# The Bott Periodicity Theorem

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The Bott periodicity theorem is of fundamental importance in many areas of mathematics, from algebraic topology to functional analysis. It appears unexpectedly in different guises and I would like to explain some of these as well as the influence it has had on the development of different fields. I will concentrate on two roles that periodicity plays. First, periodicity allows one to deloop classifying spaces and thus define cohomology theories. Second, using periodicity, "wrong way" functoriality maps can be defined and these are of integral importance in the index theorem.

Let me begin with the original statement of Bott, [5]. Consider the infinite versions of the matrix groups $\mathrm{U} = \lim_n \mathrm{U}(n)$, and $\mathrm{GL}(\mathbf{C}) = \lim_n \mathrm{GL}(n, \mathbf{C})$ with the limit topology, as well as the real groups $O$ and $\mathrm{GL}(\mathbf{R})$. Then Bott proved

**Theorem 0.1 (The Bott Periodicity Theorem)**

$$\pi_n(U) = \pi_n(GL(\mathbf{C})) = \{ \begin{array}{ll} 0 & \textit{for } n \textit{ even} \\ \mathbf{Z} & \textit{for } n \textit{ odd} \end{array}$$

*Thus the homotopy groups are periodic of period two. In the real case, where one considers the infinite orthogonal group $O$, there is a period eight periodicity where the groups are*

$$\pi_n(O) = \pi_n(GL(\mathbf{R})) = \mathbf{Z}/2, \mathbf{Z}/2, 0, \mathbf{Z}, 0, 0, 0, \mathbf{Z} \textit{ for } n \equiv 0, \cdots, 7 \textit{ mod } 8$$

Bott's original proof used Morse theory. An alternate statement is that there is a weak homotopy equivalence $\mathbf{Z} \times \mathrm{BU} \to \Omega\mathrm{U}$, the loop space of $U$. That $\mathrm{U} \to \Omega(\mathbf{Z} \times \mathrm{BU})$ is a weak homotopy equivalence is a very general but easy statement, and combined with the previous map provides us with a weak homotopy equivalence $\mathbf{Z} \times \mathrm{BU} \to \Omega^2(\mathbf{Z} \times \mathrm{BU})$. (The corresponding real statement is that there exists a weak homotopy equivalence $\mathbf{Z} \times BO \to \Omega^8(\mathbf{Z} \times BO)$.) Since $\mathbf{Z} \times \mathrm{BU}$ (resp. $\mathbf{Z} \times \mathrm{BO}$ ) classifies stable isomorphism classes of complex (resp. real) vector bundles, the equivalence $\mathbf{Z} \times \mathrm{BU} \to \Omega^2(\mathbf{Z} \times \mathrm{BU})$ is reflected in an isomorphism of the Grothendieck $K$ group of such stable isomorphism classes. Namely, $K(\Sigma^2 X) \cong K(X)$ (resp. $KO(\Sigma^8 X) \cong KO(X)$ ) for a compact space $X$, or equivalently, $K(X \times \mathbf{R}^2) \cong K(X)$ where $K$ for non-compact spaces means $K$-theory with compact supports, which can be described by finite complexes of vector bundles $\cdots \to E_{i+1} \to E_i \to E_{i-1} \to \cdots$ that are exact off of a compact set. This form of the theorem was proved directly in [2]. Hence the Bott periodicity theorem is the key step in setting up $K$-theory as an extraordinary cohomology theory, with $K^i(X) = K(\Sigma^{|i|} X)$. It is in this way that Bott periodicity plays a heroic role in extending the definition of $K$-theory to Banach algebras. Thus, for a complex Banach algebra $A$, $K_i(A) \equiv \pi_{i-1}(\mathrm{GL}(A))$ for $i \geq 1$ and periodicity states $K_i(A) \cong K_{i+2}(A)$. $K_0$ is defined, as usual, using projective modules but still satisfies the periodicity above. The proof in the Banach algebra case is modelled on the proof in [2]. The $K$-theory of Banach algebras so defined has many of the wonderful properties that $K$-theory of spaces has, and deserves to be thought of as a cohomology of an underlying space (following the Gelfand-Grothendieck philosophy of shifting the focus from spaces to algebras of functions on the spaces ). This was one of the early inspirations in the geometric study of non-commuative algebras and in the use of non-commutative algebras to study geometric problems, a field of current interest that falls under the rubric of "non-commutative differential geometry", [7]. It is the lack of periodicity that makes defining algebraic $K$-theory for abstract rings so difficult in addition to not having such desirable properties, for example excision. It is intersting to point out that once periodicity is "imposed" on algebraic $K$-theory, that its character increasingly resembles that of topological $K$-theory,[9].

In the embedding proof of the index theorem, [3], Bott periodicity emerges as a powerful tool for producing the essential "wrong way" funtoriality. In fact, in some approaches to the index theorem, periodicity is practically the same as the index theorem itself. The isomorphism $K^0(X) \to K^0(X \times \mathbf{R}^2)$ is given by multiplication by a canonical element in $K^0(X \times \mathbf{R}^2)$, called the Bott element. Before describing this element, let me describe the Bott periodicity theorem in its generalized form: the Thom isomorphism in $K$-theory. Let $\pi : E \to X$ be a complex vector bundle of rank $k$ over $X$. The complex structure is more than enough to guarantee that the bundle is oriented for $K$-theory, and the Thom isomorphism theorem says

**Theorem 0.2 (Bott-Thom isomorphism theorem)** *There is a canonical isomorphism $K^0(X) \to K^0(E)$ given by multiplication by an element $b_E \in K^0(E)$*

$b_E$ is described by the Koszul complex on $E$. That is, consider the complex of bundles

$$0 \to \Lambda^k \pi^* E' \to \Lambda^{k-1} \pi^* E' \to \ldots \to \Lambda^1 \pi^* E' \to \pi^* E' \to 0$$

where $E'$ denotes the dual vector bundle and the maps in the complex are described over a point $e \in E$ by contracting with $e$. The appearance of the Koszul complex in the description of the Bott element makes contact with Grothendieck's formulation and proof of the Riemann-Roch theorem, suggesting a connection between the Grothendieck-Riemann-Roch Theorem and topological $K$-theory. This was carried out by Baum-Fulton-MacPherson in the proof of the Riemann-Roch Theorem for singular varieties, [4].

An elegant proof of the Bott-Thom isomorphism theorem was given by Atiyah in [1]. Besides being elegant, it has the advantage of generalizing in several directions, especially to the equivariant versions, for which, as far as I know, there are still no other proofs. It also established definitively the close ties between the index theorem and periodicity. I will describe the proof in its modern formulation. For a topological space $X$, Atiyah proposed a definition of $K_*(X)$, the $K$-homology of $X$. This is the homology theory corresponding to the generalized cohomology theory $K$-theory. His proposed definition was based on some notion of "generalized" elliptic operators that live on any space $X$, in such a way that the familiar elliptic operators of geometry would define elements in $K_*(X)$ when $X$ is a manifold. This idea was successfully carried out by Brown, Douglas and Filmore and by Kasparov, [6], [8]. These generalized elliptic operators are covariant under maps and the push forward to a point $K_0(X) \to K_0(\text{point}) = \mathbf{Z}$ is the index. For a space $X$ there is a "cap" product $K^0(X) \times K_0(X) \to K_0(X)$ which pairs a vector bundle with a generalized elliptic operator and returns the elliptic operator twisted by the vector bundle. The key idea to the proof of the Bott-Thom isomorphism theorem is to construct the inverse to multiplication by the Bott element as a family of elliptic operators. Then by the existence of products (more general than the cap product defined above) the periodicity theorem reduces to a calculation of an index of a specific differential operator. In the case of a complex vector bundle, this family is a family of Dolbeault operators while for real vector bundles it is a family of Dirac operators.

Once the Bott-Thom isomorphism is established, it can be used in a classical way to define "wrong way" functorialities, which in this case is of essential importance in the index theorem. An extremely quick sketch of the index theorem goes as follows. Let $M$ be an even dimensional Spin$^c$ manifold. Spin$^c$ guarentees the existence of a Dirac operator $[D] \in K_0(M)$ and the left vertical arrow below is capping with this class. This map is in fact surjective and corresponds to the deformation step in the proof [3]. It is the statement of Poincare duality in $K$-theory. Embed $M$ into $R^n$ for large even $n$. The normal bundle is then endowed with Spin$^c$ which is enough to define a Bott element. The left horizontal arrow is just the Thom-Bott isomorphism (combined with extending from a tubular neighborhood to all of $R^n$.

$$\begin{array}{ccccc} K^0(M) & \rightarrow & K^0(\mathbf{R}^n) & \cong \mathbf{Z} \\ \downarrow & & \downarrow & \downarrow \\ K_0(M) & \rightarrow & K_0(\mathbf{R}^n) & \rightarrow \mathbf{Z} \end{array}$$

The left commuting square reduces the index theorem to the case of $\mathbf{R}^n$. Hence we use the Bott periodicity theorem to reduce the index theorem to the case of $\mathbf{R}^n$ and then use it again to handle this case.

I think it is interesting to point out that subsequent to the original periodicity theorem of Bott, there have appeared other important analogous periodicity phenomena. I would like to point out a relationship between Bott periodicity and periodicity phenomena that occur in the classification of topological manifolds (Siebenmann periodicity). Let $M$ be a topological manifold with boundary $\partial M$. Consider the structure set $\mathcal{S}(M, \partial M)$; this is the set of topological manifolds with boundary $(N, \partial N)$ together with a homotopy equivalence $h : N \rightarrow M$ which is a homeomorphism on the boundary. Equivalence is homeomorphism commuting up to homotopy. $\mathcal{S}(M, \partial M)$ is a good first approximation to the set of homeomorphisms in the same homotopy type as $M$. (The actual set of homeomorphism types is a quotient of $\mathcal{S}(M, \partial M)$.) It turns out that $\mathcal{S}(M, \partial M)$ is in fact an abelian group. Siebenmann periodicity states that for $M$ a closed manifold $\mathcal{S}(M) \cong \mathcal{S}(M \times I^4, M \times \partial I^4)$. There is actually a factor of $\mathbf{Z}$ missing which is akin to the fact that $\Omega U \cong \mathbf{Z} \times \mathrm{BU}$ and not just BU. Now we relate this to Bott periodicity. Let $h : N \rightarrow M$ define an element in the structure set. Define an element in $KO[\frac{1}{2}]_*(M)$ by $h_*([d_N + d_N^*]) - [d_M + d_M^*]$. The operator $d + d^*$ denotes the Telemann signature operator, and can be defined for a manifold with only a Lipschitz structure, which every topological manifold has (in dimension greater than four). This defines a map $\mathcal{S}(M) \rightarrow KO[\frac{1}{2}]_*(M)$ which intertwines the two periodicities. It should be noted that $KO[\frac{1}{2}]$ is four periodic with the signature operator playing the same role as the inverse of the Bott element as the Dirac operator does for complex (or real) periodicity. Let me finish by saying that the periodicity theorems here serve many of the same roles as I have been emphasizing. Namely, because of periodicity, the classifying spaces used in manifold classification can be delooped to form cohomology theories, and important non-trivial functorialities can be established.

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