1.7 Exercises

The properties of area in this set of exercises are to be deduced from the axioms for area stated in the foregoing section.

1. Prove that each of the following sets is measurable and has zero area: (a) A set consisting of a single point. (b) A set consisting of a finite number of points in a plane. (c) The union of a finite collection of line segments in a plane.
2. Every right triangular region is measurable because it can be obtained as the intersection of two rectangles. Prove that every triangular region is measurable and that its area is one half the product of its base and altitude.
3. Prove that every trapezoid and every parallelogram is measurable and derive the usual formulas for their areas.
4. A point (x, y) in the plane is called a *lattice point* if both coordinates x and y are integers. Let P be a polygon whose vertices are lattice points. The area of P is $I + \frac{1}{2}B - 1$, where I denotes the number of lattice points inside the polygon and B denotes the number on the boundary.
   (a) Prove that the formula is valid for rectangles with sides parallel to the coordinate axes.
   (b) Prove that the formula is valid for right triangles and parallelograms.
   (c) Use induction on the number of edges to construct a proof for general polygons.
5. Prove that a triangle whose vertices are lattice points cannot be equilateral.

   [*Hint:* Assume there is such a triangle and compute its area in two ways, using Exercises 2 and 4.]

6. Let $A = \{1, 2, 3, 4, 5\}$, and let $\mathscr{M}$ denote the class of all subsets of A. (There are 32 altogether, counting A itself and the empty set $\varnothing$.) For each set S in $\mathscr{M}$, let $n(S)$ denote the number of distinct elements in S. If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$, compute $n(S \cup T)$, $n(S \cap T)$, $n(S - T)$, and $n(T - S)$. Prove that the set function n satisfies the first three axioms for area.

1.8 Intervals and ordinate sets

In the theory of integration we are concerned primarily with real functions whose domains are intervals on the x-axis. Sometimes it is important to distinguish between intervals which include their endpoints and those which do not. This distinction is made by introducing the following definitions.

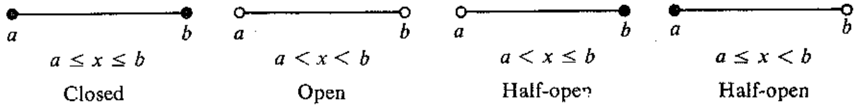


FIGURE 1.16 Examples of intervals.

If $a < b$, we denote by $[a, b]$ the set of all x satisfying the inequalities $a \le x \le b$ and refer to this set as the *closed interval* from a to b. The corresponding *open interval*, written (a, b), is the set of all x satisfying $a < x < b$. The closed interval $[a, b]$ includes the endpoints a and b, whereas the open interval does not. (See Figure 1.16.) The open interval (a, b) is also called the *interior* of $[a, b]$. Half-open intervals $(a, b]$ and $[a, b)$, which include just one endpoint are defined by the inequalities $a < x \le b$ and $a \le x < b$, respectively.

Let f be a nonnegative function whose domain is a closed interval $[a, b]$. The portion of the plane between the graph of f and the x-axis is called the *ordinate set* of f. More precisely, the ordinate set of f is the collection of all points (x, y) satisfying the inequalities

$$a \le x \le b, \qquad 0 \le y \le f(x).$$

In each of the examples shown in Figure 1.17 the shaded portion represents the ordinate set of the corresponding function.

Ordinate sets are the geometric objects whose areas we want to compute by means of the integral calculus. We shall define the concept of integral first for step functions and then use the integral of a step function to formulate the definition of integral for more general functions. Integration theory for step functions is extremely simple and leads in a natural way to the corresponding theory for more general functions. To start this program, it is necessary to have an analytic definition of a step function. This may be given most simply in terms of the concept of a *partition*, to which we turn now.

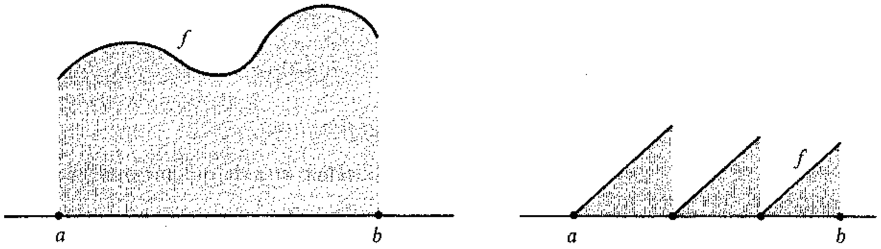


FIGURE 1.17 Examples of ordinate sets.

1.9 Partitions and step functions

Suppose we decompose a given closed interval $[a, b]$ into n subintervals by inserting $n - 1$ points of subdivision, say $x_1, x_2, \ldots, x_{n-1}$, subject only to the restriction

$$a < x_1 < x_2 < \cdots < x_{n-1} < b. \tag{1.2}$$

It is convenient to denote the point a itself by x_0 and the point b by x_n. A collection of points satisfying (1.2) is called a *partition* P of $[a, b]$, and we use the symbol

$$P = \{x_0, x_1, \ldots, x_n\}$$

to designate this partition. The partition P determines n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

A typical closed subinterval is $[x_{k-1}, x_k]$, and it is referred to as the kth closed subinterval of P; an example is shown in Figure 1.18. The corresponding open interval (x_{k-1}, x_k) is called the kth open subinterval of P.

Now we are ready to formulate an analytic definition of a step function.

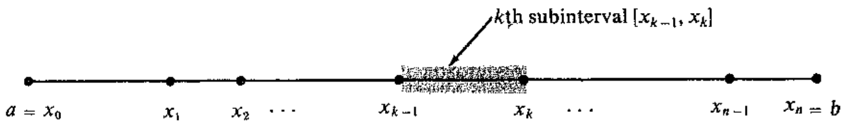


FIGURE 1.18 An example of a partition of $[a, b]$.

DEFINITION OF A STEP FUNCTION. *A function s, whose domain is a closed interval $[a, b]$, is called a step function if there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that s is constant on each open subinterval of P. That is to say, for each $k = 1, 2, \ldots, n$, there is a real number s_k such that*

$$s(x) = s_k \qquad \text{if} \qquad x_{k-1} < x < x_k .$$

Step functions are sometimes called piecewise constant functions.

> *Note:* At each of the endpoints x_{k-1} and x_k the function must have some well-defined value, but this need not be the same as s_k.

EXAMPLE. A familiar example of a step function is the "postage function," whose graph is shown in Figure 1.19. Assume that the charge for first-class mail for parcels weighing up to 20 pounds is 5 cents for every ounce or fraction thereof. The graph shows the number of 5-cent stamps required for mail weighing up to 4 ounces. In this case the line segments on the graph are half-open intervals containing their right endpoints. The domain of the function is the interval [0, 320].

From a given partition P of $[a, b]$, we can always form a new partition P' by adjoining more subdivision points to those already in P. Such a partition P' is called a *refinement* of P and is said to be *finer than* P. For example, $P = \{0, 1, 2, 3, 4\}$ is a partition of the interval [0, 4]. If we adjoint the points 3/4, $\sqrt{2}$, and 7/2, we obtain a new partition P' of [0, 4], namely, $P' = \{0, 3/4, 1, \sqrt{2}, 2, 3, 7/2, 4\}$, which is a refinement of P. (See Figure 1.20.) If a step function is constant on the open subintervals of P, then it is also constant on the open subintervals of every refinement P'.

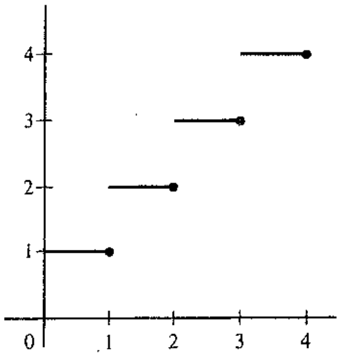


FIGURE 1.19 The postage function.

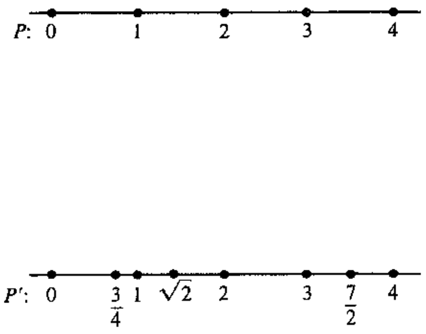


FIGURE 1.20 A partition P of [0, 4] and a refinement P'.

1.10 Sum and product of step functions

New step functions may be formed from given step functions by adding corresponding function values. For example, suppose s and t are step functions, both defined on the same interval $[a, b]$. Let P_1 and P_2 be partitions of $[a, b]$ such that s is constant on the open subintervals of P_1 and t is constant on the open subintervals of P_2. Let $u = s + t$ be the function defined by the equation

$$u(x) = s(x) + t(x) \qquad \text{if} \qquad a \le x \le b .$$

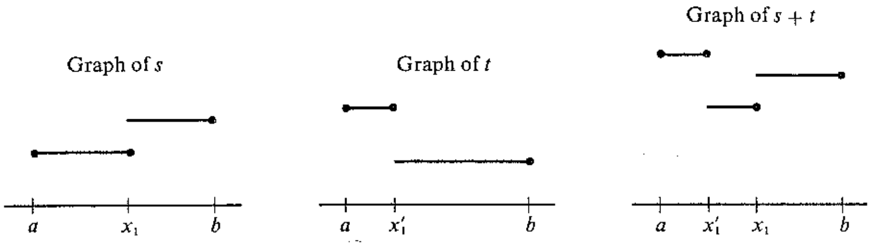


FIGURE 1.21 The sum of two step functions.

To show that u is actually a step function, we must exhibit a partition P such that u is constant on the open subintervals of P. For the new partition P, we take all the points of P_1 along with all the points of P_2. This partition, the union of P_1 and P_2, is called the *common refinement* of P_1 and P_2. Since both s and t are constant on the open subintervals of the common refinement, the same is true of u. An example is illustrated in Figure 1.21. The partition P_1 is $\{a, x_1, b\}$, the partition P_2 is $\{a, x_1', b\}$, and the common refinement is $\{a, x_1', x_1, b\}$.

Similarly, the product $v = s \cdot t$ of two step functions is another step function. An important special case occurs when one of the factors, say t, is constant throughout $[a, b]$. If $t(x) = c$ for each x in $[a, b]$, then each function value $v(x)$ is obtained by multiplying the step function $s(x)$ by the constant c.

1.11 Exercises

In this set of exercises, $[x]$ denotes the greatest integer $\le x$.

1. Let $f(x) = [x]$ and let $g(x) = [2x]$ for all real x. In each case, draw the graph of the function h defined over the interval $[-1, 2]$ by the formula given.
   (a) $h(x) = f(x) + g(x)$. (c) $h(x) = f(x)g(x)$.
   (b) $h(x) = f(x) + g(x/2)$. (d) $h(x) = \frac{1}{4}f(2x)g(x/2)$.
2. In each case, f is a function defined over the interval $[-2, 2]$ by the formula given. Draw the graph of f. If f is a step function, find a partition P of $[-2, 2]$ such that f is constant on the open subintervals of P.

(a) $f(x) = x + [x]$. (d) $f(x) = 2[x]$.
(b) $f(x) = x - [x]$. (e) $f(x) = [x + \frac{1}{2}]$.
(c) $f(x) = [-x]$. (f) $f(x) = [x] + [x + \frac{1}{2}]$.

3. In each case, sketch the graph of the function f defined by the formula given.
   (a) $f(x) = [\sqrt{x}]$ for $0 \le x \le 10$. (c) $f(x) = \sqrt{[x]}$ for $0 \le x \le 10$.
   (b) $f(x) = [x^2]$ for $0 \le x \le 3$. (d) $f(x) = [x]^2$ for $0 \le x \le 3$.
4. Prove that the greatest-integer function has the properties indicated.
   (a) $[x + n] = [x] + n$ for every integer n.
   (b) $[-x] = \begin{cases} -[x] & \text{if } x \text{ is an integer,} \\ -[x] - 1 & \text{otherwise.} \end{cases}$
   (c) $[x + y] = [x] + [y]$ or $[x] + [y] + 1$.
   (d) $[2x] = [x] + [x + \frac{1}{2}]$.
   (e) $[3x] = [x] + [x + \frac{1}{3}] + [x + \frac{2}{3}]$.

Optional exercises.

5. The formulas in Exercises 4(d) and 4(e) suggest a generalization for $[nx]$. State and prove such a generalization.
6. Recall that a lattice point (x, y) in the plane is one whose coordinates are integers. Let f be a nonnegative function whose domain is the interval $[a, b]$, where a and b are integers, $a < b$. Let S denote the set of points (x, y) satisfying $a \le x \le b$, $0 < y \le f(x)$. Prove that the number of lattice points in S is equal to the sum

$$\sum_{n=a}^{b} [f(n)].$$

7. If a and b are positive integers with no common factor, we have the formula

$$\sum_{n=1}^{b-1} \left[\frac{na}{b}\right] = \frac{(a-1)(b-1)}{2}.$$

   When $b = 1$, the sum on the left is understood to be 0.
   (a) Derive this result by a geometric argument, counting lattice points in a right triangle.
   (b) Derive the result analytically as follows: By changing the index of summation, note that $\sum_{n=1}^{b-1} [na/b] = \sum_{n=1}^{b-1} [a(b - n)/b]$. Now apply Exercises 4(a) and (b) to the bracket on the right.
8. Let S be a set of points on the real line. The *characteristic function* of S is, by definition, the function χ_S such that $\chi_S(x) = 1$ for every x in S, and $\chi_S(x) = 0$ for those x not in S. Let f be a step function which takes the constant value c_k on the kth open subinterval I_k of some partition of an interval $[a, b]$. Prove that for each x in the union $I_1 \cup I_2 \cup \cdots \cup I_n$ we have

$$f(x) = \sum_{k=1}^{n} c_k \chi_{I_k}(x).$$

   This property is described by saying that every step function is a linear combination of characteristic functions of intervals.

1.12 The definition of the integral for step functions

In this section we introduce the integral for step functions. The definition is constructed so that the integral of a nonnegative step function is equal to the area of its ordinate set.

Let s be a step function defined on $[a, b]$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that s is constant on the open subintervals of P. Denote by s_k the constant value that s takes in the kth open subinterval, so that

$$s(x) = s_k \quad \text{if} \quad x_{k-1} < x < x_k, \qquad k = 1, 2, \ldots, n.$$

DEFINITION OF THE INTEGRAL OF STEP FUNCTIONS. *The integral of s from a to b, denoted by the symbol $\int_a^b s(x)\,dx$, is defined by the following formula:*

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}). \tag{1.3}$$

That is to say, to compute the integral, we multiply each constant value s_k by the length of the kth subinterval, and then we add together all these products.

Note that the values of s at the subdivision points are immaterial since they do not appear on the right-hand side of (1.3). In particular, if s is constant on the open interval (a, b), say $s(x) = c$ if $a < x < b$, then we have

$$\int_a^b s(x)\,dx = c\sum_{k=1}^{n} (x_k - x_{k-1}) = c(b - a),$$

regardless of the values $s(a)$ and $s(b)$. If $c > 0$ and if $s(x) = c$ for all x in the closed interval $[a, b]$, the ordinate set of s is a rectangle of base $b - a$ and altitude c; the integral of s is $c(b - a)$, the area of this rectangle. Changing the value of s at one or both endpoints a or b changes the ordinate set but does not alter the integral of s or the area of its ordinate set. For example, the two ordinate sets shown in Figure 1.22 have equal areas.

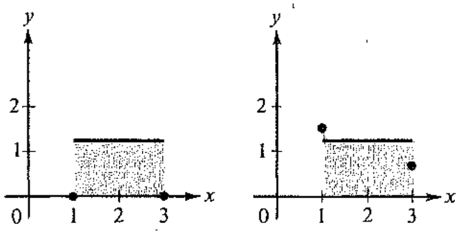


FIGURE 1.22 Changes in function values at two points do not alter area of ordinate set.

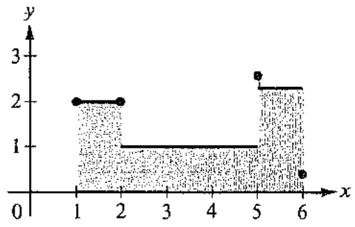


FIGURE 1.23 The ordinate set of a step function.

The ordinate set of any nonnegative step function s consists of a finite number of rectangles, one for each interval of constancy; the ordinate set may also contain or lack certain vertical line segments, depending on how s is defined at the subdivision points. The integral of s is equal to the sum of the areas of the individual rectangles, regardless of the values s takes at the subdivision points. This is consistent with the fact that the vertical segments have zero area and make no contribution to the area of the ordinate set. In Figure 1.23, the step function s takes the constant values 2, 1, and $\frac{9}{4}$ in the open intervals (1, 2), (2, 5), and (5, 6), respectively. Its integral is equal to

$$\int_1^6 s(x)\,dx = 2 \cdot (2 - 1) + 1 \cdot (5 - 2) + \tfrac{9}{4} \cdot (6 - 5) = \tfrac{29}{4}.$$

It should be noted that the formula for the integral in (1.3) is independent of the choice of the partition P as long as s is constant on the open subintervals of P. For example, suppose we change from P to a finer partition P' by inserting exactly one new subdivision point t, where $x_0 < t < x_1$. Then the first term on the right of (1.3) is replaced by the two terms $s_1 \cdot (t - x_0)$ and $s_1 \cdot (x_1 - t)$, and the rest of the terms are unchanged. Since

$$s_1 \cdot (t - x_0) + s_1 \cdot (x_1 - t) = s_1 \cdot (x_1 - x_0),$$

the value of the entire sum is unchanged. We can proceed from P to any finer partition P' by inserting the new subdivision points one at a time. At each stage, the sum in (1.3) remains unchanged, so the integral is the same for all refinements of P.

1.13 Properties of the integral of a step function

In this section we describe a number of fundamental properties satisfied by the integral of a step function. Most of these properties seem obvious when they are interpreted geometrically, and some of them may even seem trivial. All these properties carry over to integrals of more general functions, and it will be a simple matter to prove them in the general case once we have established them for step functions. The properties are listed below as theorems, and in each case a geometric interpretation for nonnegative step functions is given in terms of areas. Analytic proofs of the theorems are outlined in Section 1.15.

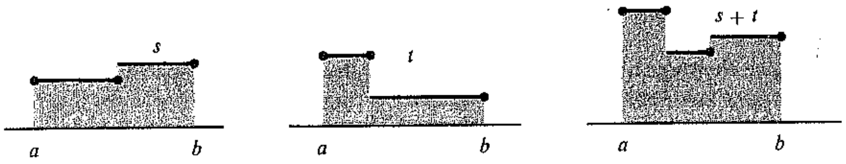


FIGURE 1.24 Illustrating the additive property of the integral.

The first property states that the integral of a sum of two step functions is equal to the sum of the integrals. This is known as the *additive* property and it is illustrated in Figure 1.24.

THEOREM 1.2. ADDITIVE PROPERTY.

$$\int_a^b [s(x) + t(x)]\, dx = \int_a^b s(x)\, dx + \int_a^b t(x)\, dx\,.$$

The next property, illustrated in Figure 1.25, is called the *homogeneous* property. It states that if all the function values are multiplied by a constant c, then the integral is also multiplied by c.

THEOREM 1.3. HOMOGENEOUS PROPERTY. *For every real number c, we have*

$$\int_a^b c \cdot s(x)\, dx = c \int_a^b s(x)\, dx\,.$$

These two theorems can be combined into one formula known as the linearity property.

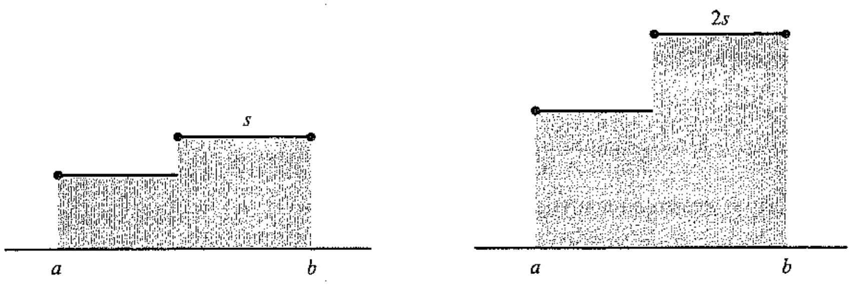


FIGURE 1.25 Illustrating the homogeneous property of the integral (with $c = 2$).

THEOREM 1.4. LINEARITY PROPERTY. *For every real c_1 and c_2, we have*

$$\int_a^b [c_1 s(x) + c_2 t(x)]\, dx = c_1 \int_a^b s(x)\, dx + c_2 \int_a^b t(x)\, dx\,.$$

Next, we have a *comparison* theorem which tells us that if one step function has larger values than another throughout $[a, b]$, its integral over this interval is also larger.

THEOREM 1.5. COMPARISON THEOREM. *If $s(x) < t(x)$ for every x in $[a, b]$, then*

$$\int_a^b s(x)\, dx < \int_a^b t(x)\, dx\,.$$

Interpreted geometrically, this theorem reflects the monotone property of area. If the ordinate set of a nonnegative step function lies inside another, the area of the smaller region is less than that of the larger.

The foregoing properties all refer to step functions defined on a common interval. The integral has further important properties that relate integrals over different intervals. Among these we have the following.

THEOREM 1.6. ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION.

$$\int_a^c s(x)\, dx + \int_c^b s(x)\, dx = \int_a^b s(x)\, dx \qquad \text{if} \qquad a < c < b\,.$$

This theorem reflects the additive property of area, illustrated in Figure 1.26. If an ordinate set is decomposed into two ordinate sets, the sum of the areas of the two parts is equal to the area of the whole.

The next theorem may be described as *invariance under translation*. If the ordinate set of a step function s is "shifted" by an amount c, the resulting ordinate set is that of another step function t related to s by the equation $t(x) = s(x - c)$. If s is defined on $[a, b]$, then t is defined on $[a + c, b + c]$, and their ordinate sets, being congruent, have equal areas.

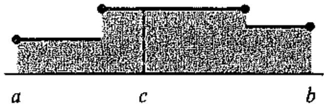


FIGURE 1.26 Additivity with respect to the interval of integration.

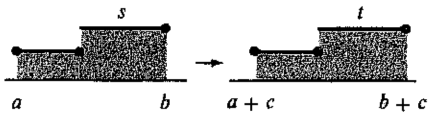


FIGURE 1.27 Illustrating invariance of the integral under translation: $t(x) = s(x - c)$.

This property is expressed analytically as follows:

THEOREM 1.7. INVARIANCE UNDER TRANSLATION.

$$\int_a^b s(x)\,dx = \int_{a+c}^{b+c} s(x-c)\,dx \qquad \textit{for every real } c\,.$$

Its geometric meaning is illustrated in Figure 1.27 for $c > 0$. When $c < 0$, the ordinate set is shifted to the left.

The homogeneous property (Theorem 1.3) explains what happens to an integral under a change of scale on the y-axis. The following theorem deals with a change of scale on the x-axis. If s is a step function defined on an interval $[a, b]$ and if we distort the scale in the horizontal direction by multiplying all x-coordinates by a factor $k > 0$, then the new graph is that of another step function t defined on the interval $[ka, kb]$ and related to s by the equation

$$t(x) = s\left(\frac{x}{k}\right) \qquad \text{if} \qquad ka \le x \le kb\,.$$

An example with $k = 2$ is shown in Figure 1.28 and it suggests that the distorted figure has an area twice that of the original figure. More generally, distortion by a positive factor k has the effect of multiplying the integral by k. Expressed analytically, this property assumes the following form:

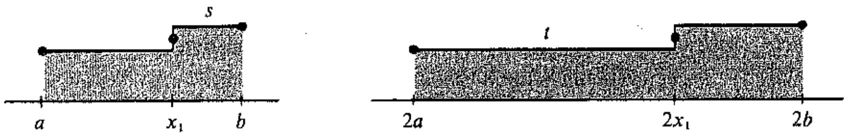


FIGURE 1.28 Change of scale on the x-axis: $t(x) = s(x/2)$.

THEOREM 1.8. EXPANSION OR CONTRACTION OF THE INTERVAL OF INTEGRATION.

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right) dx = k\int_a^b s(x)\,dx \qquad \textit{for every } k > 0\,.$$

Until now, when we have used the symbol $\int_a^b$, it has been understood that the lower limit a was less than the upper limit b. It is convenient to extend our ideas somewhat and consider integrals with a lower limit larger than the upper limit. This is done by defining

(1.4) $$\int_b^a s(x)\,dx = -\int_a^b s(x)\,dx \qquad \text{if} \qquad a < b\,.$$

We also define

$$\int_a^a s(x)\,dx = 0\,,$$

a definition that is suggested by putting $a = b$ in (1.4). These conventions allow us to conclude that Theorem 1.6 is valid not only when c is between a and b but for any arrangement of the points a, b, c. Theorem 1.6 is sometimes written in the form

$$\int_a^c s(x)\,dx + \int_c^b s(x)\,dx + \int_b^a s(x)\,dx = 0\,.$$

Similarly, we can extend the range of validity of Theorem 1.8 and allow the constant k to be negative. In particular, when $k = -1$, Theorem 1.8 and Equation (1.4) give us

$$\int_a^b s(x)\,dx = \int_{-b}^{-a} s(-x)\,dx\,.$$

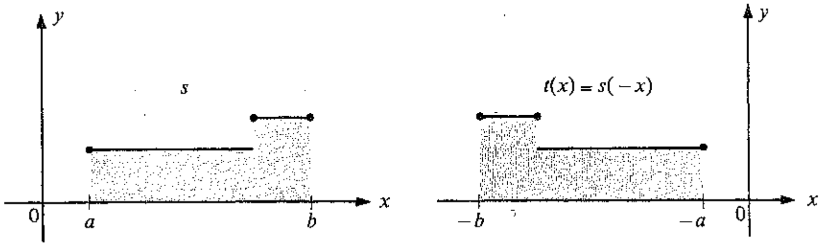


FIGURE 1.29 Illustrating the reflection property of the integral.

We shall refer to this as the *reflection property* of the integral, since the graph of the function t given by $t(x) = s(-x)$ is obtained from that of s by reflection through the y-axis. An example is shown in Figure 1.29.

1.14 Other notations for integrals

The letter x that appears in the symbol $\int_a^b s(x)\,dx$ plays no essential role in the definition of the integral. Any other letter would serve equally well. The letters t, u, v, z are frequently used for this purpose, and it is agreed that instead of $\int_a^b s(x)\,dx$ we may write $\int_a^b s(t)\,dt$, $\int_a^b s(u)\,du$, etc., all these being considered as alternative notations for the same thing. The symbols x, t, u, etc. that are used in this way are called "dummy variables." They are analogous to dummy indices used in the summation notation.

There is a tendency among some authors of calculus textbooks to omit the dummy variable and the d-symbol altogether and to write simply $\int_a^b s$ for the integral. One good reason for using this abbreviated symbol is that it suggests more strongly that the integral depends only on the *function* s and on the *interval* $[a, b]$. Also, certain formulas appear simpler in this notation. For example, the additive property becomes $\int_a^b (s + t) = \int_a^b s + \int_a^b t$. On the other hand, it becomes awkward to write formulas like Theorems 1.7 and 1.8 in the abbreviated notation. More important than this, we shall find later that the original Leibniz notation has certain practical advantages. The symbol dx, which appears to be rather superfluous at this stage, turns out to be an extremely useful computational device in connection with many routine calculations with integrals.

1.15 Exercises

1. Compute the value of each of the following integrals. You may use the theorems of Section 1.13 whenever it is convenient to do so. The notation $[x]$ denotes the greatest integer $\leq x$.
   (a) $\int_{-1}^{3} [x]\,dx.$ (d) $\int_{-1}^{3} 2[x]\,dx.$
   (b) $\int_{-1}^{3} [x + \tfrac{1}{2}]\,dx.$ (e) $\int_{-1}^{3} [2x]\,dx.$
   (c) $\int_{-1}^{3} ([x] + [x + \tfrac{1}{2}])\,dx.$ (f) $\int_{-1}^{3} [-x]\,dx.$
2. Give an example of a step function s, defined on the closed interval $[0, 5]$, which has the following properties: $\int_0^2 s(x)\,dx = 5$, $\int_0^5 s(x)\,dx = 2$.
3. Show that $\int_a^b [x]\,dx + \int_a^b [-x]\,dx = a - b$.
4. (a) If n is a positive integer, prove that $\int_0^n [t]\,dt = n(n-1)/2$.
   (b) If $f(x) = \int_0^x [t]\,dt$ for $x \geq 0$, draw the graph of f over the interval $[0, 4]$.
5. (a) Prove that $\int_0^2 [t^2]\,dt = 5 - \sqrt{2} - \sqrt{3}$.
   (b) Compute $\int_{-3}^{3} [t^2]\,dt$.
6. (a) If n is a positive integer, prove that $\int_0^n [t]^2\,dt = n(n-1)(2n-1)/6$.
   (b) If $f(x) = \int_0^x [t]^2\,dt$ for $x \geq 0$, draw the graph of f over the interval $[0, 3]$.
   (c) Find all $x > 0$ for which $\int_0^x [t]^2\,dt = 2(x - 1)$.
7. (a) Compute $\int_0^9 [\sqrt{t}]\,dt$.
   (b) If n is a positive integer, prove that $\int_0^{n^2} [\sqrt{t}]\,dt = n(n-1)(4n+1)/6$.
8. Show that the translation property (Theorem 1.7) may be expressed in the equivalent form

$$\int_{a+c}^{b+c} f(x)\,dx = \int_a^b f(x + c)\,dx\,.$$

9. Show that the following property is equivalent to Theorem 1.8:

$$\int_{ka}^{kb} f(x)\,dx = k\int_a^b f(kx)\,dx\,.$$

10. Given a positive integer p. A step function s is defined on the interval $[0, p]$ as follows: $s(x) = (-1)^n n$ if x lies in the interval $n \leq x < n + 1$, where $n = 0, 1, 2, \ldots, p - 1$; $s(p) = 0$. Let $f(p) = \int_0^p s(x)\,dx$.
    (a) Calculate $f(3)$, $f(4)$, and $f(f(3))$.
    (b) For what value (or values) of p is $|f(p)| = 7$?
11. If, instead of defining integrals of step functions by using formula (1.3), we used the definition

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k^3 \cdot (x_k - x_{k-1})\,,$$

a new and different theory of integration would result. Which of the following properties would

remain valid in this new theory?
   (a) $\int_a^b s + \int_b^c s = \int_a^c s.$ (c) $\int_a^b c \cdot s = c\int_a^b s.$
   (b) $\int_a^b (s + t) = \int_a^b s + \int_a^b t.$ (d) $\int_{a+c}^{b+c} s(x)\,dx = \int_a^b s(x + c)\,dx.$
   (e) If $s(x) < t(x)$ for each x in $[a, b]$, then $\int_a^b s < \int_a^b t.$
12. Solve Exercise 11 if we use the definition

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k \cdot (x_k^2 - x_{k-1}^2)\,.$$

Analytic proofs of the properties of the integral given in Section 1.13 are requested in t following exercises. The proofs of Theorems 1.3 and 1.8 are worked out here as sampl Hints are given for the others.

Proof of Theorem 1.3: $\int_a^b c \cdot s(x)\,dx = c\int_a^b s(x)\,dx$ for every real c.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that s is constant on the open subinterv of P. Assume $s(x) = s_k$ if $x_{k-1} < x < x_k$ $(k = 1, 2, \ldots, n)$. Then $c \cdot s(x) = c \cdot s_k$ if x_{k-1} $x < x_k$, and hence by the definition of an integral we have

$$\int_a^b c \cdot s(x)\,dx = \sum_{k=1}^{n} c \cdot s_k \cdot (x_k - x_{k-1}) = c\sum_{k=1}^{n} s_k \cdot (x_k - x_{k-1}) = c\int_a^b s(x)\,dx\,.$$

Proof of Theorem 1.8:

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)dx = k\int_a^b s(x)\,dx \qquad \text{if} \quad k > 0\,.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of the interval $[a, b]$ such that s is constant on open subintervals of P. Assume that $s(x) = s_i$ if $x_{i-1} < x < x_i$. Let $t(x) = s(x/k)$ if ka $x \leq kb$. Then $t(x) = s_i$ if x lies in the open interval (kx_{i-1}, kx_i); hence $P' = \{kx_0, kx_1, \ldots$ $kx_n\}$ is a partition of $[ka, kb]$ and t is constant on the open subintervals of P'. Therefore a step function whose integral is

$$\int_{ka}^{kb} t(x)\,dx = \sum_{i=1}^{n} s_i \cdot (kx_i - kx_{i-1}) = k\sum_{i=1}^{n} s_i \cdot (x_i - x_{i-1}) = k\int_a^b s(x)\,dx\,.$$

13. Prove Theorem 1.2 (the additive property).

    [*Hint:* Use the additive property for sums: $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$.]

14. Prove Theorem 1.4 (the linearity property).

    [*Hint:* Use the additive property and the homogeneous property.]

15. Prove Theorem 1.5 (the comparison theorem).

    [*Hint:* Use the corresponding property for sums: $\sum_{k=1}^n a_k < \sum_{k=1}^n b_k$ if $a_k < b_k$ for $k = 1, 2, \ldots, n$.]

16. Prove Theorem 1.6 (additivity with respect to the interval).

[*Hint:* If P_1 is a partition of $[a, c]$ and P_2 a partiton of $[c, b]$, then the points of P_1 along with those of P_2 form a partition of $[a, b]$.]

17. Prove Theorem 1.7 (invariance under translation).

[*Hint:* If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, then $P' = \{x_0 + c, x_1 + c, \ldots, x_n + c\}$ is a partition of $[a + c, b + c]$.]

1.16 The integral of more general functions

The integral $\int_a^b s(x)\,dx$ has been defined when s is a step function. In this section we shall formulate a definition of $\int_a^b f(x)\,dx$ that will apply to more general functions f. The definition will be constructed so that the resulting integral has all the properties listed in Section 1.13.

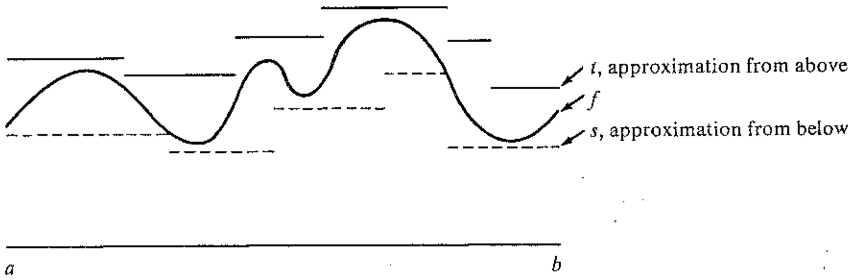


FIGURE 1.30 Approximating a function f from above and below by step functions.

The approach will be patterned somewhat after the method of Archimedes, which was explained above in Section I 1.3. The idea is simply this: We begin by approximating the function f from below and from above by step functions, as suggested in Figure 1.30. That is, we choose an arbitrary step function, say s, whose graph lies below that of f, and a arbitrary step function, say t, whose graph lies above that of f. Next, we consider the collection of all the numbers $\int_a^b s(x)\,dx$ and $\int_a^b t(x)\,dx$ obtained by choosing s and t in all possible ways. In general, we have

$$\int_a^b s(x)\,dx < \int_a^b t(x)\,dx$$

because of the comparison theorem. If the integral of f is to obey the comparison theorem, then it must be a number which falls between $\int_a^b s(x)\,dx$ and $\int_a^b t(x)\,dx$ for every pair of approximating functions s and t. If there is *only one* number which has this property we define the integral of f to be this number.

There is only one thing that can cause trouble in this procedure, and it occurs in the very first step. Unfortunately, it is not possible to approximate *every* function from above and from below by step functions. For example, the function f given by the equations

$$f(x) = \frac{1}{x} \quad \text{if} \quad x \neq 0\,, \qquad f(0) = 0\,,$$

is defined for all real x, but on any interval $[a, b]$ containing the origin we cannot surround f by step functions. This is due to the fact that f has arbitrarily large values near the origin or, as we say, f is *unbounded* in every neighborhood of the origin (see Figure 1.31). Therefore, we shall first restrict ourselves to those functions that are *bounded* on $[a, b]$, that is, to those functions f for which there exists a number $M > 0$ such that

$$-M \leq f(x) \leq M \tag{1.5}$$

for every x in $[a, b]$. Geometrically, the graph of such a function lies between the graphs of two constant step functions s and t having the values $-M$ and $+M$, respectively. (See Figure 1.32.) In a case like this, we say that f is bounded by M. The two inequalities in (1.5) can also be written as

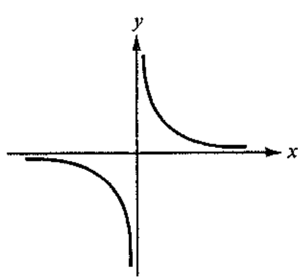


FIGURE 1.31 An unbounded function.

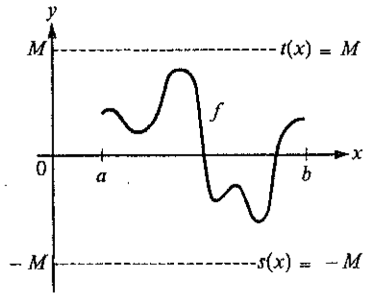


FIGURE 1.32 A bounded function.

$$|f(x)| \leq M\,.$$

With this point taken care of, we can proceed to carry out the plan described above and to formulate the definition of the integral.

DEFINITION OF THE INTEGRAL OF A BOUNDED FUNCTION. *Let f be a function defined and bounded on $[a, b]$. Let s and t denote arbitrary step functions defined on $[a, b]$ such that*

$$s(x) \leq f(x) \leq t(x) \tag{1.6}$$

for every x in $[a, b]$. If there is one and only one number I such that

$$\int_a^b s(x)\,dx \leq I \leq \int_a^b t(x)\,dx \tag{1.7}$$

for every pair of step functions s and t satisfying (1.6), then this number I is called the integral of f from a to b, and is denoted by the symbol $\int_a^b f(x)\,dx$ or by $\int_a^b f$. When such an I exists, the function f is said to be integrable on $[a, b]$.

If $a < b$, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$, provided f is integrable on $[a, b]$. We also define $\int_a^a f(x)\,dx = 0$. If f is integrable on $[a, b]$, we say that the integral $\int_a^b f(x)\,dx$ *exists*. The function f is called the *integrand*, the numbers a and b are called the *limits of integration*, and the interval $[a, b]$ the *interval of integration*.

1.17 Upper and lower integrals

Assume f is bounded on $[a, b]$. If s and t are step functions satisfying (1.6), we say s is *below* f, and t is *above* f, and we write $s \le f \le t$.

Let S denote the set of all numbers $\int_a^b s(x)\,dx$ obtained as s runs through all step functions below f, and let T be the set of all numbers $\int_a^b t(x)\,dx$ obtained as t runs through all step functions above f. That is, let

$$S = \left\{\int_a^b s(x)\,dx \;\middle|\; s \le f\right\}, \qquad T = \left\{\int_a^b t(x)\,dx \;\middle|\; f \le t\right\}.$$

Both sets S and T are nonempty since f is bounded. Also, $\int_a^b s(x)\,dx \le \int_a^b t(x)\,dx$ if $s \le f \le t$, so every number in S is less than every number in T. Therefore, by Theorem I.34, S has a supremum, and T has an infimum, and they satisfy the inequalities

$$\int_a^b s(x)\,dx \le \sup S \le \inf T \le \int_a^b t(x)\,dx$$

for all s and t satisfying $s \le f \le t$. This shows that both numbers $\sup S$ and $\inf T$ satisfy (1.7). Therefore, f is integrable on $[a, b]$ if and only if $\sup S = \inf T$, in which case we have

$$\int_a^b f(x)\,dx = \sup S = \inf T.$$

The number $\sup S$ is called the *lower integral* of f and is denoted by $\underline{I}(f)$. The number $\inf T$ is called the *upper* integral of f and is denoted by $\bar{I}(f)$. Thus, we have

$$\underline{I}(f) = \sup\left\{\int_a^b s(x)\,dx \;\middle|\; s \le f\right\}, \qquad \bar{I}(f) = \inf\left\{\int_a^b t(x)\,dx \;\middle|\; f \le t\right\}.$$

The foregoing argument proves the following theorem.

THEOREM 1.9. *Every function f which is bounded on $[a, b]$ has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ satisfying the inequalities*

$$\int_a^b s(x)\,dx \le \underline{I}(f) \le \bar{I}(f) \le \int_a^b t(x)\,dx$$

for all step functions s and t with $s \le f \le t$. The function f is integrable on $[a, b]$ if and only if its upper and lower integrals are equal, in which case we have

$$\int_a^b f(x)\,dx = \underline{I}(f) = \bar{I}(f).$$

1.18 The area of an ordinate set expressed as an integral

The concept of area was introduced axiomatically in Section 1.6 as a set function having certain properties. From these properties we proved that the area of the ordinate set of a nonnegative step function is equal to the integral of the function. Now we show that the same is true for any integrable nonnegative function. We recall that the ordinate set of a nonnegative function f over an interval $[a, b]$ is the set of all points (x, y) satisfying the inequalities $0 \le y \le f(x)$, $a \le x \le b$.

THEOREM 1.10. *Let f be a nonnegative function, integrable on an interval $[a, b]$, and let Q denote the ordinate set of f over $[a, b]$. Then Q is measurable and its area is equal to the integral $\int_a^b f(x)\,dx$.*

Proof. Let S and T be two step regions satisfying $S \subseteq Q \subseteq T$. Then there are two step functions s and t satisfying $s \le f \le t$ on $[a, b]$, such that

$$a(S) = \int_a^b s(x)\,dx \qquad \text{and} \qquad a(T) = \int_a^b t(x)\,dx.$$

Since f is integrable on $[a, b]$, the number $I = \int_a^b f(x)\,dx$ is the only number satisfying the inequalities

$$\int_a^b s(x)\,dx \le I \le \int_a^b t(x)\,dx$$

for all step functions s and t with $s \le f \le t$. Therefore this is also the only number satisfying $a(S) \le I \le a(T)$ for all step regions S and T with $S \subseteq Q \subseteq T$. By the exhaustion property, this proves that Q is measurable and that $a(Q) = I$.

Let Q denote the ordinate set of Theorem 1.10, and let Q' denote the set that remains if we remove from Q those points on the graph of f. That is, let

$$Q' = \{(x, y) \mid a \le x \le b,\, 0 \le y < f(x)\}.$$

The argument used to prove Theorem 1.10 also shows that Q' is measurable and that $a(Q') = a(Q)$. Therefore, by the difference property of area, the set $Q - Q'$ is measurable and

$$a(Q - Q') = a(Q) - a(Q') = 0.$$

In other words, we have proved the following theorem.

THEOREM 1.11. *Let f be a nonnegative function, integrable on an interval $[a, b]$. Then the graph of f, that is, the set*

$$\{(x, y) \mid a \le x \le b,\, y = f(x)\},$$

is measurable and has area equal to 0.

1.19 Informal remarks on the theory and technique of integration

Two fundamental questions arise at this stage: (1) *Which bounded functions are integrable?* (2) *Given that a function f is integrable, how do we compute the integral of f?*

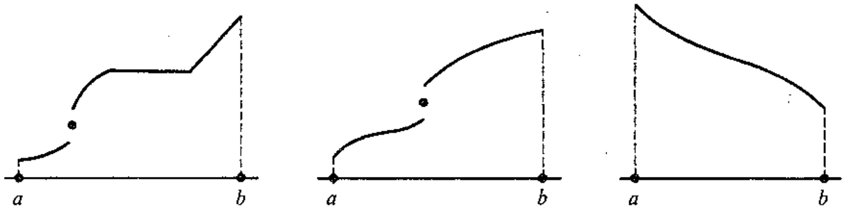
The first question comes under the heading "Theory of Integration" and the second under the heading "Technique of Integration." A complete answer to question (1) lies beyond the scope of an introductory course and will not be given in this book. Instead, we shall give partial answers which require only elementary ideas.

First we introduce an important class of functions known as *monotonic functions.* In the following section we define these functions and give a number of examples. Then we prove that all bounded monotonic functions are integrable. Fortunately, most of the functions that occur in practice are monotonic or sums of monotonic functions, so the results of this miniature theory of integration are quite comprehensive.

The discussion of "Technique of Integration" begins in Section 1.23, where we calculate the integral $\int_0^b x^p\,dx$, when p is a positive integer. Then we develop general properties of the integral, such as linearity and additivity, and show how these properties help us to extend our knowledge of integrals of specific functions.

1.20 Monotonic and piecewise monotonic functions. Definitions and examples

A function f is said to be *increasing* on a set S if $f(x) \le f(y)$ for every pair of points x and y in S with $x < y$. If the strict inequality $f(x) < f(y)$ holds for all $x < y$ in S, the function is said to be *strictly increasing* on S. Similarly, f is called *decreasing* on S if $f(x) \ge f(y)$ for all $x < y$ in S. If $f(x) > f(y)$ for all $x < y$ in S, then f is called *strictly decreasing* on S. A function is called *monotonic* on S if it is increasing on S or if it is decreasing on S. The term *strictly monotonic* means that f is strictly increasing on S or strictly decreasing on S. Ordinarily, the set S under consideration is either an open interval or a closed interval. Examples are shown in Figure 1.33.



Increasing Strictly increasing Strictly decreasing

FIGURE 1.33 Monotonic functions.

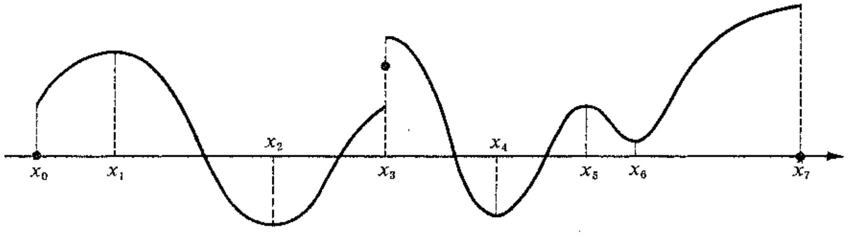


FIGURE 1.34 A piecewise monotonic function.

A function f is said to be *piecewise monotonic* on an interval if its graph consists o finite number of monotonic pieces. That is to say, f is piecewise monotonic on $[a, b$ there is a partition P of $[a, b]$ such that f is monotonic on each of the open subintervals P. In particular, step functions are piecewise monotonic, as are all the examples shown Figures 1.33 and 1.34.

EXAMPLE 1. *The power functions.* If p is a positive integer, we have the inequality

$$x^p < y^p \qquad \text{if} \quad 0 \le x < y,$$

which is easily proved by mathematical induction. This shows that the power functio defined for all real x by the equation $f(x) = x^p$, is strictly increasing on the nonnegat real axis. It is also strictly monotonic on the negative real axis (it is decreasing if p is e and increasing if p is odd). Therefore, f is piecewise monotonic on every finite interval

EXAMPLE 2. *The square-root function.* Let $f(x) = \sqrt{x}$ for $x \ge 0$. This function is stric increasing on the nonnegative real axis. In fact, if $0 \le x < y$, we have

$$\sqrt{y} - \sqrt{x} = \frac{y - x}{\sqrt{y} + \sqrt{x}};$$

hence, $\sqrt{y} - \sqrt{x} > 0$.

EXAMPLE 3. The graph of the function g defined by the equation

$$g(x) = \sqrt{r^2 - x^2} \qquad \text{if} \quad -r \le x \le r$$

is a semicircle of radius r. This function is strictly increasing on the interval $-r \le x$ and strictly decreasing on the interval $0 \le x \le r$. Hence, g is piecewise monotonic $[-r, r]$.

1.21 Integrability of bounded monotonic functions

The importance of monotonic functions in integration theory is due to the follow theorem.

THEOREM 1.12. *If f is monotonic on a closed interval $[a, b]$, then f is integrable on $[a,$*

Proof. We shall prove the theorem for increasing functions. The proof for decreas functions is analogous. Assume f is increasing and let $\underline{I}(f)$ and $\bar{I}(f)$ denote its lower upper integrals, respectively. We shall prove that $\underline{I}(f) = \bar{I}(f)$.

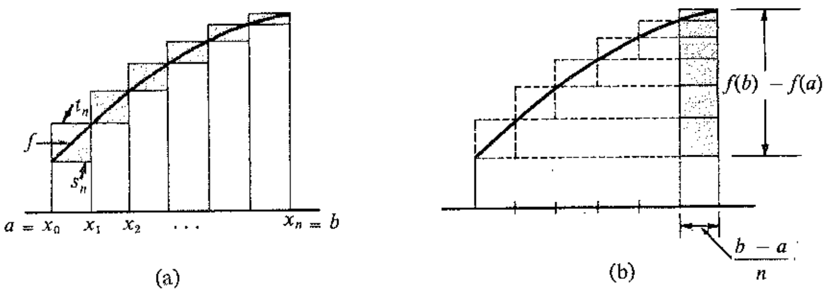
Let n be a positive integer and construct two special approximating step functions s_n t_n as follows: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ into *n equal* subintervals, t is, subintervals $[x_{k-1}, x_k]$ with $x_k - x_{k-1} = (b - a)/n$ for each k. Now define s_n and t_n the formulas

$$s_n(x) = f(x_{k-1}), \qquad t_n(x) = f(x_k) \qquad \text{if} \quad x_{k-1} < x < x_k.$$

At the subdivision points, define s_n and t_n so as to preserve the relations $s_n(x) \le f(x) \le t_n(x)$ throughout $[a, b]$. An example is shown in Figure 1.35(a). For this choice of step functions, we have

$$\begin{aligned}\int_a^b t_n - \int_a^b s_n &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) \\ &= \frac{b-a}{n}\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})] = \frac{(b-a)[f(b) - f(a)]}{n},\end{aligned}$$

where the last equation is a consequence of the telescoping property of finite sums. This last relation has a simple geometric interpretation. The difference $\int_a^b t_n - \int_a^b s_n$ is equal to the sum of the areas of the shaded rectangles in Figure 1.35(a). By sliding these rectangles to the right so that they rest on a common base as in Figure 1.35(b), we see that they fill out a rectangle of base $(b - a)/n$ and altitude $f(b) - f(a)$; the sum of the areas is therefore C/n, where $C = (b - a)[f(b) - f(a)]$.



(a) (b)

FIGURE 1.35 Proof of integrability of an increasing function.

Now we rewrite the foregoing relation in the form

$$\int_a^b t_n - \int_a^b s_n = \frac{C}{n}. \tag{1.8}$$

The lower and upper integrals of f satisfy the inequalities

$$\int_a^b s_n \le \underline{I}(f) \le \int_a^b t_n \qquad \text{and} \qquad \int_a^b s_n \le \overline{I}(f) \le \int_a^b t_n.$$

Multiplying the first set of inequalities by (-1) and adding the result to the second set, we obtain

$$\overline{I}(f) - \underline{I}(f) \le \int_a^b t_n - \int_a^b s_n.$$

Using (1.8) and the relation $\underline{I}(f) \le \overline{I}(f)$, we obtain

$$0 \le \overline{I}(f) - \underline{I}(f) \le \frac{C}{n}$$

for every integer $n \ge 1$. Therefore, by Theorem I.31, we must have $\underline{I}(f) = \overline{I}(f)$. This proves that f is integrable on $[a, b]$.

1.22 Calculation of the integral of a bounded monotonic function

The proof of Theorem 1.12 not only shows that the integral of a bounded increasing function *exists*, but it also suggests a method for computing the value of the integral. This is described by the following theorem.

THEOREM 1.13. *Assume f is increasing on a closed interval $[a, b]$. Let $x_k = a + k(b - a)/n$ for $k = 0, 1, \ldots, n$. If I is any number which satisfies the inequalities*

$$\frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k) \le I \le \frac{b-a}{n}\sum_{k=1}^{n} f(x_k) \tag{1.9}$$

for every integer $n \ge 1$, then $I = \int_a^b f(x)\,dx$.

Proof. Let s_n and t_n be the special approximating step functions obtained by subdivision of the interval $[a, b]$ into n equal parts, as described in the proof of Theorem 1.12. Then, inequalities (1.9) state that

$$\int_a^b s_n \le I \le \int_a^b t_n$$

for every $n \ge 1$. But the integral $\int_a^b f(x)\,dx$ satisfies the same inequalities as I. Using Equation (1.8) we see that

$$0 \le \left| I - \int_a^b f(x)\,dx \right| \le \frac{C}{n}$$

for every integer $n \ge 1$. Therefore, by Theorem I.31, we have $I = \int_a^b f(x)\,dx$, as asserted.

An analogous argument gives a proof of the corresponding theorem for decreasing functions.

THEOREM 1.14. *Assume f is decreasing on $[a, b]$. Let $x_k = a + k(b - a)/n$ for $k = 0, 1, \ldots, n$. If I is any number which satisfies the inequalities*

$$\frac{b-a}{n}\sum_{k=1}^{n} f(x_k) \le I \le \frac{b-a}{n}\sum_{k=0}^{n-1} f(x_k)$$

for every integer $n \ge 1$, then $I = \int_a^b f(x)\,dx$.

1.23 Calculation of the integral $\int_0^b x^p\,dx$ when p is a positive integer

To illustrate the use of Theorem 1.13 we shall calculate the integral $\int_0^b x^p\,dx$ where $b > 0$ and p is any positive integer. The integral exists because the integrand is bounded and increasing on $[0, b]$.

THEOREM 1.15. *If p is a positive integer and $b > 0$, we have*

$$\int_0^b x^p\,dx = \frac{b^{p+1}}{p+1}.$$

Proof. We begin with the inequalities

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$$

valid for every integer $n \geq 1$ and every integer $p \geq 1$. These inequalities may be easily proved by mathematical induction. (A proof is outlined in Exercise 13 of Section I 4.10.) Multiplication of these inequalities by b^{p+1}/n^{p+1} gives us

$$\frac{b}{n}\sum_{k=1}^{n-1}\left(\frac{kb}{n}\right)^p < \frac{b^{p+1}}{p+1} < \frac{b}{n}\sum_{k=1}^{n}\left(\frac{kb}{n}\right)^p.$$

If we let $f(x) = x^p$ and $x_k = kb/n$, for $k = 0, 1, 2, \ldots, n$, these inequalities become

$$\frac{b}{n}\sum_{k=0}^{n-1} f(x_k) < \frac{b^{p+1}}{p+1} < \frac{b}{n}\sum_{k=1}^{n} f(x_k).$$

Therefore, the inequalities (1.9) of Theorem 1.13 are satisfied with $f(x) = x^p$, $a = 0$, and $I = b^{p+1}/(p + 1)$. It follows that $\int_0^b x^p\,dx = b^{p+1}/(p + 1)$.

1.24 The basic properties of the integral

From the definition of the integral, it is possible to deduce the following properties. Proofs are given in Section 1.27.

THEOREM 1.16. LINEARITY WITH RESPECT TO THE INTEGRAND. *If both f and g are integrable on $[a, b]$, so is $c_1 f + c_2 g$ for every pair of constants c_1 and c_2. Furthermore, we have*

$$\int_a^b [c_1 f(x) + c_2 g(x)]\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx.$$

Note: By use of mathematical induction, the linearity property can be generalized as follows: If $f_1, \ldots, f_n$ are integrable on $[a, b]$, then so is $c_1 f_1 + \cdots + c_n f_n$ for all real $c_1, \ldots, c_n$, and

$$\int_a^b \sum_{k=1}^{n} c_k f_k(x)\,dx = \sum_{k=1}^{n} c_k \int_a^b f_k(x)\,dx.$$

THEOREM 1.17. ADDITIVITY WITH RESPECT TO THE INTERVAL OF INTEGRATION. *If two of the following three integrals exist, the third also exists, and we have*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Note: In particular, if f is monotonic on $[a, b]$ and also on $[b, c]$, then both integrals $\int_a^b f$ and $\int_b^c f$ exist, so $\int_a^c f$ also exists and is equal to the sum of the other two integrals.

THEOREM 1.18. INVARIANCE UNDER TRANSLATION. *If f is integrable on $[a, b]$, then fo every real c we have*

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x - c)\,dx.$$

THEOREM 1.19. EXPANSION OR CONTRACTION OF THE INTERVAL OF INTEGRATION. *If f i integrable on $[a, b]$, then for every real $k \neq 0$ we have*

$$\int_a^b f(x)\,dx = \frac{1}{k}\int_{ka}^{kb} f\left(\frac{x}{k}\right)dx.$$

Note: In both Theorems 1.18 and 1.19, the existence of one of the integrals implies th existence of the other. When $k = -1$, Theorem 1.19 is called the *reflection property.*

THEOREM 1.20. COMPARISON THEOREM. *If both f and g are integrable on $[a, b]$ and i $g(x) \leq f(x)$ for every x in $[a, b]$, then we have*

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

An important special case of Theorem 1.20 occurs when $g(x) = 0$ for every x. In thi case, the theorem states that if $f(x) \geq 0$ everywhere on $[a, b]$, then $\int_a^b f(x)\,dx \geq 0$. I other words, a nonnegative function has a nonnegative integral. It can also be show that if we have the *strict* inequality $g(x) < f(x)$ for all x in $[a, b]$, then the same stric inequality holds for the integrals, but the proof is not easy to give at this stage.

In Chapter 5 we shall discuss various methods for calculating the value of an integra without the necessity of using the definition in each case. These methods, however, ar applicable to only a relatively small number of functions, and for most integrable function the actual numerical value of the integral can only be estimated. This is usually done b approximating the integrand above and below by step functions or by other simple function whose integrals can be evaluated exactly. Then the comparison theorem is used to obtai corresponding approximations for the integral of the function in question. This idea wi be explored more fully in Chapter 7.

1.25 Integration of polynomials

In Section 1.23 we established the integration formula

$$\int_0^b x^p\,dx = \frac{b^{p+1}}{p+1} \tag{1.10}$$

for $b > 0$ and p any positive integer. The formula is also valid if $b = 0$, since both membe are zero. We can use Theorem 1.19 to show that (1.10) also holds for negative b. We simply take $k = -1$ in Theorem 1.19 to obtain

$$\int_0^{-b} x^p\,dx = -\int_0^b (-x)^p\,dx = (-1)^{p+1}\int_0^b x^p\,dx = \frac{(-b)^{p+1}}{p+1},$$

which shows that (1.10) holds for negative b. The additive property $\int_a^b x^p\,dx = \int_0^b x^p\,dx - \int_0^a x^p\,dx$ now leads to the more general formula

$$\int_a^b x^p\,dx = \frac{b^{p+1} - a^{p+1}}{p+1},$$

valid for all real a and b, and any integer $p \geq 0$.

Sometimes the special symbol

$$P(x)\Big|_a^b$$

is used to designate the difference $P(b) - P(a)$. Thus the foregoing formula may also be written as follows:

$$\int_a^b x^p\,dx = \frac{x^{p+1}}{p+1}\Big|_a^b = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

This formula, along with the linearity property, enables us to integrate every polynomial. For example, to compute the integral $\int_1^3 (x^2 - 3x + 5)\,dx$, we find the integral of each term and then add the results. Thus, we have

$$\int_1^3 (x^2 - 3x + 5)\,dx = \int_1^3 x^2\,dx - 3\int_1^3 x\,dx + 5\int_1^3 dx = \frac{x^3}{3}\Big|_1^3 - 3\,\frac{x^2}{2}\Big|_1^3 + 5x\Big|_1^3$$

$$= \frac{3^3 - 1^3}{3} - 3\,\frac{3^2 - 1^2}{2} + 5\,\frac{3^1 - 1^1}{1} = \frac{26}{3} - 12 + 10 = \frac{20}{3}.$$

More generally, to compute the integral of any polynomial we integrate term by term:

$$\int_a^b \sum_{k=0}^{n} c_k x^k\,dx = \sum_{k=0}^{n} c_k \int_a^b x^k\,dx = \sum_{k=0}^{n} c_k \frac{b^{k+1} - a^{k+1}}{k+1}.$$

We can also integrate more complicated functions formed by piecing together various polynomials. For example, consider the integral $\int_0^1 |x(2x - 1)|\,dx$. Because of the absolute-value signs, the integrand is not a polynomial. However, by considering the sign of $x(2x - 1)$, we can split the interval $[0, 1]$ into two subintervals, in each of which the integrand is a polynomial. As x varies from 0 to 1, the product $x(2x - 1)$ changes sign at the point $x = \frac{1}{2}$; it is negative if $0 < x < \frac{1}{2}$ and positive if $\frac{1}{2} < x < 1$. Therefore, we use the additive property to write

$$\int_0^1 |x(2x - 1)|\,dx = -\int_0^{1/2} x(2x - 1)\,dx + \int_{1/2}^1 x(2x - 1)\,dx$$

$$= \int_0^{1/2} (x - 2x^2)\,dx + \int_{1/2}^1 (2x^2 - x)\,dx$$

$$= \left(\tfrac{1}{8} - \tfrac{1}{12}\right) + \left(\tfrac{7}{12} - \tfrac{3}{8}\right) = \tfrac{1}{4}.$$

1.26 Exercises

Compute each of the following integrals.

1. $\int_0^3 x^2\,dx.$
2. $\int_{-3}^3 x^2\,dx.$
3. $\int_0^2 4x^3\,dx.$
4. $\int_{-2}^2 4x^3\,dx.$
5. $\int_0^1 5t^4\,dt.$
6. $\int_{-1}^1 5t^4\,dt.$
7. $\int_0^1 (5x^4 - 4x^3)\,dx.$
8. $\int_{-1}^1 (5x^4 - 4x^3)\,dx.$
9. $\int_{-1}^2 (t^2 + 1)\,dt.$
10. $\int_2^3 (3x^2 - 4x + 2)\,dx.$
11. $\int_0^{1/2} (8t^3 + 6t^2 - 2t + 5)\,dt.$
12. $\int_{-2}^4 (u - 1)(u - 2)\,du.$
13. $\int_{-1}^0 (x + 1)^2\,dx.$
14. $\int_0^{-1} (x + 1)^2\,dx.$
15. $\int_0^2 (x - 1)(3x - 1)\,dx.$
16. $\int_0^2 |(x - 1)(3x - 1)|\,dx.$
17. $\int_0^3 (2x - 5)^3\,dx.$
18. $\int_{-3}^3 (x^2 - 3)^3\,dx.$
19. $\int_0^5 x^2(x - 5)^4\,dx.$
20. $\int_{-2}^{-4} (x + 4)^{10}\,dx.$ [*Hint:* Theorem 1.18.]
21. Find all values of c for which

    (a) $\int_0^c x(1 - x)\,dx = 0,$ (b) $\int_0^c |x(1 - x)|\,dx = 0.$
22. Compute each of the following integrals. Draw the graph of f in each case.

    (a) $\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1, \\ 2 - x & \text{if } 1 \leq x \leq 2. \end{cases}$

    (b) $\int_0^1 f(x)\,dx$ where $f(x) = \begin{cases} x & \text{if } 0 \leq x \leq c, \\ c\,\dfrac{1 - x}{1 - c} & \text{if } c \leq x \leq 1; \end{cases}$

    c is a fixed real number, $0 < c < 1$.
23. Find a quadratic polynomial P for which $P(0) = P(1) = 0$ and $\int_0^1 P(x)\,dx = 1$.
24. Find a cubic polynomial P for which $P(0) = P(-2) = 0$, $P(1) = 15$, and $3\int_{-2}^0 P(x)\,dx = 4$

Optional exercises

25. Let f be a function whose domain contains $-x$ whenever it contains x. We say that f is an *even* function if $f(-x) = f(x)$ and an *odd* function if $f(-x) = -f(x)$ for all x in the domain of f. If f is integrable on $[0, b]$, prove that
    (a) $\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx$ if f is even;
    (b) $\int_{-b}^{b} f(x)\,dx = 0$ if f is odd.
26. Use Theorems 1.18 and 1.19 to derive the formula

$$\int_a^b f(x)\,dx = (b-a)\int_0^1 f[a + (b-a)x]\,dx\,.$$

27. Theorems 1.18 and 1.19 suggest a common generalization for the integral $\int_a^b f(Ax+B)\,dx$. Guess the formula suggested and prove it with the help of Theorems 1.18 and 1.19. Discuss also the case $A = 0$.
28. Use Theorems 1.18 and 1.19 to derive the formula

$$\int_a^b f(c-x)\,dx = \int_{c-b}^{c-a} f(x)\,dx\,.$$

1.27 Proofs of the basic properties of the integral

This section contains proofs of the basic properties of the integral listed in Theorems 1.16 through 1.20 in Section 1.24. We make repeated use of the fact that every function f which is bounded on an interval $[a, b]$ has a lower integral $\underline{I}(f)$ and an upper integral $\bar{I}(f)$ given by

$$\underline{I}(f) = \sup\left\{\int_a^b s \;\middle|\; s \le f\right\}, \qquad \bar{I}(g) = \inf\left\{\int_a^b t \;\middle|\; f \le t\right\},$$

where s and t denote arbitrary step functions below and above f, respectively. We know, by Theorem 1.9, that f is integrable if and only if $\underline{I}(f) = \bar{I}(f)$, in which case the value of the ntegral of f is the common value of the upper and lower integrals.

Proof of the Linearity Property (*Theorem* 1.16). We decompose the linearity property into two parts:

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g\,, \tag{A}$$

$$\int_a^b cf = c\int_a^b f\,. \tag{B}$$

To prove (A), let $I(f) = \int_a^b f$ and let $I(g) = \int_a^b g$. We shall prove that $\underline{I}(f+g) = \bar{I}(f+g) = I(f) + I(g)$.

Let s_1 and s_2 denote arbitrary step functions below f and g, respectively. Since f and g are integrable, we have

$$I(f) = \sup\left\{\int_a^b s_1 \;\middle|\; s_1 \le f\right\}, \qquad I(g) = \sup\left\{\int_a^b s_2 \;\middle|\; s_2 \le g\right\}.$$

By the additive property of the supremum (Theorem I.33), we also have

$$I(f) + I(g) = \sup\left\{\int_a^b s_1 + \int_a^b s_2 \;\middle|\; s_1 \le f,\ s_2 \le g\right\}. \tag{1.11}$$

But if $s_1 \le f$ and $s_2 \le g$, then the sum $s = s_1 + s_2$ is a step function below $f + g$, and we have

$$\int_a^b s_1 + \int_a^b s_2 = \int_a^b s \le \underline{I}(f+g)\,.$$

Therefore, the number $\underline{I}(f+g)$ is an upper bound for the set appearing on the right of (1.11). This upper bound cannot be less than the least upper bound of the set, so we have

$$I(f) + I(g) \le \underline{I}(f+g)\,. \tag{1.12}$$

Similarly, if we use the relations

$$I(f) = \inf\left\{\int_a^b t_1 \;\middle|\; f \le t_1\right\}, \qquad I(g) = \inf\left\{\int_a^b t_2 \;\middle|\; g \le t_2\right\},$$

where t_1 and t_2 denote arbitrary step functions above f and g, respectively, we obtain the inequality

$$\bar{I}(f+g) \le I(f) + I(g)\,. \tag{1.13}$$

Inequalities (1.12) and (1.13) together show that $\underline{I}(f+g) = \bar{I}(f+g) = I(f) + I(g)$. Therefore $f + g$ is integrable and relation (A) holds.

Relation (B) is trivial if $c = 0$. If $c > 0$, we note that every step function s_1 below cf is of the form $s_1 = cs$, where s is a step function below f. Similarly, every step function t_1 above cf is of the form $t_1 = ct$, where t is a step function above f. Therefore we have

$$\underline{I}(cf) = \sup\left\{\int_a^b s_1 \;\middle|\; s_1 \le cf\right\} = \sup\left\{c\int_a^b s \;\middle|\; s \le f\right\} = cI(f)$$

and

$$\bar{I}(cf) = \inf\left\{\int_a^b t_1 \;\middle|\; cf \le t_1\right\} = \inf\left\{c\int_a^b t \;\middle|\; f \le t\right\} = cI(f)\,.$$

Therefore $\underline{I}(cf) = \bar{I}(cf) = cI(f)$. Here we have used the following properties of the supremum and infimum:

$$\sup\{cx \mid x \in A\} = c\sup\{x \mid x \in A\}, \qquad \inf\{cx \mid x \in A\} = c\inf\{x \mid x \in A\}, \tag{1.14}$$

which hold if $c > 0$. This proves (B) if $c > 0$.

If $c < 0$, the proof of (B) is basically the same, except that every step function s_1 below cf is of the form $s_1 = ct$, where t is a step function *above* f, and every step function t_1 above cf is of the form $t_1 = cs$, where s is a step function *below* f. Also, instead of (1.14) we use the relations

$$\sup\{cx \mid x \in A\} = c\inf\{x \mid x \in A\}, \qquad \inf\{cx \mid x \in A\} = c\sup\{x \mid x \in A\},$$

which hold if $c < 0$. We now have

$$\underline{I}(cf) = \sup\left\{\int_a^b s_1 \,\middle|\, s_1 \le cf\right\} = \sup\left\{c\int_a^b t \,\middle|\, f \le t\right\} = c\inf\left\{\int_a^b t \,\middle|\, f \le t\right\} = c\bar{I}(f)\,.$$

Similarly, we find $\bar{I}(cf) = c\underline{I}(f)$. Therefore (B) holds for all real c.

Proof of Additivity with Respect to the Interval of Integration (*Theorem* 1.17). Suppose that $a < b < c$, and assume that the two integrals $\int_a^b f$ and $\int_b^c f$ exist. Let $\underline{I}(f)$ and $\bar{I}(f)$ denote the upper and lower integrals of f over the interval $[a, c]$. We shall prove that

$$\underline{I}(f) = \bar{I}(f) = \int_a^b f + \int_b^c f\,. \tag{1.15}$$

If s is any step function below f on $[a, c]$, we have

$$\int_a^c s = \int_a^b s + \int_b^c s\,.$$

Conversely, if s_1 and s_2 are step functions below f on $[a, b]$ and on $[b, c]$, respectively, then the function s which is equal to s_1 on $[a, b)$ and equal to s_2 on $[b, c]$ is a step function below f on $[a, c]$ for which we have

$$\int_a^c s = \int_a^b s_1 + \int_b^c s_2\,.$$

Therefore, by the additive property of the supremum (Theorem 1.33), we have

$$\underline{I}(f) = \sup\left\{\int_a^c s \,\middle|\, s \le f\right\} = \sup\left\{\int_a^b s_1 \,\middle|\, s_1 \le f\right\} + \sup\left\{\int_b^c s_2 \,\middle|\, s_2 \le f\right\} = \int_a^b f + \int_b^c f\,.$$

Similarly, we find

$$\bar{I}(f) = \int_a^b f + \int_b^c f\,,$$

which proves (1.15) when $a < b < c$. The proof is similar for any other arrangement of the points a, b, c.

Proof of the Translation Property (*Theorem* 1.18). Let g be the function defined on the interval $[a + c, b + c]$ by the equation $g(x) = f(x - c)$. Let $\underline{I}(g)$ and $\bar{I}(g)$ denote the lower and upper integrals of g on the interval $[a + c, b + c]$. We shall prove that

$$\underline{I}(g) = \bar{I}(g) = \int_a^b f(x)\,dx\,. \tag{1.16}$$

Let s be any step function below g on the interval $[a + c, b + c]$. Then the function s_1 defined on $[a, b]$ by the equation $s_1(x) = s(x + c)$ is a step function below f on $[a, b]$. Moreover, every step function s_1 below f on $[a, b]$ has this form for some s below g. Also, by the translation property for integrals of step functions, we have

$$\int_{a+c}^{b+c} s(x)\,dx = \int_a^b s(x + c)\,dx = \int_a^b s_1(x)\,dx\,.$$

Therefore we have

$$\underline{I}(g) = \sup\left\{\int_{a+c}^{b+c} s \,\middle|\, s \le g\right\} = \sup\left\{\int_a^b s_1 \,\middle|\, s_1 \le f\right\} = \int_a^b f(x)\,dx\,.$$

Similarly, we find $\bar{I}(g) = \int_a^b f(x)\,dx$, which proves (1.16).

Proof of the Expansion Property (*Theorem* 1.19). Assume $k > 0$ and define g on the interval $[ka, kb]$ by the equation $g(x) = f(x/k)$. Let $\underline{I}(g)$ and $\bar{I}(g)$ denote the lower and upper integrals of g on $[ka, kb]$. We shall prove that

$$\underline{I}(g) = \bar{I}(g) = k\int_a^b f(x)\,dx\,. \tag{1.17}$$

Let s be any step function below g on $[ka, kb]$. Then the function s_1 defined on $[a, b]$ by the equation $s_1(x) = s(kx)$ is a step function below f on $[a, b]$. Moreover, every step function s_1 below f on $[a, b]$ has this form. Also, by the expansion property for integrals of step functions, we have

$$\int_{ka}^{kb} s(x)\,dx = k\int_a^b s(kx)\,dx = k\int_a^b s_1(x)\,dx\,.$$

Therefore we have

$$\underline{I}(g) = \sup\left\{\int_{ka}^{kb} s \,\middle|\, s \le g\right\} = \sup\left\{k\int_a^b s_1 \,\middle|\, s_1 \le f\right\} = k\int_a^b f(x)\,dx\,.$$

Similarly, we find $\bar{I}(g) = k\int_a^b f(x)\,dx$, which proves (1.17) if $k > 0$. The same type of proof can be used if $k < 0$.

Proof of the Comparison Theorem (*Theorem* 1.20). Assume $g \le f$ on the interval $[a, b]$. Let s be any step function below g, and let t be any step function above f. Then we have $\int_a^b s \le \int_a^b t$, and hence Theorem I.34 gives us

$$\int_a^b g = \sup\left\{\int_a^b s \,\middle|\, s \le g\right\} \le \inf\left\{\int_a^b t \,\middle|\, f \le t\right\} = \int_a^b f\,.$$

This proves that $\int_a^b g \le \int_a^b f$, as required.