Lecture 2 - Vector Spaces, Norms, and Cauchy Sequences

September 13, 2011

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1 Vector Spaces

1.1 Definitions

A set $\mathcal{X}$ is called a vector space if it has an addition operation, denoted $x + y$ for $x, y \in \mathcal{X}$, that satisfies

- Closure: $x + y \in \mathcal{V}$ when $x, y \in \mathcal{X}$
- Commutativity: $x + y = y + x$
- An origin: There is an element $0_\mathcal{X} \in \mathcal{X}$ with $x + 0_\mathcal{X} = x$ whenever $x \in \mathcal{X}$
- Additive inverses: If $x \in \mathcal{X}$, there is some $y \in \mathcal{X}$ with $x + y = 0$ (this "y" is often denoted $-x$)

and a constant multiplication operation, denoted $\lambda \cdot x$ (or just λx) for $\lambda \in \mathbb{R}$ and $x \in \mathcal{X}$, that satisfies

- Clusure: if $\lambda \in \mathbb{R}$ and $x \in \mathcal{X}$ then $\lambda \cdot x \in \mathcal{X}$
- Distributivity: $\lambda \cdot (x + y) = \lambda \cdot x \, + \, \lambda \cdot y$ when $\lambda \in \mathbb{R}$ and $x, y \in \mathcal{X}$
- Associativity: $\lambda \cdot (\nu \cdot x) = (\lambda\nu) \cdot x$ when $\lambda, \nu \in \mathbb{R}$ and $x \in \mathcal{X}$
- Zeros: $0 \cdot x = 0_\mathcal{X}$ whenever $x \in \mathcal{X}$.

The basic examples of vector spaces are the Euclidean spaces $\mathbb{R}^k$. This is the normal subject of a typical linear algebra course. Even more interesting are the infinite dimensional cases.

1.2 Examples

1.2.1 The vector space $\mathcal{V}$ of lists

The first example of an infinite dimensional vector space is the space $\mathcal{V}$ of lists of real numbers. We define

$$\mathcal{V} \;=\; \left\{\, (x_1,\, x_2,\, x_3,\, \dots) \;\middle|\; x_i \in \mathbb{R} \text{ for all } i \,\right\}. \tag{1}$$

Addition is defined pointwise: setting $X = (x_1, x_2, \dots)$ and $Y = (y_1, y_2, \dots)$, we define

$$X + Y = (x_1 + y_1,\, x_2 + y_2,\, \dots) \tag{2}$$

Constant multiplication is likewise defined pointwise: if $\lambda \in \mathbb{R}$ we define

$$\lambda X = (\lambda x_1,\, \lambda x_2,\, \dots)\,. \tag{3}$$

It is simple exercise to verify the vector space axioms.

1.2.2 The vector space $\mathcal{V}_F$ of lists that terminate

A second example is the space $\mathcal{V}_F$ of sequences that eventually terminate in zeros. Specifically, we define

$$\mathcal{V}_F = \{\, X \in \mathcal{V} \;\mid\; X = (x_1, x_2, \dots) \text{ where only finitely many of the } x_i \text{ are nonzero} \,\}\,. \tag{4}$$

Clearly $\mathcal{V}_F \subset \mathcal{V}$, but $\mathcal{V}_F \neq \mathcal{V}$. To prove that $\mathcal{V}_F$ is a vector space in its own right, we only have to prove that the addition operation is closed; when that is proved, the other vector space axioms hold because they hold in the larger space $\mathcal{V}$. That is, if $x, y \in \mathcal{V}_F$, we have to show that $x + y \in \mathcal{V}_F$. But this is simple: assuming $X, Y \in \mathcal{V}$, they can be expressed as

$$X = (x_1,\, \dots,\, x_k, 0,\, 0,\, \dots) \quad \textit{and} \quad Y = (y_1,\, \dots,\, y_l, 0,\, 0,\, \dots) \tag{5}$$

where, without loss of generality, we can assume that $l \geq k$. Then

$$X + Y = (x_1 + y_1,\, \dots,\, x_k + y_k, y_{k+1}, \dots, y + l,\, 0,\, 0,\, \dots)\,,$$

so that $X + Y$ terminates in zeros as well.

2 Normed Vector Spaces

2.1 Definitions

<u>Definition</u>. If $\mathcal{X}$ is a vector space, a function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is called a **norm** on $\mathcal{X}$ provided the following hold:

i) Positivity: $\|x\| \geq 0$, with equality if and only if $x = 0_{\mathcal{X}}$

ii) Compatibility with constant multiplication: $\|\lambda x\| = |\lambda| \|x\|$

iii) Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

□

Definition. A pair $(\mathcal{X}, \|\cdot\|)$ is called a **normed vector space** if $\mathcal{X}$ is a vector space and $\|\cdot\|$ is a norm on $\mathcal{X}$. □

Proposition 2.1 *If $(\mathcal{X}, \|\cdot\|)$ is a normed vector space, then the function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric on $\mathcal{X}$.*

Proof. We have to prove that d is positive (except for $d(x, x)$ which we have to prove is zero), that d is symmetric, and that it obeys the triangle inequality. The positivity of d follows directly from that of $\|\cdot\|$. Symmetry follows by

$$\begin{aligned} d(x, y) &= \|x - y\| && (6)\\ &= \|(-1)(y - x)\| && (7)\\ &= |-1| \cdot \|y - x\| && (8)\\ &= \|y - x\| && (9)\\ &= d(y, x). && (10) \end{aligned}$$

Also, $d(x, x) = \|x - x\| = 0$. The triangle inequality for d follows from the triangle inequality for $\|\cdot\|$. To see this, note

$$\begin{aligned} d(x, z) &= \|x - y\| && (11)\\ &= \|x - z + z - y\| && (12)\\ &\leq \|x - z\| + \|z - y\| && (13)\\ &= d(x, y) + d(y, z). && (14) \end{aligned}$$

□

2.2 Examples

2.2.1 The *sup*-norm on $\mathcal{V}_F$

We can give $\mathcal{V}_F$ the following norm. If $X \in \mathcal{V}_F$, we can write $X = (x_1, x_2, \ldots)$ where all but finitely many of the x_i are zero. Define

$$\|X\| = \sup_i \{|x_i|\}. \tag{15}$$

One has to verify the appropriate axioms to prove that $\|\cdot\|$ is a norm.

2.2.2 The l^p norms on $\mathcal{V}_F$

Given any p, we can define a functional $\|\cdot\|_p$ that takes $\mathcal{V}_F$ into the reals. If $X = (x_1, x_2, \dots)$, then we define

$$\|X\|_p \;\triangleq\; \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}. \tag{16}$$

Note that $0 \le \|X\|_p < \infty$ for any p, for any $X \in \mathcal{V}_F$; the fact that $\|X\|_p < \infty$ can be seen by noting that the sum on the right is a finite sum.

We claim that the functional $\|\cdot\|_p$ is a norm as long as $1 \le p < \infty$. This is called the l^p-norm on $\mathcal{V}_F$. One must verify the three axioms for norms; the only difficult axiom to verify is the triangle inequality. The triangle inequality for the l^p-norm is so important that it is given a special name: the Minkowski inequality.

Theorem 2.2 (Minkowski's Inequality) *If $1 \le p < \infty$, then whenever $X, Y \in \mathcal{V}_F$ we have*

$$\|X + Y\|_p \;\le\; \|X\|_p + \|Y\|_p. \tag{17}$$

Proof. The proof is a little off the beaten path, so it is in a suplimentary set of notes. □

3 Banach Spaces

3.1 Definitions

Recall that a norm on a vector space $\mathcal{X}$ determines a distance function, so that any normed vector space is also a metric space. Therefore we have the ability to determine if a sequence is a Cauchy sequence.

Proposition 3.1 *If $(\mathcal{X}, \|\cdot\|)$ is a normed vector space, then a sequence of points $\{X_i\}_{i=1}^{\infty} \subset \mathcal{X}$ is a Cauchy sequence iff given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $i, j > N$ implies*

$$\|X_i - X_j\| \;<\; \epsilon.$$

Proof. Simple exercise in verifying the definitions. □

A vector spaces will never have a "boundary" in the sense that there is some kind of wall that cannot be moved past. Still, it is *not* always the case that Cauchy sequences are convergent. It is therefore important to distinguish between those normed vector spaces that are complete and those that are not complete. The former are known as Banach spaces.

<u>Definition</u>. A normed vector space $(\mathcal{X}, \|\cdot\|)$ is called a **Banach space** if it is complete, in the sense that whenever a sequence is Cauchy with respect to the norm $\|\cdot\|$, it is convergent. □

3.2 Examples

3.2.1 A Cauchy sequence in $(\mathcal{V}_F, \|\cdot\|_{sup})$ that is not convergent.

Let $(\mathcal{V}_F, \|\cdot\|_{\text{sup}})$ be the vector space of sequences of real numbers that terminate in all zeros, along with the *sup*-norm. Let $X_i \in \mathcal{V}_F$ be the sequence

$$X_i = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{i}, 0, \ldots\right) \tag{18}$$

meaning that the k^{th} entry of X_i is $\frac{1}{k}$ when $k \leq i$, and is 0 when $k > i$. To prove the sequence $\{X_i\}_{i=1}^{\infty}$ is Cauchy, choose any $\epsilon > 0$, and then select some $N > \frac{1}{\epsilon}$. Then if $i, j > N$, where without loss of generality we assume $j > i$, we have

$$\|X_i - X_j\|_{sup} = \left\|\left(0, \ldots, 0, -\frac{1}{i+1}, \ldots, -\frac{1}{j}, 0, \ldots\right)\right\|_{sup} \tag{19}$$

$$= \frac{1}{i+1} < \frac{1}{N} < \epsilon. \tag{20}$$

Thus $\{X_i\}_{i=1}^{\infty}$ is Cauchy.

To prove that $\{X_i\}$ has no limit, assume the contrary: that there is some $X \in \mathcal{V}_F$ with $X = \lim_{i\to\infty} X_i$. Write $X = (x_1, x_2, \ldots)$. Since $X \in \mathcal{V}_F$, it eventually terminates in all zeros. Therefore $x_k = 0$ whenever k is large enough, so we can choose a specific k for which $x_k = 0$. However, whenever $n > k$, we have that the k^{th} entry of X_n is $\frac{1}{k}$. Therefore the k^{th} entry of $X_n - X$ is also $\frac{1}{k}$, so that

$$\|X_n - X\|_{sup} \geq \frac{1}{k}. \tag{21}$$

This contradicts that eventually $\|X_n - X\|_{sup} < \epsilon$.

3.2.2 A Cauchy sequence in $(\mathcal{V}_F, \|\cdot\|_2)$ that is not convergent.

Recall the l^2 norm: if $X = (x_1, x_2, \ldots) \in \mathcal{V}_F$, then

$$\|X\|_2 = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}. \tag{22}$$

Again, the sum on the right converges because it is actually a finite sum (as the x_i are eventually all zero). Consider the sequence $\{X_i\}$ from the previous example, where $X_i =$

$(1, \frac{1}{2}, \ldots, \frac{1}{i}, 0, \ldots)$. To see that X_i is Cauchy with respect to the l^2 norm, we again assume that $j > i$ and compute

$$\|X_i - X_j\|_2 = \left\| \left(0, \ldots, 0, -\frac{1}{i+1}, \ldots, -\frac{1}{j}, 0, \ldots\right) \right\|_2 \quad (23)$$

$$= \left(\sum_{k=i+1}^{j} \frac{1}{k^2} \right)^{\frac{1}{2}} < \left(\sum_{k=i+1}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{2}}. \quad (24)$$

Since the sum $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent, the tail $\sum_{k=i+1}^{\infty} \frac{1}{k^2}$ can be made as small as desired by choosing i sufficiently large. Therefore $\{X_i\}$ is Cauchy.

To see that $\{X_i\}$ has no limit, we argue as in the previous example. If $X \in \mathcal{V}_F$ satisfies $\lim X_i = X$, then after writing $X = (x_1, x_2, \ldots)$ there must be some k with $x_k = 0$. However, that implies that for sufficiently large n we always have that the k^{th} entry of $X_n - X$ is $1/k$, so that

$$\|X_n - X\|_2 = \|(\ldots, 1/k \ldots)\|_2 \quad (25)$$

$$= \left(\ldots + \frac{1}{k^2} + \ldots \right)^2 \geq \frac{1}{k}. \quad (26)$$

Therefore the l^2-norm of $X_n - X$ is always bigger that $\frac{1}{k}$, contradicting the assumption that X is the limit of the sequence $\{X_i\}$.

3.2.3 A sequence in $\mathcal{V}_F$ that is Cauchy in the l^2 norm but not the l^1 norm.

We have shown that the sequence $\{X_i\}$ from the previous two examples is Cauchy in both the l^2 and *sup* norms. To show that it is not Cauchy in the l^1 norm, chose $i, j \in \mathbb{N}$ (where again $j > i$) and consider $\|X_i - X_j\|_1$. We have

$$\|X_i - X_j\|_1 = \left\| \left(0, \ldots, 0, -\frac{1}{i+1}, \ldots, -\frac{1}{j}, 0, \ldots\right) \right\|_1 \quad (27)$$

$$= \sum_{k=i+1}^{j} \frac{1}{k} \quad (28)$$

Since the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, the tail $\sum_{k=i+1}^{\infty} \frac{1}{k}$ is also divergent, meaning

$$\lim_{j \to \infty} \sum_{k=i+1}^{j} \frac{1}{k} = +\infty. \quad (29)$$

Thus regardless of how large i is, we can choose some j larger still to make $\|X_i - X_j\|_1$ large. Thus $\{X_i\}$ is not a Cauchy sequence.

4 The l^p and l^∞ spaces

We have considered the spaces $\mathcal{V}_F$ with the various l^p norms and the *sup*-norm, and we have seen that these spaces are not complete.

Recall the vector space $\mathcal{V}$ that consists of infinite lists of real numbers $X = (x_1, x_2, \ldots)$, where the list does not necessarily terminate in zeros. If we make no other restrictions, the functionals $\|\cdot\|_p$ and $\|\cdot\|_{sup}$ cannot be considered norms on $\mathcal{V}$ because many elements would have infinite lengths. For instance, if $X = (x_1, x_2, \ldots)$ where $x_i = i$, then of course $\|X\|_{sup} = \infty$.

<u>**Definition**</u>. (The l^p spaces for $1 \leq p < \infty$.) Given $1 \leq p < \infty$, we define $l^p \subset \mathcal{V}$ to be the normed vector space of elements $X = (x_1, x_2, \ldots) \in \mathcal{V}$ that are p-summable, meaning

$$\sum_{i=1}^{\infty} |x_i|^p \ < \ \infty. \tag{30}$$

To make l^p a normed vector space, we give it the l^p-norm $\|\cdot\|_p$, meaning

$$\|X\|_p \ = \ \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}. \tag{31}$$

□

<u>**Definition**</u>. (The l^∞ space.) We define the space l^∞ to be the subspace of $\mathcal{V}$ of elements X that are bounded in the *sup*-norm. That is, $X \in l^\infty$ iff $X \in \mathcal{V}$ and, after writing $X = (x_1, x_2, \ldots)$, we have

$$\sup_i |x_i| \ < \ \infty. \tag{32}$$

To make l^∞ a normed space, we give it the *sup*-norm (which we now also call the l^∞ norm).

Theorem 4.1 (Riesz-Fischer) *For $1 \leq p \leq \infty$, the space l^p is a Banach space.*

<u>*Proof*</u>. Again, the proof is a little more than we will need in the future. However, you might try to prove it on your own; the argument needed is quite elementary. □