we form the combination $t_1 v_1 + t_2 v_2$, we find the particular solution $y_1(x) = x^3 - 6x$, as before. In this case, the use of variation of parameters required the evaluation of the integrals $\int x^3 \sin x \, dx$ and $\int x^3 \cos x \, dx$. With the method of undetermined coefficients, no integration is required.

If the coefficient b is zero, the equation $y'' + ay' = R$ cannot be satisfied by a polynomial of degree n, but it can be satisfied by a polynomial of degree $n + 1$ if $a \neq 0$. If both a and b are zero, the equation becomes $y'' = R$; its general solution is a polynomial of degree $n + 2$ obtained by two successive integrations.

CASE 2. *The right-hand member has the form $R(x) = p(x)e^{mx}$, where p is a polynomial of degree n, and m is constant.*

In this case the change of variable $y = u(x)e^{mx}$ transforms the differential equation $y'' + ay' + by = R$ to a new equation,

$$u'' + (2m + a)u' + (m^2 + am + b)u = p .$$

This is the type discussed in Case 1 so it always has a polynomial solution u_1. Hence, the original equation has a particular solution of the form $y_1 = u_1(x)e^{mx}$, where u_1 is a polynomial. If $m^2 + am + b \neq 0$, the degree of u_1 is the same as the degree of p. If $m^2 + am + b = 0$ but $2m + a \neq 0$, the degree of u_1 is one greater than that of p. If both $m^2 + am + b = 0$ and $2m + a = 0$, the degree of u_1 is two greater than the degree of p.

EXAMPLE 2. Find a particular solution of the equation $y'' + y = xe^{3x}$.

Solution. The change of variable $y = ue^{3x}$ leads to the new equation $u'' + 6u' + 10u = x$. Trying $u_1(x) = Ax + B$, we find the particular solution $u_1(x) = (5x - 3)/50$, so a particular solution of the original equation is $y_1 = e^{3x}(5x - 3)/50$.

The method of undetermined coefficients can also be used if R has the form $R(x) = p(x)e^{mx} \cos \alpha x$, or $R(x) = p(x)e^{mx} \sin \alpha x$, where p is a polynomial and m and α are constants. In either case, there is always a particular solution of the form $y_1(x) = e^{mx}[q(x) \cos \alpha x + r(x) \sin \alpha x]$, where q and r are polynomials.

8.17 Exercises

Find the general solution of each of the differential equations in Exercises 1 through 17. If the solution is not valid over the entire real axis, describe an interval over which it is valid.

1. $y'' - y = x$.
2. $y'' - y' = x^2$.
3. $y'' + y' = x^2 + 2x$.
4. $y'' - 2y' + 3y = x^3$.
5. $y'' - 5y' + 4y = x^2 - 2x + 1$.
6. $y'' + y' - 6y = 2x^3 + 5x^2 - 7x + 2$.
7. $y'' - 4y = e^{2x}$.
8. $y'' + 4y = e^{-2x}$.
9. $y'' + y' - 2y = e^x$.
10. $y'' + y' - 2y = e^{2x}$.
11. $y'' + y' - 2y = e^x + e^{2x}$.
12. $y'' - 2y' + y = x + 2x e^x$.
13. $y'' + 2y' + y = e^{-x}/x^2$.
14. $y'' + y = \cot^2 x$.
15. $y'' - y = 2/(1 + e^x)$.
16. $y'' + y' - 2y = e^x/(1 + e^x)$.
17. $y'' + 6y' + 9y = f(x)$, where $f(x) = 1$ for $1 \leq x \leq 2$, and $f(x) = 0$ for all other x.
18. If k is a nonzero constant, prove that the equation $y'' - k^2 y = R(x)$ has a particular solution y_1 given by

$$y_1 = \frac{1}{k} \int_0^x R(t) \sinh k(x - t) \, dt .$$

Find the general solution of the equation $y'' - 9y = e^{3x}$.

19. If k is a nonzero constant, prove that the equation $y'' + k^2y = R(x)$ has a particular solution y_1 given by

$$y_1 = \frac{1}{k}\int_0^x R(t)\sin k(x - t)\,dt\,.$$

Find the general solution of the equation $y'' + 9y = \sin 3x$.

In each of Exercises 20 through 25, determine the general solution.

20. $y'' + y = \sin x$.
21. $y'' + y = \cos x$.
22. $y'' + 4y = 3x \cos x$.
23. $y'' + 4y = 3x \sin x$.
24. $y'' - 3y' = 2e^{2x} \sin x$.
25. $y'' + y = e^{2x} \cos 3x$.

8.18 Examples of physical problems leading to linear second-order equations with constant coefficients

EXAMPLE 1. *Simple harmonic motion.* Suppose a particle is constrained to move in a straight line with its acceleration directed toward a fixed point of the line and proportional to the displacement from that point. If we take the origin as the fixed point and let y be the displacement at time x, then the acceleration y'' must be negative when y is positive, and positive when y is negative. Therefore we can write $y'' = -k^2y$, or

$$y'' + k^2y = 0\,,$$

where k^2 is a positive constant. This is called the differential equation of *simple harmonic motion*. It is often used as the mathematical model for the motion of a point on a vibrating mechanism such as a plucked string or a vibrating tuning fork. The same equation arises in electric circuit theory where it is called the equation of the harmonic oscillator.

Theorem 8.6 tells us that all solutions have the form

$$y = A \sin kx + B \cos kx\,, \tag{8.34}$$

where A and B are arbitrary constants. We can express the solutions in terms of the sine or cosine alone. For example, we can introduce new constants C and α, where

$$C = \sqrt{A^2 + B^2} \qquad \text{and} \qquad \alpha = \arctan\frac{B}{A}\,,$$

then we have (see Figure 8.4) $A = C\cos\alpha$, $B = C \sin \alpha$, and Equation (8.34) becomes

$$y = C\cos\alpha \sin kx + C \sin\alpha \cos kx = C\sin(kx + \alpha)\,.$$

When the solution is written in this way, the constants C and α have a simple geometric interpretation (see Figure 8.5). The extreme values of y, which occur when $\sin(kx + \alpha) = \pm 1$, are $\pm C$. When $x = 0$, the initial displacement is $C \sin \alpha$. As x increases, the particle oscillates between the extreme values $+C$ and $-C$ with period $2\pi/k$. The angle $kx + \alpha$ is called the *phase angle* and α itself is called the initial value of the phase angle.

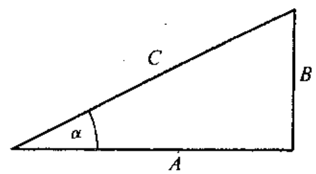


FIGURE 8.4

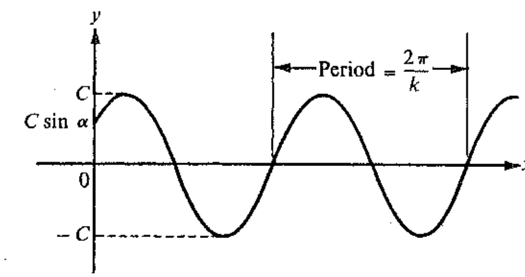


FIGURE 8.5 Simple harmonic motion.

EXAMPLE 2. *Damped vibrations.* If a particle undergoing simple harmonic motion is suddenly subjected to an external force proportional to its velocity, the new motion satisfies a differential equation of the form

$$y'' + 2cy' + k^2y = 0\,,$$

where c and k^2 are constants, $c \neq 0$, $k > 0$. If $c > 0$, we will show that all solutions tend to zero as $x \to +\infty$. In this case, the differential equation is said to be *stable*. The external force causes *damping* of the motion. If $c < 0$, we will show that some solutions have arbitrarily large absolute values as $x \to +\infty$. In this case, the equation is said to be *unstable*.

Since the discriminant of the equation is $d = (2c)^2 - 4k^2 = 4(c^2 - k^2)$, the nature of the solutions is determined by the relative sizes of c^2 and k^2. The three cases $d = 0$, $d > 0$, and $d < 0$ may be analyzed as follows:

(a) *Zero discriminant:* $c^2 = k^2$. In this case, all solutions have the form

$$y = e^{-cx}(A + Bx).$$

If $c > 0$, all solutions tend to 0 as $x \to +\infty$. This case is referred to as *critical damping*. If $B \neq 0$, each solution will change sign exactly once because of the linear factor $A + Bx$. An example is shown in Figure 8.6(a). If $c < 0$, each nontrivial solution tends to $+\infty$ or to $-\infty$ as $x \to +\infty$.

(b) *Positive discriminant:* $c^2 > k^2$. By Theorem 8.7 all solutions have the form

$$y = e^{-cx}(Ae^{hx} + Be^{-hx}) = Ae^{(h-c)x} + Be^{-(h+c)x}\,,$$

where $h = \frac{1}{2}\sqrt{d} = \sqrt{c^2 - k^2}$. Since $h^2 = c^2 - k^2$, we have $h^2 - c^2 < 0$ so $(h - c)(h + c) < 0$. Therefore, the numbers $h - c$ and $h + c$ have opposite signs. If $c > 0$, then $h + c$ is positive so $h - c$ is negative, and hence both exponentials $e^{(h-c)x}$ and $e^{-(h+c)x}$ tend to zero as $x \to +\infty$. In this case, referred to as *overcritical damping*, all solutions tend to 0 for large x. An example is shown in Figure 8.6(a). Each solution can change sign at most once.

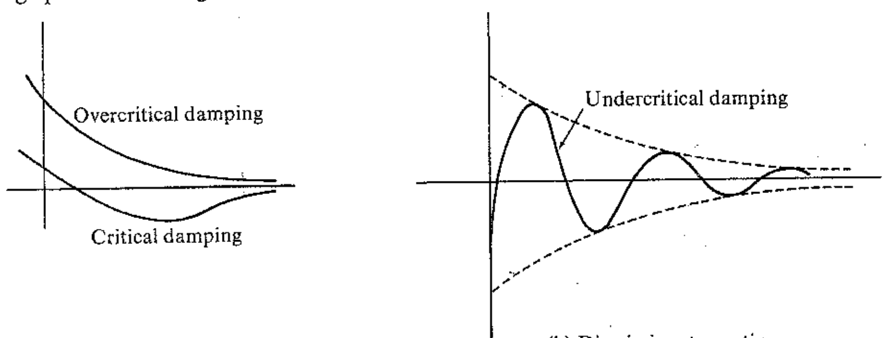
If $c < 0$, then $h - c$ is positive but $h + c$ is negative. Thus, both exponentials $e^{(h-c)x}$

and $e^{-(h+c)x}$ tend to $+\infty$ for large x, so again there are solutions with arbitrarily large absolute values.

(c) *Negative discriminant:* $c^2 < k^2$. In this case, all solutions have the form

$$y = Ce^{-cx} \sin (hx + \alpha),$$

where $h = \frac{1}{2}\sqrt{-d} = \sqrt{k^2 - c^2}$. If $c > 0$, every nontrivial solution oscillates, but the amplitude of the oscillation decreases to 0 as $x \to +\infty$. This case is called *undercritical damping* and is illustrated in Figure 8.6(b). If $c < 0$, all nontrivial solutions take arbitrarily large positive and negative values as $x \to +\infty$.



(a) Discriminant 0 or positive

(b) Discriminant negative

FIGURE 8.6 Damped vibrations occurring as solutions of $y'' + 2cy' + k^2y = 0$, with $c > 0$, and discriminant $4(c^2 - k^2)$.

EXAMPLE 3. *Electric circuits.* If we insert a capacitor in the electric circuit of Example 5 in Section 8.6, the differential equation which serves as a model for this circuit is given by

$$LI'(t) + RI(t) + \frac{1}{C}\int I(t)\,dt = V(t),$$

where C is a positive constant called the *capacitance*. Differentiation of this equation gives a second-order linear equation of the form

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = V'(t).$$

If the impressed voltage $V(t)$ is constant, the right member is zero and the equation takes the form

$$I''(t) + \frac{R}{L}I'(t) + \frac{1}{LC}I(t) = 0.$$

This is the same type of equation analyzed in Example 2 except that $2c$ is replaced by R/L, and k^2 is replaced by $1/(LC)$. In this case, the coefficient c is positive so the equation is always stable. In other words, the current $I(t)$ always tends to 0 as $t \to +\infty$. The terminology of Example 2 is also used here. The current is said to be critically damped when the discriminant is zero ($CR^2 = 4L$), overcritically damped when the discriminant is positive ($CR^2 > 4L$), and undercritically damped when the discriminant is negative ($CR^2 < 4L$).

EXAMPLE 4. *Motion of a rocket with variable mass.* A rocket is propelled by burning fuel in a combustion chamber, allowing the products of combustion to be expelled backward. Assume the rocket starts from rest and moves vertically upward along a straight line. Designate the altitude of the rocket at time t by $r(t)$, the mass of the rocket (including fuel) by $m(t)$, and the velocity of the exhaust matter, relative to the rocket, by $c(t)$. In the absence of external forces, the equation

$$m(t)r''(t) = m'(t)c(t) \tag{8.35}$$

is used as a mathematical model for discussing the motion. The left member, $m(t)r''(t)$, is the product of the mass of the rocket and its acceleration. The right member, $m'(t)c(t)$, is the accelerating force on the rocket caused by the thrust developed by the rocket engine. In the examples to be considered here, $m(t)$ and $c(t)$ are known or can be prescribed in terms of $r(t)$ or its derivative $r'(t)$ (the velocity of the rocket). Equation (8.35) then becomes a second-order differential equation for the position function r.

If external forces are also present, such as gravitational attraction, then, instead of (8.35), we use the equation

$$m(t)r''(t) = m'(t)c(t) + F(t), \tag{8.36}$$

where $F(t)$ represents the sum of all external forces acting on the rocket at time t.

Before we consider a specific example, we will give an argument which may serve to motivate the Equation (8.35). For this purpose we consider first a rocket that fires its exhaust matter intermittently, like bullets from a gun. Specifically, we consider a time interval $[t, t + h]$, where h is a small positive number; we assume that some exhaust matter is expelled at time t, and that no further exhaust matter is expelled in the half-open interval $(t, t + h]$. On the basis of this assumption, we obtain a formula whose limit, as $h \to 0$, is Equation (8.35).

Just before the exhaust material is expelled at time t, the rocket has mass $m(t)$ and velocity $v(t)$. At the end of the time interval $[t, t + h]$, the rocket has mass $m(t + h)$ and velocity $v(t + h)$. The mass of the expelled matter is $m(t) - m(t + h)$, and its velocity during the interval is $v(t) + c(t)$, since $c(t)$ is the velocity of the exhaust relative to the rocket. Just before the exhaust material is expelled at time t, the rocket is a system with momentum $m(t)v(t)$. At time $t + h$, this system consists of two parts, a rocket with momentum $m(t + h)v(t + h)$ and exhaust matter with momentum $[m(t) - m(t + h)][v(t) + c(t)]$. The law of conservation of momentum states that the momentum of the new system must be equal to that of the old. Therefore, we have

$$m(t)v(t) = m(t + h)v(t + h) + [m(t) - m(t + h)][v(t) + c(t)],$$

from which we obtain

$$m(t + h)[v(t + h) - v(t)] = [m(t + h) - m(t)]c(t).$$

Dividing by h and letting $h \to 0$, we find that

$$m(t)v'(t) = m'(t)c(t),$$

which is equivalent to Equation (8.35).

Consider a special case in which the rocket starts from rest with an initial weight of w pounds (including b pounds of fuel) and moves vertically upward along a straight line. Assume the fuel is consumed at a constant rate of k pounds per second and that the products of combustion are discharged directly backward with a constant speed of c feet per second relative to the rocket. Assume the only external force acting on the rocket is the earth's gravitational attraction. We want to know how high the rocket will travel before all its fuel is consumed.

Since all the fuel is consumed when $kt = b$, we restrict t to the interval $0 \le t \le b/k$. The only external force acting on the rocket is $-m(t)g$, the velocity $c(t) = -c$, so Equation (8.36) becomes

$$m(t)r''(t) = -m'(t)c - m(t)g.$$

The weight of the rocket at time t is $w - kt$, and its mass $m(t)$ is $(w - kt)/g$; hence we have $m'(t) = -k/g$ and the foregoing equation becomes

$$r''(t) = -\frac{m'(t)}{m(t)}c - g = \frac{kc}{w - kt} - g.$$

Integrating, and using the initial condition $r'(0) = 0$, we find

$$r'(t) = -c \log \frac{w - kt}{w} - gt.$$

Integrating again and using the initial condition $r(0) = 0$, we obtain the relation

$$r(t) = \frac{c(w - kt)}{k} \log \frac{w - kt}{w} - \frac{1}{2}gt^2 + ct.$$

All the fuel is consumed when $t = b/k$. At that instant the altitude is

$$r\left(\frac{b}{k}\right) = \frac{c(w - b)}{k} \log \frac{w - b}{w} - \frac{1}{2}\frac{gb^2}{k^2} + \frac{cb}{k}. \tag{8.37}$$

This formula is valid if $b < w$. For some rockets, the weight of the carrier is negligible compared to the weight of the fuel, and it is of interest to consider the limiting case $b = w$. We cannot put $b = w$ in (8.37) because of the presence of the term $\log (w - b)/w$. However, if we let $b \to w$, the first term in (8.37) is an indeterminate form with limit 0. Therefore, when $b \to w$, the limiting value of the right member of (8.37) is

$$\lim_{b \to w} r\left(\frac{b}{k}\right) = -\frac{1}{2}\frac{gw^2}{k^2} + \frac{cw}{k} = -\frac{1}{2}gT^2 + cT,$$

where $T = w/k$ is the time required for the entire weight w to be consumed.

8.19 Exercises

In Exercises 1 through 5, a particle is assumed to be moving in simple harmonic motion, according to the equation $y = C \sin (kx + \alpha)$. The *velocity* of the particle is defined to be the derivative y'. The *frequency* of the motion is the reciprocal of the period. (Period $= 2\pi/k$; frequency $= k/2\pi$.) The frequency represents the number of cycles completed in unit time, provided $k > 0$.

1. Find the amplitude C if the frequency is $1/\pi$ and if the initial values of y and y' (when $x = 0$) are 2 and 4, respectively.
2. Find the velocity when y is zero, given that the amplitude is 7 and the frequency is 10.
3. Show that the equation of motion can also be written as follows:
$$y = A \cos (mx + \beta).$$
Find equations that relate the constants A, m, β, and C, k, α.
4. Find the equation of motion given that $y = 3$ and $y' = 0$ when $x = 0$ and that the period is $\frac{1}{2}$.
5. Find the amplitude of the motion if the period is 2π and the velocity is $\pm v_0$ when $y = y_0$.
6. A particle undergoes simple harmonic motion. Initially its displacement is 1, its velocity is 2 and its acceleration is -12. Compute its displacement and acceleration when the velocity is $\sqrt{8}$.
7. For a certain positive number k, the differential equation of simple harmonic motion $y'' + k^2y = 0$ has solutions of the form $y = f(x)$ with $f(0) = f(3) = 0$ and $f(x) < 0$ for all x in the open interval $0 < x < 3$. Compute k and find all solutions.
8. The current $I(t)$ at time t flowing in an electric circuit obeys the differential equation $I''(t) + I(t) = G(t)$, where G is a step function given by $G(t) = 1$ if $0 \le t \le 2\pi$, $G(t) = 0$ for all other t. Determine the solution which satisfies the initial conditions $I(0) = 0$, $I'(0) = 1$.
9. The current $I(t)$ at time t flowing in an electric circuit obeys the differential equation
$$I''(t) + RI'(t) + I(t) = \sin \omega t,$$
where R and ω are positive constants. The solution can be expressed in the form $I(t) = F(t) + A \sin (\omega t + \alpha)$, where $F(t) \to 0$ as $t \to +\infty$, and A and α are constants depending on R and ω, with $A > 0$. If there is a value of ω which makes A as large as possible, then $\omega/(2\pi)$ is called a *resonance frequency* of the circuit.
(a) Find all resonance frequencies when $R = 1$.
(b) Find those values of R for which the circuit will have a resonance frequency.
10. A spaceship is returning to earth. Assume that the only external force acting on it is the action of gravity, and that it falls along a straight line toward the center of the earth. The effect of gravity is partly overcome by firing a rocket directly downward. The rocket fuel is consumed at a constant rate of k pounds per second and the exhaust material has a constant speed of c feet per second relative to the rocket. Find a formula for the distance the spaceship falls in time t if it starts from rest at time $t = 0$ with an initial weight of w pounds.
11. A rocket of initial weight w pounds starts from rest in free space (no external forces) and moves along a straight line. The fuel is consumed at a constant rate of k pounds per second and the products of combustion are discharged directly backward at a constant speed of c feet per second relative to the rocket. Find the distance traveled at time t.
12. Solve Exercise 11 if the initial speed of the rocket is v_0 and if the products of combustion are fired at such a speed that the discharged material remains at rest in space.

8.20 Remarks concerning nonlinear differential equations

Since second-order linear differential equations with constant coefficients occur in such a wide variety of scientific problems, it is indeed fortunate that we have systematic methods for solving these equations. Many nonlinear equations also arise naturally from both physical and geometrical problems, but there is no comprehensive theory comparable to that for linear equations. In the introduction to this chapter we mentioned a classic "bag of tricks" that has been developed for treating many special cases of nonlinear equations. We conclude this chapter with a discussion of some of these tricks and some of the problems they help to solve. We shall consider only first-order equations which can be solved for the derivative y' and expressed in the form

$$y' = f(x, y)\,. \tag{8.38}$$

We recall that a solution of (8.38) on an interval I is any function, say $y = Y(x)$, which is differentiable on I and satisfies the relation $Y'(x) = f[x, Y(x)]$ for all x in I. In the linear case, we proved an existence-uniqueness theorem which tells us that one and only one solution exists satisfying a prescribed initial condition. Moreover, we have an explicit formula for determining this solution.

This is not typical of the general case. A nonlinear equation may have *no* solution satisfying a given initial condition, or it may have *more than one*. For example, the equation $(y')^2 - xy' + y + 1 = 0$ has no solution with $y = 0$ when $x = 0$, since this would require that $(y')^2 = -1$ when $x = 0$. On the other hand, the equation $y' = 3y^{2/3}$ has two distinct solutions, $Y_1(x) = 0$ and $Y_2(x) = x^3$, satisfying the initial condition $y = 0$ when $x = 0$.

Thus, the study of nonlinear equations is more difficult because of the possible nonexistence or nonuniqueness of solutions. Also, even when solutions exist, it may not be possible to determine them explicitly in terms of familiar functions. Sometimes we can eliminate the derivative y' from the differential equation and arrive at a relation of the form

$$F(x, y) = 0$$

satisfied by some, or perhaps all, solutions. If this equation can be solved for y in terms of x, we get an explicit formula for the solution. More often than not, however, the equation is too complicated to solve for y. For example, in a later section we shall study the differential equation

$$y' = \frac{y - x}{y + x},$$

and we shall find that every solution necessarily satisfies the relation

$$\frac{1}{2}\log(x^2 + y^2) + \arctan\frac{y}{x} + C = 0 \tag{8.39}$$

for some constant C. It would be hopeless to try to solve this equation for y in terms of x. In a case like this, we say that the relation (8.39) is an *implicit formula* for the solutions. It is common practice to say that the differential equation has been "solved" or "integrated" when we arrive at an implicit formula such as $F(x, y) = 0$ in which no derivatives of the unknown function appear. Sometimes this formula reveals useful information about the solutions. On the other hand, the reader should realize that such an implicit relation may be less helpful than the differential equation itself for studying properties of the solutions.

In the next section we show how qualitative information about the solutions can often be obtained directly from the differential equation without a knowledge of explicit or implicit formulas for the solutions.

8.21 Integral curves and direction fields

Consider a differential equation of first order, say $y' = f(x, y)$, and suppose some of the solutions satisfy an implicit relation of the form

$$F(x, y, C) = 0\,, \tag{8.40}$$

where C denotes a constant. If we introduce a rectangular coordinate system and plot all the points (x, y) whose coordinates satisfy (8.40) for a particular C, we obtain a curve called an *integral curve* of the differential equation. Different values of C usually give different integral curves, but all of them share a common geometric property. The differential equation $y' = f(x, y)$ relates the slope y' at each point (x, y) of the curve to the coordinates x and y. As C takes on all its values, the collection of integral curves obtained is called a *one-parameter family* of curves.

For example, when the differential equation is $y' = 3$, integration gives us $y = 3x + C$, and the integral curves form a family of straight lines, all having slope 3. The arbitrary constant C represents the y-intercept of these lines.

If the differential equation is $y' = x$, integration yields $y = \frac{1}{2}x^2 + C$, and the integral curves form a family of parabolas as shown in Figure 8.7. Again, the constant C tells us where the various curves cross the y-axis. Figure 8.8 illustrates the family of exponential

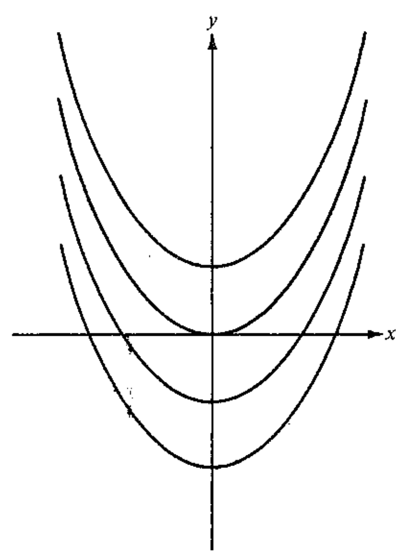


FIGURE 8.7 Integral curves of the differential equation $y' = x$.

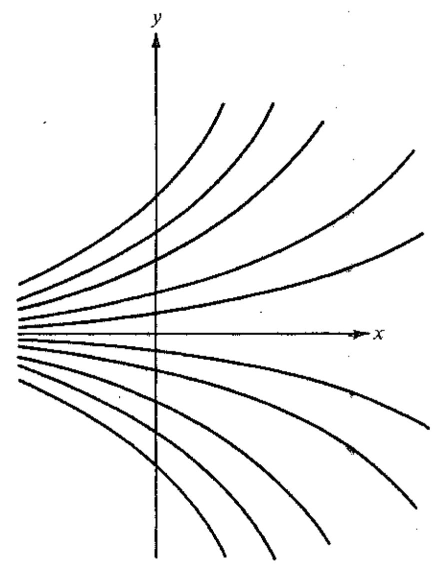


FIGURE 8.8 Integral curves of the differential equation $y' = y$.

curves, $y = Ce^x$, which are integral curves of the differential equation $y' = y$. Once more, C represents the y-intercept. In this case, C is also equal to the slope of the curve at the point where it crosses the y-axis.

A family of nonparallel straight lines is shown in Figure 8.9. These are integral curves of the differential equation

$$y = x\frac{dy}{dx} - \frac{1}{4}\left(\frac{dy}{dx}\right)^2, \tag{8.41}$$

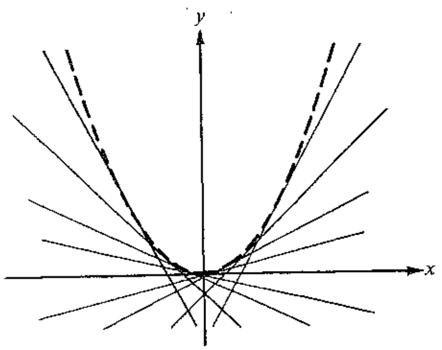


FIGURE 8.9 Integral curves of the differential equation $y = x\dfrac{dy}{dx} - \dfrac{1}{4}\left(\dfrac{dy}{dx}\right)^2$.

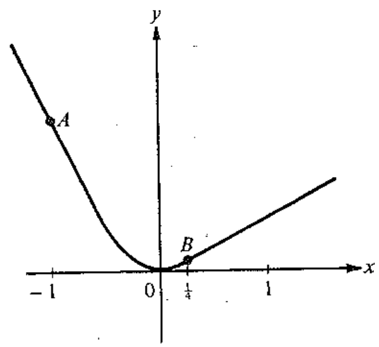


FIGURE 8.10 A solution of Equation (8.41) that is not a member of the family in Equation (8.42).

and a one-parameter family of solutions is given by

$$y = Cx - \tfrac{1}{4}C^2 . \tag{8.42}$$

This family is one which possesses an *envelope*, that is, a curve having the property that at each of its points it is tangent to one of the members of the family.† The envelope here is $y = x^2$ and its graph is indicated by the dotted curve in Figure 8.9. The envelope of a family of integral curves is itself an integral curve because the slope and coordinates at a point of the envelope are the same as those of one of the integral curves of the family. In this example, it is easy to verify directly that $y = x^2$ is a solution of (8.41). Note that this particular solution is not a member of the family in (8.42). Further solutions, not members of the family, may be obtained by piecing together members of the family with portions of the envelope. An example is shown in Figure 8.10. The tangent line at A comes from taking $C = -2$ in (8.42) and the tangent at B comes from $C = \frac{1}{2}$. The resulting solution, $y = f(x)$, is given as follows:

$$f(x) = \begin{cases} -2x - 1 & \text{if } x \le -1 , \\ x^2 & \text{if } -1 \le x \le \frac{1}{4} , \\ \frac{1}{2}x - \frac{1}{16} & \text{if } x \ge \frac{1}{4} . \end{cases}$$

† And conversely, each member of the family is tangent to the envelope.

This function has a derivative and satisfies the differential equation in (8.41) for every real x. It is clear that an infinite number of similar examples could be constructed in the same way. This example shows that it may not be easy to exhibit all possible solutions of a differential equation.

Sometimes it is possible to find a first-order differential equation satisfied by all members of a one-parameter family of curves. We illustrate with two examples.

EXAMPLE 1. Find a first-order differential equation satisfied by all circles with center at the origin.

Solution. A circle with center at the origin and radius C satisfies the equation $x^2 + y^2 = C^2$. As C varies over all positive numbers, we obtain every circle with center at the origin. To find a first-order differential equation having these circles as integral curves, we simply differentiate the Cartesian equation to obtain $2x + 2yy' = 0$. Thus, each circle satisfies the differential equation $y' = -x/y$.

EXAMPLE 2. Find a first-order differential equation for the family of all circles passing through the origin and having their centers on the x-axis.

Solution. If the center of a circle is at $(C, 0)$ and if it passes through the origin, the theorem of Pythagoras tells us that each point (x, y) on the circle satisfies the Cartesian equation $(x - C)^2 + y^2 = C^2$, which can be written as

$$x^2 + y^2 - 2Cx = 0 . \tag{8.43}$$

To find a differential equation having these circles as integral curves, we differentiate (8.43) to obtain $2x + 2yy' - 2C = 0$, or

$$x + yy' = C . \tag{8.44}$$

Since this equation contains C, it is satisfied only by that circle in (8.43) corresponding to the same C. To obtain one differential equation satisfied by all the curves in (8.43), we must eliminate C. We could differentiate (8.44) to obtain $1 + yy'' + (y')^2 = 0$. This is a second-order differential equation satisfied by all the curves in (8.43). We can obtain a first-order equation by eliminating C algebraically from (8.43) and (8.44). Substituting $x + yy'$ for C in (8.43), we obtain $x^2 + y^2 - 2x(x + yy')$, a first-order equation which can be solved for y' and written as $y' = (y^2 - x^2)/(2xy)$.

Figure 8.11 illustrates what is called a *direction field* of a differential equation. This is simply a collection of short line segments drawn tangent to the various integral curves. The particular example shown in Figure 8.11 is a direction field of the equation $y' = y$.

A direction field can be constructed without solving the differential equation. Choose a point, say (a, b), and compute the number $f(a, b)$ obtained by substituting in the righthand side of the differential equation $y' = f(x, y)$. If there is an integral curve through this point, its slope there must be equal to $f(a, b)$. Therefore, if we draw a short line segment through (a, b) having this slope, it will be part of a direction field of the differential equation. By drawing several of these line segments, we can get a fair idea of the general behavior of the

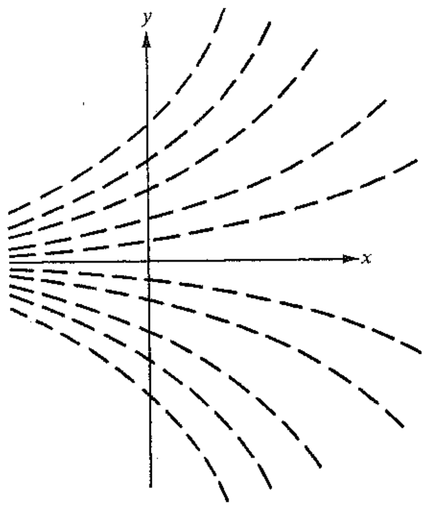


FIGURE 8.11 A direction field for the differential equation $y' = y$.

integral curves. Sometimes such qualitative information about the solution may be all that is needed. Notice that different points $(0, b)$ on the y-axis yield different integral curves. This gives us a geometric reason for expecting an arbitrary constant to appear when we integrate a first-order equation.

8.22 Exercises

In Exercises 1 through 12, find a first-order differential equation having the given family of curves as integral curves.

1. $2x + 3y = C$.
2. $y = Ce^{-2x}$.
3. $x^2 - y^2 = C$.
4. $xy = C$.
5. $y^2 = Cx$.
6. $x^2 + y^2 + 2Cy = 1$.
7. $y = C(x - 1)e^x$.
8. $y^4(x + 2) = C(x - 2)$.
9. $y = C\cos x$.
10. $\arctan y + \arcsin x = C$.
11. All circles through the points $(1, 0)$ and $(-1, 0)$.
12. All circles through the points $(1, 1)$ and $(-1, -1)$.

In the construction of a direction field of a differential equation, sometimes the work may be speeded considerably if we first locate those points at which the slope y' has a constant value C. For each C, these points lie on a curve called an *isocline*.

13. Plot the isoclines corresponding to the constant slopes $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2 for the differential equation $y' = x^2 + y^2$. With the aid of the isoclines, construct a direction field for the equation and try to determine the shape of the integral curve passing through the origin.
14. Show that the isoclines of the differential equation $y' = x + y$ form a one-parameter family of straight lines. Plot the isoclines corresponding to the constant slopes $0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2$. With the aid of the isoclines, construct a direction field and sketch the integral curve passing through the origin. One of the integral curves is also an isocline; find this curve.

15. Plot a number of isoclines and construct a direction field for the equation

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

If you draw the direction field carefully, you should be able to determine a one-parameter family of solutions of this equation from the appearance of the direction field.

8.23 First-order separable equations

A first-order differential equation of the form $y' = f(x, y)$ in which the right member $f(x, y)$ splits into a product of two factors, one depending on x alone and the other depending on y alone, is said to be a *separable equation*. Examples are $y' = x^3$, $y' = y$, $y' = \sin y \log x$, $y' = x/\tan y$, etc. Thus each separable equation can be expressed in the form

$$y' = Q(x)R(y),$$

where Q and R are given functions. When $R(y) \neq 0$, we can divide by $R(y)$ and rewrite this differential equation in the form

$$A(y)y' = Q(x),$$

where $A(y) = 1/R(y)$. The next theorem tells us how to find an implicit formula satisfied by every solution of such an equation.

THEOREM 8.10. *Let $y = Y(x)$ be any solution of the separable differential equation*

$$A(y)y' = Q(x) \tag{8.45}$$

such that Y' is continuous on an open interval I. Assume that both Q and the composite function $A \circ Y$ are continuous on I. Let G be any primitive of A, that is, any function such that $G' = A$. Then the solution Y satisfies the implicit formula

$$G(y) = \int Q(x)\, dx + C \tag{8.46}$$

for some constant C. Conversely, if y satisfies (8.46) *then y is a solution of* (8.45).

Proof. Since Y is a solution of (8.45), we must have

$$A[Y(x)]Y'(x) = Q(x) \tag{8.47}$$

for each x in I. Since $G' = A$, this equation becomes

$$G'[Y(x)]Y'(x) = Q(x).$$

But, by the chain rule, the left member is the derivative of the composite function $G \circ Y$.

Therefore $G \circ Y$ is a primitive of Q, which means that

$$G[Y(x)] = \int Q(x)\,dx + C \tag{8.48}$$

for some constant C. This is the relation (8.46). Conversely, if $y = Y(x)$ satisfies (8.46), differentiation gives us (8.47), which shows that Y is a solution of the differential equation (8.45).

Note: The implicit formula (8.46) can also be expressed in terms of A. From (8.47) we have

$$\int A[Y(x)]\,Y'(x)\,dx = \int Q(x)\,dx + C.$$

If we make the substitution $y = Y(x)$, $dy = Y'(x)\,dx$ in the integral on the left, the equation becomes

$$\int A(y)\,dy = \int Q(x)\,dx + C. \tag{8.49}$$

Since the indefinite integral $\int A(y)\,dy$ represents any primitive of A, Equation (8.49) is an alternative way of writing (8.46).

In practice, formula (8.49) is obtained directly from (8.45) by a mechanical process. In the differential equation (8.45) we write dy/dx for the derivative y' and then treat dy/dx as a fraction to obtain the relation $A(y)\,dy = Q(x)\,dx$. Now we simply attach integral signs to both sides of this equation and add the constant C to obtain (8.49). The justification for this mechanical process is provided by Theorem 8.10. This process is another example illustrating the effectiveness of the Leibniz notation.

EXAMPLE. The nonlinear equation $xy' + y = y^2$ is separable since it can be written in the form

$$\frac{y'}{y(y-1)} = \frac{1}{x}, \tag{8.50}$$

provided that $y(y-1) \neq 0$ and $x \neq 0$. Now the two constant functions $y = 0$ and $y = 1$ are clearly solutions of $xy' + y = y^2$. The remaining solutions, if any exist, satisfy (8.50) and, hence, by Theorem 8.10 they also satisfy

$$\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} + K$$

for some constant K. Since the integrand on the left is $1/(y-1) - 1/y$, when we integrate, we find that

$$\log|y-1| - \log|y| = \log|x| + K.$$

This gives us $|(y-1)/y| = |x|\,e^K$ or $(y-1)/y = Cx$ for some constant C. Solving for y, we obtain the explicit formula

$$y = \frac{1}{1 - Cx}. \tag{8.51}$$

Theorem 8.10 tells us that for any choice of C this y is a solution; therefore, in this example we have determined all solutions: the constant functions $y = 0$ and $y = 1$ and all the functions defined by (8.51). Note that the choice $C = 0$ gives the constant solution $y = 1$.

8.24 Exercises

In Exercises 1 through 12, assume solutions exist and find an implicit formula satisfied by the solutions.

1. $y' = x^3/y^2$.
2. $\tan x \cos y = -y' \tan y$.
3. $(x+1)y' + y^2 = 0$.
4. $y' = (y-1)(y-2)$.
5. $y\sqrt{1-x^2}\,y' = x$.
6. $(x-1)y' = xy$.
7. $(1-x^2)^{1/2}y' + 1 + y^2 = 0$.
8. $xy(1+x^2)y' - (1+y^2) = 0$.
9. $(x^2-4)y' = y$.
10. $xyy' = 1 + x^2 + y^2 + x^2y^2$.
11. $yy' = e^{x+2y}\sin x$.
12. $x\,dx + y\,dy = xy(x\,dy - y\,dx)$.

In Exercises 13 through 16, find functions f, continuous on the whole real axis, which satisfy the conditions given. When it is easy to enumerate all of them, do so; in any case, find as many as you can.

13. $f(x) = 2 + \int_1^x f(t)\,dt$.
14. $f(x)f'(x) = 5x, \quad f(0) = 1$.
15. $f'(x) + 2xe^{f(x)} = 0, \quad f(0) = 0$.
16. $f^2(x) + [f'(x)]^2 = 1$. *Note:* $f(x) = -1$ is one solution.
17. A nonnegative function f, continuous on the whole real axis, has the property that its ordinate set over an arbitrary interval has an area proportional to the length of the interval. Find f.
18. Solve Exercise 17 if the area is proportional to the difference of the function values at the end-points of the interval.
19. Solve Exercise 18 when "difference" is replaced by "sum."
20. Solve Exercise 18 when "difference" is replaced by "product."

8.25 Homogeneous first-order equations

We consider now a special kind of first-order equation,

$$y' = f(x, y), \tag{8.52}$$

in which the right-hand side has a special property known as *homogeneity*. This means that

$$f(tx, ty) = f(x, y) \tag{8.53}$$

for all x, y, and all $t \neq 0$. In other words, replacement of x by tx and y by ty has no effect on the value of $f(x, y)$. Equations of the form (8.52) which have this property are called *homogeneous* (sometimes called *homogeneous of degree zero*). Examples are the following:

$$y' = \frac{y-x}{y+x}, \qquad y' = \left(\frac{x^2+y^2}{xy}\right)^3, \qquad y' = \frac{x}{y}\sin\frac{x^2+y^2}{x^2-y^2}, \qquad y' = \log x - \log y.$$

If we use (8.53) with $t = 1/x$, the differential equation in (8.52) becomes

$$y' = f\left(1, \frac{y}{x}\right). \tag{8.54}$$

The appearance of the quotient y/x on the right suggests that we introduce a new unknown function v where $v = y/x$. Then $y = vx$, $y' = v'x + v$, and this substitution transforms (8.54) into

$$v'x + v = f(1, v) \qquad \text{or} \qquad x\frac{dv}{dx} = f(1, v) - v\,.$$

This last equation is a first-order separable equation for v. We may use Theorem 8.10 to obtain an implicit formula for v and then replace v by y/x to obtain an implicit formula for y.

EXAMPLE. Solve the differential equation $y' = (y - x)/(y + x)$.

Solution. We rewrite the equation as follows:

$$y' = \frac{y/x - 1}{y/x + 1}.$$

The substitution $v = y/x$ transforms this into

$$x\frac{dv}{dx} = \frac{v - 1}{v + 1} - v = -\frac{1 + v^2}{v + 1}.$$

Applying Theorem 8.10, we get

$$\int \frac{v}{1 + v^2}\,dv + \int \frac{1}{1 + v^2}\,dv = -\int \frac{dx}{x} + C\,.$$

Integration yields

$$\tfrac{1}{2}\log(1 + v^2) + \arctan v = -\log|x| + C\,.$$

Replacing v by y/x, we have

$$\tfrac{1}{2}\log(x^2 + y^2) - \tfrac{1}{2}\log x^2 + \arctan\frac{y}{x} = -\log|x| + C\,,$$

and since $\log x^2 = 2\log|x|$, this simplifies to

$$\tfrac{1}{2}\log(x^2 + y^2) + \arctan\frac{y}{x} = C\,.$$

There are some interesting geometric properties possessed by the solutions of a homogeneous equation $y' = f(x, y)$. First of all, it is easy to show that straight lines through the origin are isoclines of the equation. We recall that an isocline of $y' = f(x, y)$ is a curve along which the slope y' is constant. This property is illustrated in Figure 8.12 which shows a direction field of the differential equation $y' = -2y/x$. The isocline corresponding

to slope c has the equation $-2y/x = c$, or $y = -\tfrac{1}{2}cx$ and is therefore a line of slope $-\tfrac{1}{2}c$ through the origin. To prove the property in general, consider a line of slope m through the origin. Then $y = mx$ for all (x, y) on this line; in particular, the point $(1, m)$ is on the line. Suppose now, for the sake of simplicity, that there is an integral curve through each point of the line $y = mx$. The slope of the integral curve through a point (a, b) on this line is $f(a, b) = f(a, ma)$. If $a \neq 0$, we may use the homogeneity property in (8.53) to

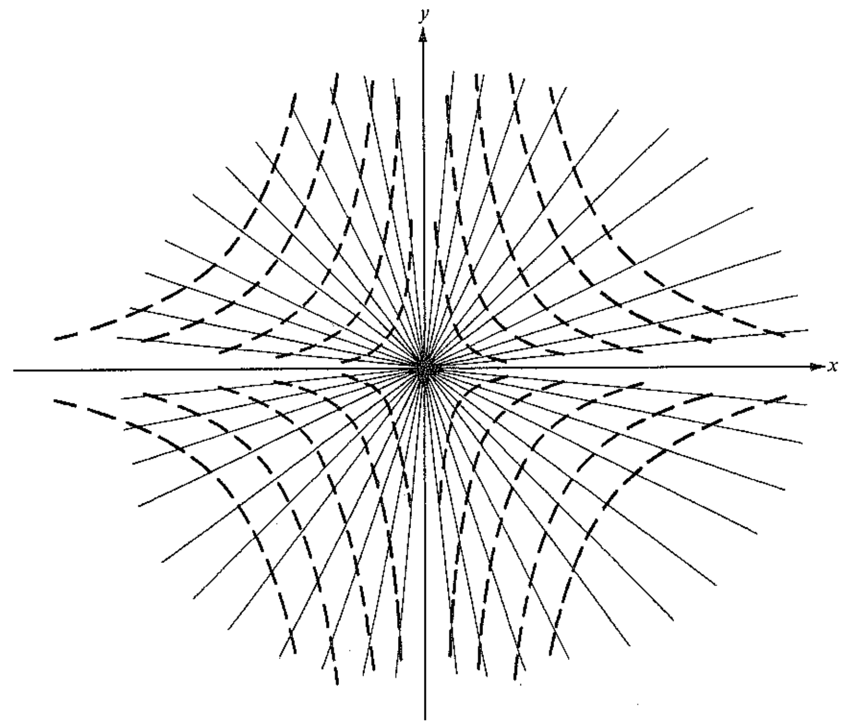


FIGURE 8.12 A direction field for the differential equation $y' = -2y/x$. The isoclines are straight lines through the origin.

write $f(a, ma) = f(1, m)$. In other words, if $(a, b) \neq (0, 0)$, the integral curve through (a, b) has the same slope as the integral curve through $(1, m)$. Therefore the line $y = mx$ is an isocline, as asserted. (It can also be shown that these are the only isoclines of a homogeneous equation.)

This property of the isoclines suggests a property of the integral curves known as *invariance under similarity transformations*. We recall that a similarity transformation carries a set S into a new set kS obtained by multiplying the coordinates of each point of S by a constant factor $k > 0$. Every line through the origin remains fixed under a similarity transformation. Therefore, the isoclines of a homogeneous equation do not change under a similarity transformation; hence the appearance of the direction field does not change either. This suggests that similarity transformations carry integral curves into integral curves. To prove this analytically, let us assume that S is an integral curve described by an explicit formula of the form

$$y = F(x)\,. \tag{8.55}$$

To say that S is an integral curve of $y' = f(x, y)$ means that we have

$$F'(x) = f(x, F(x)) \tag{8.56}$$

for all x under consideration. Now choose any point (x, y) on kS. Then the point $(x/k, y/k)$ lies on S and hence its coordinates satisfy (8.55), so we have $y/k = F(x/k)$ or $y = kF(x/k)$. In other words, the curve kS is described by the equation $y = G(x)$, where $G(x) = kF(x/k)$. Note that the derivative of G is given by

$$G'(x) = kF'\left(\frac{x}{k}\right)\cdot\frac{1}{k} = F'\left(\frac{x}{k}\right).$$

To prove that kS is an integral curve of $y' = f(x, y)$ it will suffice to show that $G'(x) = f(x, G(x))$ or, what is the same thing, that

$$F'\left(\frac{x}{k}\right) = f\left(x, kF\left(\frac{x}{k}\right)\right). \tag{8.57}$$

But if we replace x by x/k in Equation (8.56) and then use the homogeneity property with $t = k$, we obtain

$$F'\left(\frac{x}{k}\right) = f\left(\frac{x}{k}, F\left(\frac{x}{k}\right)\right) = f\left(x, kF\left(\frac{x}{k}\right)\right),$$

and this proves (8.57). In other words, we have shown that kS is an integral curve whenever S is. A simple example in which this geometric property is quite obvious is the homogeneous equation $y' = -x/y$ whose integral curves form a one-parameter family of concentric circles given by the equation $x^2 + y^2 = C$.

It can also be shown that if the integral curves of a first-order equation $y' = f(x, y)$ are invariant under similarity transformations, then the differential equation is necessarily homogeneous.

8.26 Exercises

1. Show that the substitution $y = x/v$ transforms a homogeneous equation $y' = f(x, y)$ into a first-order equation for v which is separable. Sometimes this substitution leads to integrals that are easier to evaluate than those obtained by the substitution $y = xv$ discussed in the text.

Integrate the differential equations in Exercises 2 through 11.

2. $y' = \dfrac{-x}{y}$.
3. $y' = 1 + \dfrac{y}{x}$.
4. $y' = \dfrac{x^2 + 2y^2}{xy}$.
5. $(2y^2 - x^2)y' + 3xy = 0$.

6. $xy' = y - \sqrt{x^2 + y^2}$.
7. $x^2y' + xy + 2y^2 = 0$.
8. $y^2 + (x^2 - xy + y^2)y' = 0$.
9. $y' = \dfrac{y(x^2 + xy + y^2)}{x(x^2 + 3xy + y^2)}$.
10. $y' = \dfrac{y}{x} + \sin\dfrac{y}{x}$.
11. $x(y + 4x)y' + y(x + 4y) = 0$.

8.27 Some geometrical and physical problems leading to first-order equations

We discuss next some examples of geometrical and physical problems that lead to first-order differential equations that are either separable or homogeneous.

Orthogonal trajectories. Two curves are said to intersect *orthogonally* at a point if their tangent lines are perpendicular at that point. A curve which intersects every member of a family of curves orthogonally is called an orthogonal trajectory for the family. Figure 8.13 shows some examples. Problems involving orthogonal trajectories are of importance in both pure and applied mathematics. For example, in the theory of fluid flow, two orthogonal families of curves are called the *equipotential lines* and the *stream lines*, respectively. In the theory of heat, they are known as *isothermal lines* and *lines of flow*.

Suppose a given family of curves satisfies a first-order differential equation, say

$$y' = f(x, y)\,. \tag{8.58}$$

The number $f(x, y)$ is the slope of an integral curve passing through (x, y). The slope of each orthogonal trajectory through this point is the negative reciprocal $-1/f(x, y)$, so the orthogonal trajectories satisfy the differential equation

$$y' = -\frac{1}{f(x, y)}\,. \tag{8.59}$$

If (8.58) is separable, then (8.59) is also separable. If (8.58) is homogeneous, then (8.59) is also homogeneous.

EXAMPLE 1. Find the orthogonal trajectories of the family of all circles through the origin with their centers on the x-axis.

Solution. In Example 2 of Section 8.21 we found that this family is given by the Cartesian equation $x^2 + y^2 - 2Cx = 0$ and that it satisfies the differential equation $y' = (y^2 - x^2)/(2xy)$. Replacing the right member by its negative reciprocal, we find that the orthogonal trajectories satisfy the differential equation

$$y' = \frac{2xy}{x^2 - y^2}\,.$$

This homogeneous equation may be integrated by the substitution $y = vx$, and it leads to the family of integral curves

$$x^2 + y^2 - 2Cy = 0\,.$$

This is a family of circles passing through the origin and having their centers on the y-axis. Examples are shown in Figure 8.13.

Pursuit problems. A point Q is constrained to move along a prescribed plane curve C_1. Another point P in the same plane "pursues" the point Q. That is, P moves in such a manner that its direction of motion is always toward Q. The point P thereby traces out another curve C_2 called a *curve of pursuit*. An example is shown in Figure 8.14 where C_1 is

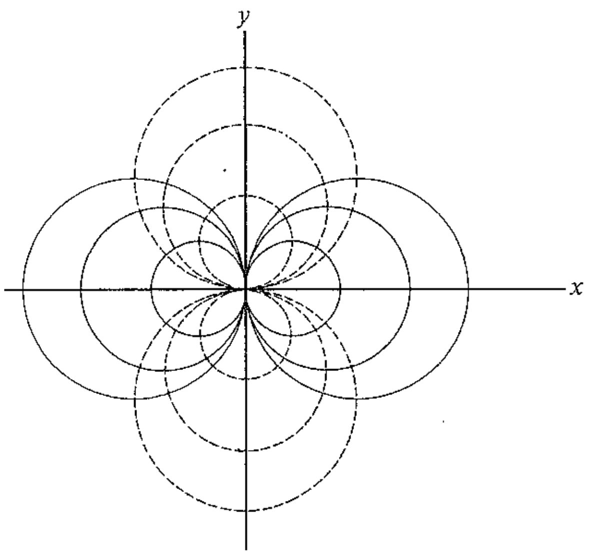


FIGURE 8.13 Orthogonal circles.

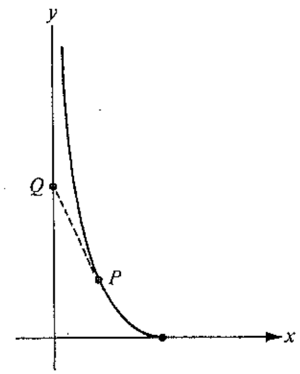


FIGURE 8.14 The tractrix as a curve of pursuit. The distance from P to Q is constant.

the y-axis. In a typical problem of pursuit we seek to determine the curve C_2 when the curve C_1 is known and some additional piece of information is given concerning P and Q, for example, a relation between their positions or their velocities.

When we say that the direction of motion of P is always toward Q, we mean that the tangent line of C_2 through P passes through Q. Therefore, if we denote by (x, y) the rectangular coordinates of P at a given instant, and by (X, Y) those of Q at the same instant, we must have

$$y' = \frac{Y - y}{X - x}. \tag{8.60}$$

The additional piece of information usually enables us to consider X and Y as known functions of x and y, in which case Equation (8.60) becomes a first-order differential equation for y. Now we consider a specific example in which this equation is separable.

EXAMPLE 2. A point Q moves on a straight line C_1, and a point P pursues Q in such a way that the distance from P to Q has a constant value $k > 0$. If P is initially not on C_1, find the curve of pursuit.

Solution. We take C_1 to be the y-axis and place P initially at the point $(k, 0)$. Since the distance from P to Q is k, we must have $(X - x)^2 + (Y - y)^2 = k^2$. But $X = 0$ on C_1, so we have $Y - y = \sqrt{k^2 - x^2}$, and the differential equation (8.60) becomes

$$y' = \frac{\sqrt{k^2 - x^2}}{-x}.$$

Integrating this equation with the help of the substitution $x = k \cos t$ and using the fact that $y = 0$ when $x = k$, we obtain the relation

$$y = k \log \frac{k + \sqrt{k^2 - x^2}}{x} - \sqrt{k^2 - x^2}.$$

The curve of pursuit in this example is called a *tractrix;* it is shown in Figure 8.14.

Flow of fluid through an orifice. Suppose we are given a tank (not necessarily cylindrical) containing a fluid. The fluid flows from the tank through a sharp-edged orifice. If there were no friction (and hence no loss of energy) the speed of the jet would be equal to $\sqrt{2gy}$ feet per second, where y denotes the height (in feet) of the surface above the orifice.† (See Figure 8.15.) If A_0 denotes the area (in square feet) of the orifice, then $A_0\sqrt{2gy}$ represents the number of cubic feet per second of fluid flowing from the orifice. Because of friction, the jet stream contracts somewhat and the actual rate of discharge is more nearly $cA_0\sqrt{2gy}$, where c is an experimentally determined number called the *discharge coefficient*. For ordinary sharp-edged orifices, the approximate value of c is 0.60. Using this and taking $g = 32$, we find that the speed of the jet is $4.8\sqrt{y}$ feet per second, and therefore the rate of discharge of volume is $4.8A_0\sqrt{y}$ cubic feet per second.

Let $V(y)$ denote the volume of the fluid in the tank when the height of the fluid is y. If the cross-sectional area of the tank at the height u is $A(u)$, then we have $V(y) = \int_0^y A(u)\, du$, from which we obtain $dV/dy = A(y)$. The argument in the foregoing paragraph tells us that the rate of change of volume with respect to time is $dV/dt = -4.8A_0\sqrt{y}$ cubic feet per second, the minus sign coming in because the volume is decreasing. By the chain rule we have

$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\frac{dy}{dt}.$$

Combining this with the equation $dV/dt = -4.8A_0\sqrt{y}$, we obtain the differential equation

$$A(y)\frac{dy}{dt} = -4.8A_0\sqrt{y}.$$

† If a particle of mass m falls freely through a distance y and reaches a speed v, its kinetic energy $\frac{1}{2}mv^2$ must be equal to the potential energy mgy (the work done in lifting it up a distance y). Solving for v, we get $v = \sqrt{2gy}$.

This separable differential equation is used as the mathematical model for problems concerning fluid flow through an orifice. The height y of the surface is related to the time t by an equation of the form

$$\int \frac{A(y)}{\sqrt{y}}\, dy = -4.8A_0 \int dt + C. \tag{8.61}$$

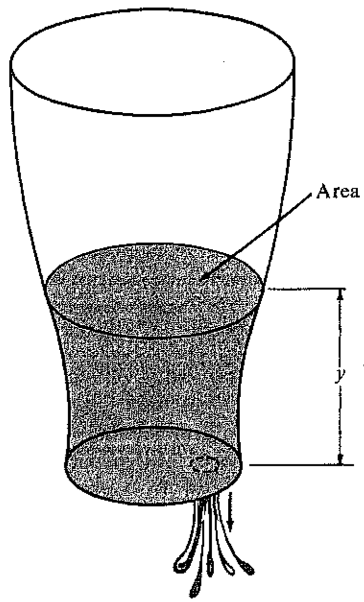


FIGURE 8.15 Flow of fluid through an orifice.

EXAMPLE 3. Consider a specific case in which the cross-sectional area of the tank is constant, say $A(y) = A$ for all y, and suppose the level of the fluid is lowered from 10 feet to 9 feet in 10 minutes (600 seconds). These data can be combined with Equation (8.61) to give us

$$-\int_{10}^{9} \frac{dy}{\sqrt{y}} = k\int_0^{600} dt,$$

where $k = 4.8A_0/A$. Using this, we can determine k and we find that

$$\frac{\sqrt{10} - \sqrt{9}}{\frac{1}{2}} = 600k \qquad \text{or} \qquad k = \frac{\sqrt{10} - 3}{300}.$$

Now we can compute the time required for the level to fall from one given value to any other. For example, if at time t_1 the level is 7 feet and at time t_2 it is 1 foot (t_1, t_2 measured in minutes, say), then we must have

$$-\int_7^1 \frac{dy}{\sqrt{y}} = k\int_{60t_1}^{60t_2} dt,$$

which yields

$$t_2 - t_1 = \frac{2(\sqrt{7} - 1)}{60k} = 10\,\frac{\sqrt{7} - 1}{\sqrt{10} - 3} = \frac{10(\sqrt{7} - 1)(\sqrt{10} + 3)}{10 - 9} = (10)(1.645)(6.162)$$
$$= 101.3 \text{ min.}$$

8.28 Miscellaneous review exercises

In each of Exercises 1 through 10 find the orthogonal trajectories of the given family of curves.

1. $2x + 3y = C.$
2. $xy = C.$
3. $x^2 + y^2 + 2Cy = 1.$
4. $y^2 = Cx.$
5. $x^2y = C.$
6. $y = Ce^{-2x}.$
7. $x^2 - y^2 = C.$
8. $y = C\cos x.$
9. All circles through the points (1, 0) and (−1, 0).
10. All circles through the points (1, 1) and (−1, −1).
11. A point Q moves upward along the positive y-axis. A point P, initially at (1, 0), pursues Q in such a way that its distance from the y-axis is $\frac{1}{2}$ the distance of Q from the origin. Find a Cartesian equation for the path of pursuit.
12. Solve Exercise 11 when the fraction $\frac{1}{2}$ is replaced by an arbitrary positive number k.
13. A curve with Cartesian equation $y = f(x)$ passes through the origin. Lines drawn parallel to the coordinate axes through an arbitrary point of the curve form a rectangle with two sides on the axes. The curve divides every such rectangle into two regions A and B, one of which has an area equal to n times the other. Find the function f.
14. Solve Exercise 13 if the two regions A and B have the property that, when rotated about the x-axis, they sweep out solids one of which has a volume n times that of the other.
15. The graph of a nonnegative differentiable function f passes through the origin and through the point $(1, 2/\pi)$. If, for every $x > 0$, the ordinate set of f above the interval $[0, x]$ sweeps out a solid of volume $x^2f(x)$ when rotated about the x-axis, find the function f.
16. A nonnegative differentiable function f is defined on the closed interval [0, 1] with $f(1) = 0$. For each a, $0 < a < 1$, the line $x = a$ cuts the ordinate set of f into two regions having areas A and B, respectively, A being the area of the leftmost region. If $A - B = 2f(a) + 3a + b$, where b is a constant independent of a, find the function f and the constant b.
17. The graph of a function f passes through the two points $P_0 = (0, 1)$ and $P_1 = (1, 0)$. For every point $P = (x, y)$ on the graph, the curve lies above the chord P_0P, and the area $A(x)$ of the region between the curve and the chord PP_0 is equal to x^3. Determine the function f.
18. A tank with vertical sides has a square cross-section of area 4 square feet. Water is leaving the tank through an orifice of area 5/3 square inches. If the water level is initially 2 feet above the orifice, find the time required for the level to drop 1 foot.
19. Refer to the preceding problem. If water also flows into the tank at the rate of 100 cubic inches per second, show that the water level approaches the value $(25/24)^2$ feet above the orifice, regardless of the initial water level.
20. A tank has the shape of a right circular cone with its vertex up. Find the time required to empty a liquid from the tank through an orifice in its base. Express your result in terms of the dimensions of the cone and the area A_0 of the orifice.

21. The equation $xy'' - y' + (1 - x)y = 0$ possesses a solution of the form $y = e^{mx}$, where m is constant. Determine this solution explicitly.
22. Solve the differential equation $(x + y^3) + 6xy^2y' = 0$ by making a suitable change of variable which converts it into a linear equation.
23. Solve the differential equation $(1 + y^2e^{2x})y' + y = 0$ by introducing a change of variable of the form $y = ue^{mx}$, where m is constant and u is a new unknown function.
24. (a) Given a function f which satisfies the relations

$$2f'(x) = f\left(\frac{1}{x}\right) \quad \text{if } x > 0, \quad f(1) = 2,$$

let $y = f(x)$ and show that y satisfies a differential equation of the form

$$x^2y'' + axy' + by = 0,$$

where a and b are constants. Determine a and b.
(b) Find a solution of the form $f(x) = Cx^n$.

25. (a) Let u be a nonzero solution of the second-order equation

$$y'' + P(x)y' + Q(x)y = 0.$$

Show that the substitution $y = uv$ converts the equation

$$y'' + P(x)y' + Q(x)y = R(x)$$

into a first-order linear equation for v'.
(b) Obtain a nonzero solution of the equation $y'' - 4y' + x^2(y' - 4y) = 0$ by inspection and use the method of part (a) to find a solution of

$$y'' - 4y' + x^2(y' - 4y) = 2xe^{-x^3/3}$$

such that $y = 0$ and $y' = 4$ when $x = 0$.
26. Scientists at the Ajax Atomics Works isolated one gram of a new radioactive element called Deteriorum. It was found to decay at a rate proportional to the *square* of the amount present. After one year, $\frac{1}{2}$ gram remained.
(a) Set up and solve the differential equation for the mass of Deteriorum remaining at time t.
(b) Evaluate the decay constant in units of $gm^{-1}\,yr^{-1}$.
27. In the preceding problem, suppose the word *square* were replaced by *square root*, the other data remaining the same. Show that in this case the substance would decay entirely within a finite time, and find this time.
28. At the beginning of the Gold Rush, the population of Coyote Gulch, Arizona was 365. From then on, the population would have grown by a factor of e each year, except for the high rate of "accidental" death, amounting to one victim per day among every 100 citizens. By solving an appropriate differential equation determine, as functions of time, (a) the actual population of Coyote Gulch t years from the day the Gold Rush began, and (b) the cumulative number of fatalities.
29. With what speed should a rocket be fired upward so that it never returns to earth? (Neglect all forces except the earth's gravitational attraction.)

30. Let $y = f(x)$ be that solution of the differential equation

$$y' = \frac{2y^2 + x}{3y^2 + 5}$$

which satisfies the initial condition $f(0) = 0$. (Do not attempt to solve this differential equation.)
(a) The differential equation shows that $f'(0) = 0$. Discuss whether f has a relative maximum or minimum or neither at 0.
(b) Notice that $f'(x) \geq 0$ for each $x \geq 0$ and that $f'(x) \geq \frac{2}{3}$ for each $x \geq \frac{10}{3}$. Exhibit two positive numbers a and b such that $f(x) > ax - b$ for each $x \geq \frac{10}{3}$.
(c) Show that $x/y^2 \to 0$ as $x \to +\infty$. Give full details of your reasoning.
(d) Show that y/x tends to a finite limit as $x \to +\infty$ and determine this limit.
31. Given a function f which satisfies the differential equation

$$xf''(x) + 3x[f'(x)]^2 = 1 - e^{-x}$$

for all real x. (Do not attempt to solve this differential equation.)
(a) If f has an extremum at a point $c \neq 0$, show that this extremum is a minimum.
(b) If f has an extremum at 0, is it a maximum or a minimum? Justify your conclusion.
(c) If $f(0) = f'(0) = 0$, find the *smallest* constant A such that $f(x) \leq Ax^2$ for all $x \geq 0$.