Young's, Minkowski's, and Hölder's inequalities

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1 Introduction

The Cauchy inequality is the familiar expression

$$2ab \le a^2 + b^2. \tag{1}$$

This can be proven very simply: noting that $(a-b)^2 \geq 0$, we have

$$0 \le (a-b)^2 = a^2 - 2ab - b^2 \tag{2}$$

which, after rearranging terms, is precisely the Cauchy inequality. In this note, we prove Young's inequality, which is a version of the Cauchy inequality that lets the power of 2 be replaced by the power of p for any $1 < p < \infty$. From Young's inequality follow the Minkowski inequality (the triangle inequality for the l^p-norms), and the Hölder inequalities.

2 Young's Inequality

When $1 < p < \infty$ and $a, b \geq 0$, Young's inequality is the expression

$$ab \;\leq\; \frac{p-1}{p}\, a^{\frac{p}{p-1}} \;+\; \frac{1}{p}\, b^p. \tag{3}$$

This seems strange and complicated. What good could it possibly be?

The first thing to note is Young's inequality is a far-reaching generalization of Cauchy's inequality. In particular, if $p = 2$, then $\frac{1}{p} = \frac{p-1}{p} = \frac{1}{2}$ and we have Cauchy's inequality:

$$ab \;\leq\; \frac{1}{2}a^2 \;+\; \frac{1}{2}b^2. \tag{4}$$

Normally to use Young's inequality one chooses a specific p, and a and b are free-floating quantities. For instance, if $p = 5$, we get

$$ab \;\leq\; \frac{4}{5}\, a^{5/4} \;+\; \frac{1}{5}\, b^5.$$

Before proving Young's inequality, we require a certain fact about the exponential function.

Lemma 2.1 (The interpolation inequality for e^x.) *If $t \in [0, 1]$, then*

$$e^{ta+(1-t)b} \leq te^a + (1-t)e^b. \tag{5}$$

Proof. The equation of the secant line through the points (a, e^a) and (b, e^b) on the graph of e^x is precisely

$$t \;\mapsto\; \left(ta \,+\, (1-t)b,\; te^a \,+\, (1-t)e^b\right). \tag{6}$$

Obviously the graph of this line intersects the graph of e^x at precisely two points: (b, e^b) when $t = 0$, and (a, e^a) when $t = 1$. To parametrize the graph of e^x so that the x-value of this parametrization and that of the parametrization of the secant line are the same, we use

$$t \;\mapsto\; \Big(ta \,+\, (1-t)b,\; e^{ta\,+\,(1-t)b}\Big). \tag{7}$$

But because e^x is concave up, any secant line lies above the graph in between the points of intersection. This means precisely that the y-values of these two parametrized curves obey

$$e^{ta\,+\,(1-t)b} \;\leq\; t\,e^a \,+\, (1-t)\,e^b, \tag{8}$$

which was to be proved. □

Theorem 2.2 (Young's Inequality) *Assume a and b are real numbers, and $p > 1$. Then*

$$a\,b \;\leq\; \frac{p-1}{p}\, a^{\frac{p}{p-1}} \;+\; \frac{1}{p}\, b^p.$$

Proof. There are a number of conceptually different ways to prove this inequality. Our method will use Lemma 2.1. Writing

$$ab = e^{\log a + \log b} \tag{9}$$

$$= Exp\left(\frac{p-1}{p}\frac{p}{p-1}\log a + \left(1-\frac{p-1}{p}\right)\left(\frac{1}{1-\frac{p-1}{p}}\right)\log b\right), \tag{10}$$

from Lemma 2.1 we get

$$ab \leq \frac{p-1}{p}Exp\left(\frac{p}{p-1}\log a\right) + \left(1-\frac{p-1}{p}\right)Exp\left(\left(\frac{1}{1-\frac{p-1}{p}}\right)\log b\right) \tag{11}$$

$$= \frac{p-1}{p}Exp\left(\frac{p}{p-1}\log a\right) + \frac{1}{p}Exp\left(p\log b\right) \tag{12}$$

$$= \frac{p-1}{p}\, a^{\frac{p}{p-1}} + \frac{1}{p}\, b^{p}. \tag{13}$$

□

3 Minkowski's Inequality

Theorem 3.1 (Minkowski's Inequality) *If* $1 \leq p < \infty$*, then whenever* $X, Y \in \mathcal{V}_F$ *we have*

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p. \tag{14}$$

Proof. To prove that $\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$, we will replace Y by tY, and use the observation that

$$\|X+Y\|_p - \|X\|_p = \int_0^1 \frac{d}{dt}\|X+tY\|_p\, dt \tag{15}$$

$$\|X\| + \|Y\|_p - \|X\|_p = \int_0^1 \frac{d}{dt}\left(\|X\|_p + t\|Y\|_p\right)\, dt \tag{16}$$

and then all we need to prove is that

$$\frac{d}{dt}\|X+tY\|_p \leq \frac{d}{dt}\left(\|X\|_p + t\|Y\|_p\right), \tag{17}$$

which is actually simpler. Note that the right side of (17) is just $\|Y\|_p$. Computing the left side is slightly tougher:

$$\frac{d}{dt}\|X+tY\|_p = \frac{d}{dt}\left(\sum_{i=1}^{\infty}|x_i - ty_i|^p\right)^{\frac{1}{p}} \tag{18}$$

$$= \left(\sum_{i=1}^{\infty}|x_i - ty_i|^p\right)^{\frac{1-p}{p}} \sum_{i=1}^{\infty}|x_i - ty_i|^{p-1}\cdot sgn(x_i - ty_i)\cdot y_i \tag{19}$$

$$= \|X-tY\|_p^{1-p}\cdot\sum_{i=1}^{\infty}|x_i - ty_i|^{p-1}\cdot sgn(x_i-ty_i)\cdot y_i. \tag{20}$$

But of course $sgn(x_i - ty_i)\cdot y_i \leq |y_i|$, so we have

$$\frac{d}{dt}\|X+tY\|_p \leq \sum_{i=1}^{\infty}\left(\frac{|x_i - ty_i|}{\|X-tY\|_p}\right)^{p-1}|y_i|. \tag{21}$$

To proceed from here, we manipulate this expression so that eventually we can use Young's inequality to our advantage. We have

$$\frac{d}{dt}\|X+tY\|_p \leq \sum_{i=1}^{\infty}\left(\frac{|x_i - ty_i|}{\|X-tY\|_p}\right)^{p-1}\frac{|y_i|}{\|Y\|_p^{\frac{p-1}{p}}}\|Y\|_p^{\frac{p-1}{p}} \tag{22}$$

$$= \sum_{i=1}^{\infty}\left(\frac{|x_i - ty_i|}{\|X-tY\|_p}\|Y\|_p^{\frac{1}{p}}\right)^{p-1}\cdot\frac{|y_i|}{\|Y\|_p^{\frac{p-1}{p}}}. \tag{23}$$

When $p=1$ we get directly that

$$\frac{d}{dt}\|X+tY\|_1 \leq \sum_{i=1}^{\infty}|y_i| \tag{24}$$

$$= \|Y\|_1 \tag{25}$$

$$= \frac{d}{dt}\left(\|X\|_1 + t\|Y\|_1\right) \tag{26}$$

as desired. When $1 < p < \infty$ we apply Young's inequality to get

$$\frac{d}{dt}\|X+tY\|_p \leq \sum_{i=1}^{\infty}\left(\frac{p-1}{p}\left(\frac{|x_i - ty_i|}{\|X-tY\|_p}\|Y\|_p^{\frac{1}{p}}\right)^{(p-1)\frac{p}{p-1}} + \frac{1}{p}\left(\frac{|y_i|}{\|Y\|_p^{\frac{p-1}{p}}}\right)^p\right) \tag{27}$$

$$= \frac{p-1}{p}\sum_{i=1}^{\infty}\frac{|x_i - ty_i|^p}{\|X-tY\|_p^p}\|Y\|_p + \frac{1}{p}\sum_{i=1}^{\infty}\frac{|y_i|^p}{\|Y\|_p^{p-1}} \tag{28}$$

$$= \frac{p-1}{p}\left(\frac{\|Y\|_p}{\|X-tY\|_p^p}\cdot\sum_{i=1}^{\infty}|x_i - ty_i|^p\right) + \frac{1}{p}\left(\frac{1}{\|Y\|_p^{p-1}}\cdot\sum_{i=1}^{\infty}|y_i|^p\right) \tag{29}$$

Finally note that $\sum_{i=1}^{\infty} |x_i - ty_i|^p$ equals precisely $\|X - tY\|_p^p$ and $\sum_{i=1}^{\infty} |y_i|^p$ equals precisely $\|Y\|_p^p$. Therefore

$$\begin{aligned}
\frac{d}{dt}\|X+tY\|_p &\leq \frac{p-1}{p}\left(\frac{\|Y\|_p}{\|X-tY\|_p^p}\cdot\|X-tY\|_p^p\right)+\frac{1}{p}\left(\frac{1}{\|Y\|_p^{p-1}}\cdot\|Y\|_p^p\right) && (30)\\
&= \frac{p-1}{p}\|Y\|_p+\frac{1}{p}\|Y\|_p && (31)\\
&= \|Y\|_p. && (32)
\end{aligned}$$

Therefore, as desired, we have proved that

$$\frac{d}{dt}\|X+tY\|_p \leq \frac{d}{dt}\left(\|X\|_p + t\|Y\|_p\right), \qquad (33)$$

so the theorem follows from (15) and (16). □

4 Hölder's inequality

Theorem 4.1 (Hölder's inequality) *If $X, Y \in \mathcal{V}_F$, then*

$$\sum_{i=1}^{\infty} x_i y_i \leq \|X\|_{\frac{p}{p-1}}\|Y\|_p. \qquad (34)$$

<u>*Proof*</u>. By Young's inequality we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{x_i}{\|X\|_{\frac{p}{p-1}}}\frac{y_i}{\|Y\|_p} &\leq \sum_{i=1}^{\infty}\frac{|x_i|}{\|X\|_{\frac{p}{p-1}}}\frac{|y_i|}{\|Y\|_p} && (35)\\
&\leq \sum_{i=1}^{\infty}\left(\frac{p-1}{p}\frac{|x_i|^{\frac{p}{p-1}}}{\|X\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}}+\frac{1}{p}\frac{|y_i|^p}{\|Y\|_p^p}\right) && (36)\\
&= \frac{p-1}{p}\frac{1}{\|X\|_{\frac{p}{p-1}}^{\frac{p}{p-1}}}\sum_{i=1}^{\infty}|x_i|^{\frac{p}{p-1}}+\frac{1}{p}\frac{1}{\|Y\|_p^p}\sum_{i=1}^{\infty}|y_i|^p && (37)\\
&= \frac{p-1}{p}+\frac{1}{p} && (38)
\end{aligned}$$

Thus we have shown that

$$\begin{aligned}
\frac{1}{\|X\|_{\frac{p}{p-1}}\|Y\|_p}\sum_{i=1}^{\infty}x_i y_i &= \sum_{i=1}^{\infty}\frac{x_i}{\|X\|_{\frac{p}{p-1}}}\frac{y_i}{\|Y\|_p} && (39)\\
&\leq 1, && (40)
\end{aligned}$$

so after multiplying both sides by $\|X\|_{\frac{p}{p-1}} \|Y\|_p$ we get

$$\sum_{i=1}^{\infty} x_i y_i \quad \leq \quad \|X\|_{\frac{p}{p-1}} \|Y\|_p \tag{41}$$

which was to be proved. □