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**MAYER-VIETORIS SEQUENCES IN CYCLIC HOMOLOGY  
OF TOPOLOGICAL ALGEBRAS**

**BY**

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# INTRODUCTION

In this paper we derive Mayer-Vietoris sequences for Hochschild and cyclic homology for certain topological algebras. We relate this to well known sequences in K-theory of Banach algebras and apply it to several computations.

In recent years it has become increasingly clear that non-commutative algebras are arising in geometric problems. For example, to a foliation on a manifold Connes [10] associates a non-commutative algebra which contains much of the geometric and analytic information of the foliation. Similarly, to a group action there is a canonically associated algebra. Along different lines, modules over the ring of differential operators on a complex variety have been studied and have been very useful in solving problems in geometry and group representations [2]. It is extremely desirable to study the "geometry" of these algebras in order to better understand the situations from which they arose.

An important role in the classical "commutative" arena has been played by de Rham Cohomology. In the non-commutative setting, cyclic homology is emerging as a good substitute. An important property of classical cohomology theories is excision. For de Rham cohomology, the best statement of excision is the existence of Mayer-Vietoris sequences. Simple calculations in cyclic homology show that excision generally fails,

in the sense that the relative cyclic homology of an algebra with respect to an ideal depends not only on the ideal but on the way it sits inside the algebra. Our Mayer-Vietoris sequence is analogous to one of Milnor's for algebraic  $K_0, K_1$  [19]. Thus

**Theorem**    *Let*

$$\begin{array}{ccc} A & \rightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \rightarrow & A_3 \end{array}$$

*be a Cartesian square of algebras satisfying:*

1.  $0 \rightarrow A \rightarrow A_1 \oplus A_2 \rightarrow A_3 \rightarrow 0$  is exact. Note that the only place where exactness doesn't follow from the Cartesian square property is the surjectivity.
2.  $H_i(A; A_j \otimes A_j) = 0$  for  $i > 0, j = 1, 2, 3$  and  $A_j \otimes_A A_j = A_j$ ,  $j = 1, 2, 3$ .

*Then there exists a long exact sequence:*

$$\dots \rightarrow H_{i+1}(A_3) \rightarrow H_i(A) \rightarrow H_i(A_1) \oplus H_i(A_2) \rightarrow H_i(A_3) \rightarrow H_{i-1}(A_3) \rightarrow \dots$$

*If either  $A_1 \rightarrow A_3$  or  $A_2 \rightarrow A_3$  is surjective, then we have a long exact sequence:*

$$\dots \rightarrow HC_i(A) \rightarrow HC_i(A_1) \oplus HC_i(A_2) \rightarrow HC_i(A_3) \rightarrow \dots$$

Here,  $H_*(A; M)$  denotes Hochschild homology of  $A$  with coefficients in the bimodule  $M$ .  $H_*(A)$  is a shorthand for  $H_*(A; A)$  and  $HC_*(A)$  is cyclic homology of  $A$ .

Notice that the sequence 1. is not a sequence of rings. That is, none of the homomorphisms are ring homomorphisms. Therefore, a priori, these maps do not even induce



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# 1 Algebraic Mayer-Vietoris

In this section by an algebra we mean an associative algebra with unit over a field  $k$  of characteristic zero.

## 1.1 Review of Cyclic Homology

In this section we quickly recall the fundamental notions of cyclic homology, mainly to fix notation. Our treatment of cyclic homology is a combination of Connes, Loday and Quillen and Goodwillie [10],[18] and [13]. Each of these papers is a good introduction to cyclic homology.

Recall that a cyclic object  $X$  in a category  $\mathcal{D}$  is a simplicial object with extra structure. Namely for each  $n \in \mathbf{Z}^+$  there is a morphism  $t_{n+1} : X[n] \rightarrow X[n]$  that satisfies

1.  $t_{n+1}^{n+1} = \text{Id}$
2.  $\partial_i t_{n+1} = t_n \partial_{i-1}$  for  $0 < i \leq n$  and  $\partial_n$  for  $i = 0$
3.  $s_i t_{n+1} = t_{n+2} s_{i-1}$  for  $0 < i \leq n$  and  $t_{n+2}^2 s_n$  for  $i = 0$ .

We will be solely interested in cyclic vector spaces. We are interested in cyclic vector spaces that arise from algebras in the following way: We let  $ZA[n] = A^{\otimes n+1}$ , where the tensor product is taken over the ground ring. We then define  $\partial_i : ZA[n] \rightarrow ZA[n-1]$  by

$$\partial_i(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n \text{ for } i = 0, \dots, n-1$$

and

$$\partial_n(a_0 \otimes \dots \otimes a_n) = a_n a_0 \otimes \dots \otimes a_{n-1}$$

$$s_i(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes 1 \otimes \dots \otimes a_n \text{ for } i = 0, \dots, n-1$$

and where the 1 lies in the  $i$ th place. And finally we define

$$t(a_0 \otimes \dots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$$

It is a simple matter to check the cyclic vector space hypotheses. We will often write  $Z_n A$  in place of  $ZA[n]$ .

Given a cyclic vector space  $X$  define three operators.

$$N = 1 + t + t^2 + \dots + t^n$$

is the "norm" operator,

$$b = \sum_{i=0}^n (-1)^i \partial_i$$

is the "Hochschild" boundary operator and

$$b' = \sum_{i=0}^{n-1} (-1)^i \partial_i$$

Form the first quadrant double complex  $C_{i,j} = X[j]$  as in the following diagram:

$$\begin{array}{ccccccc}
 & & \uparrow & & & & \\
 & & \downarrow b & & \downarrow -b' & & \downarrow b \\
 X[2] & \xleftarrow{1-t} & X[2] & \xleftarrow{N} & X[2] & \xleftarrow{1-t} & \\
 \downarrow b & & \downarrow -b' & & \downarrow b & & \\
 X[1] & \xleftarrow{1-t} & X[1] & \xleftarrow{N} & X[1] & \xleftarrow{1-t} & \\
 \downarrow b & & \downarrow -b' & & \downarrow b & & \\
 X[0] & \xleftarrow{1-t} & X[0] & \xleftarrow{N} & X[0] & \xleftarrow{1-t} & \\
 & & \downarrow & & \downarrow & & \\
 & & \leftarrow & & \leftarrow & & 
 \end{array} \tag{1}$$

The homology of this double complex is called the *cyclic homology* of  $X$  and denoted by  $HC_*(X)$ . If  $X$  comes from an algebra as above then we denote  $HC_*(ZA)$  by  $HC_*(A)$ . If  $A$  has a unit then the odd columns are acyclic, since  $s(a_0 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes \dots \otimes a_n$  defines a contracting homotopy. While in each even column is a complex whose homology is called the *Hochschild homology* of  $X$  and is denoted by  $H_*(X)$ . Define a map of double complexes by

$$S : C_{i,j} \xrightarrow{\cong} C_{i-2,j}$$

This map induces the following short exact sequence of complexes:

$$0 \rightarrow \ker(S) \rightarrow Tot_*(C) \rightarrow Tot_{*-2}(C) \rightarrow 0$$

$\ker(S)$  consists of the first two columns of the double complex and using the fact that the first column is acyclic it is very easy to see that the homology of  $\ker(S)$  is the same as that of the 0-th column, that is, to the Hochschild homology of  $X$ . Hence the short exact sequence above gives rise to a long exact sequence

$$\dots \rightarrow H_*(X) \xrightarrow{I} HC_*(X) \xrightarrow{S} HC_{*-2}(X) \xrightarrow{\tilde{B}} H_{*-1}(X) \rightarrow \dots \tag{2}$$

The map  $\tilde{B} \circ I$  will be denoted by  $B$ , and gives us a map

$$B : H_*(X) \rightarrow H_{*+1}(X)$$

It is easy to check that  $B \circ B = 0$  and hence  $(H_*(X), B)$  forms a cochain complex. We will denote its cohomology by  $H_{DR}^*(X)$  and call it the non-commutative de Rham cohomology of  $X$ . We would like to point out that this is not the same as Karoubi's non-commutative de Rham homology, but that there is relationship [17]. Of course when the cyclic object involved is  $ZA$  for some algebra  $A$ , we write  $H_*(A)$  and  $H_{DR}^*(A)$ . By chasing through the diagram (1) one sees that the map  $B$  is given by the composite of maps  $B = (1-t)sN$ .

The definition above is not Connes' original definition of cyclic homology. For an algebra  $A$  over a field of characteristic zero, he formed the complex  $C_*^\lambda(A) = Z_*(A)/(1-t)$ . Our double complex has an obvious augmentation to  $C_*^\lambda(A)$ . The rows of the double complex with augmentation are exact, since they are the homology of finite cyclic groups with coefficients in vector spaces of zero characteristic. This proves that Connes' original definition agrees with the one given by Quillen and Loday [18]. For certain purposes it is



Let  $A$  be an algebra and  $M$  an  $A$ -bimodule. Such a bimodule defines a left (or a right) module over the algebra  $A^e = A \otimes_k A^{op}$  and in fact the category of  $A$ -bimodules is equivalent to the category of left (or right)  $A^e$ -modules. The Hochschild homology of  $A$  with coefficients in  $M$  is defined by

$$H_*(A; M) = Tor_*^{A^e}(M, A)$$

where  $A$  is viewed as an  $A$  bimodule in the obvious way, and then as a left  $A^e$ -module,  $M$  a right  $A^e$ -module. By homological algebra, we may use any projective resolution of  $A$  as a left  $A^e$ -module. We will define, the following standard resolution of  $A$ . We let  $P_i = A^{\otimes i+2}$  with the action of  $A^e$  defined by

$$(a \otimes b^\circ)(a_0 \otimes \cdots \otimes a_{i+1}) = aa_0 \otimes \cdots \otimes a_{i+1}b^\circ$$

This makes  $P_i$  in fact a free  $A^e$ -module. We define maps  $\partial : P_i \rightarrow P_{i-1}$  by

$$\partial(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^i (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}$$

It is easy to check that  $s(a_0 \otimes \cdots \otimes a_{i+1}) = 1 \otimes a_0 \cdots$  is a contracting homotopy. And thus the  $P_i$  define a resolution. Therefore the Hochschild homology of  $A$ ,  $H_*(A; M)$  is given as the homology of the complex

$$M \otimes_{A^e} P_* \cong M \otimes A^{\otimes *}$$

and with differential given by

$$\begin{aligned} & \partial(m \otimes a_1 \otimes \cdots \otimes a_n) \\ &= ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & \quad + (-1)^n a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \end{aligned}$$

We denote this complex by  $Z_*(A; M)$ . Notice that the  $Z_*A$  above is the same as  $Z_*(A; A)$ . We will continue to denote this by  $Z_*A$ . And since we will be using  $H_*(A; A)$  so much we will denote this simply by  $H_*(A)$ .

## 1.2 Homotopy Invariance

There is one more property of cyclic homology of rings that we will continually take advantage of, namely, its homotopy invariance. There are several equivalent ways of stating this. The one most suited to our work is that the "Lie derivative" acts trivially on cyclic homology (at least on the image of the  $S$  map). This fact was noted both by Connes [10] and by Goodwillie [13].

Let  $\delta$  be a derivation of a ring  $A$ . We then define its Lie derivative  $\mathcal{L}_\delta : Z_*A \rightarrow Z_*A$  by

$$\mathcal{L}_\delta(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^n a_0 \otimes \cdots \otimes \delta(a_i) \otimes \cdots \otimes a_n$$

$\mathcal{L}_\delta$  commutes with the Hochschild differential  $b$  and with the  $t$  map hence defines a map of cyclic vector spaces. It thus induces maps on  $H_{DR}^*(A)$ ,  $HC_*(A)$  and on  $HP_*(A)$ . Homotopy invariance then takes the form



**Theorem 1.2** 1.  $\mathcal{L}_\delta$  acts like zero on  $H_{DR}^*(A)$ .

2.  $\mathcal{L}_\delta$  acts like zero on  $HP_*(A)$ .

3.  $\mathcal{L}_\delta \circ S$  acts like zero on  $HC_*(A)$ .

In order to give a proof of homotopy invariance, we introduce a certain construct due to Karoubi [17] and Arveson [10]. Let  $A$  be an algebra with unit, over a field  $k$  of characteristic zero. There exists a bimodule  $\Omega_1(A)$  and a derivation of  $d : A \rightarrow \Omega_1(A)$  which is universal for derivations of  $A$  into bimodules. We define  $\Omega_1(A) = Ker\{A \otimes A \xrightarrow{mult} A\}$ . Now define  $\Omega_n(A) = \Omega_1(A) \otimes_A \dots \otimes_A \Omega_1(A)$ . The differential is defined by  $d(a) = 1 \otimes a - a \otimes 1$  for  $a \in A = \Omega_0(A)$  and  $d(a_0 da_1 \dots da_n) = da_0 da_1 \dots da_n$ . Now  $\Omega_n(A) \cong A \otimes \bar{A} \otimes \dots \otimes \bar{A}$  where  $\bar{A} = A/k$ , by the identification  $a_0 da_1 \dots da_n \mapsto a_0 \otimes \dots \otimes a_n$ . Let  $A^+$  be  $A$  with an identity adjoined, even if it already has one. Then Karoubi defines

$$H_*^{DR}(A) = H_*(\Omega_*(A^+)/[\Omega_*(A^+), \Omega_*(A^+)])$$

where the commutator is the graded commutator. Both Karoubi [17] and Connes [10] discovered the following beautiful relationship between  $HC_*(A)$  and  $H_*^{DR}(A)$ . Let  $\theta : \Omega_n(A^+) \rightarrow C_n^\lambda(A)$  by

$$\theta(a_0 da_1 da_2 \dots da_n) = a'_0 \otimes a_1 \otimes \dots \otimes a_n$$

where  $a'_0 = a_0 - \epsilon(a_0)$ , here  $\epsilon$  is the augmentation of  $A^+ \rightarrow A$ .

**Theorem 1.3** [17] *There is an isomorphism  $\theta : H_i^{DR}(A) \cong Ker\{\tilde{B} : HC_i(A) \rightarrow H_{i+1}(A)\}$ .*

With this theorem in hand, many proofs, such as homotopy invariance, take on a very classical look.

*Proof:* (of homotopy invariance)

Let  $\delta : A \rightarrow A$  be a derivation of  $A$  and define  $\mathcal{L}_\delta : Z_n(A) \rightarrow Z_n(A)$ , as above. It is not hard to see that this action on  $\Omega_*(A)$  is given by  $\mathcal{L}_\delta(a_0 da_1 da_2 \dots da_n)$

$$= \delta(a_0) da_1 da_2 \dots da_n + \sum_{i=1}^n a_0 da_1 \dots d\delta(a_i) \dots da_n$$

Now define  $i_\delta(a_0 da_1 \dots da_n)$

$$\sum_{i=1}^n (-1)^i a_0 da_1 \dots da_{i-1} (a_i) da_{i+1} \dots da_n$$

It is a simple matter to check that in  $\Omega_*(A)$  one has the expression

$$di_\delta + i_\delta d = \mathcal{L}_\delta$$

Thus to show that  $\mathcal{L}_\delta$  acts by zero on the image of the  $S$  map we let  $x \in S(HC_{i+2})$ . Then  $x$  is in the kernel of the map  $\tilde{B}$ . Hence, taking  $\theta^{-1}(x)$  we find that

$$\mathcal{L}_\delta(x) = \theta(\mathcal{L}_\delta(\theta^{-1}(x))) = 0$$

□

### 1.3 Change of Rings and Mayer-Vietoris

In this section we derive the algebraic prototype of our Mayer-Vietoris sequence in Hochschild homology and the other homology theories we are considering. Thus let

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & A_1 \\ \downarrow \phi_2 & & \downarrow \psi_1 \\ A_2 & \xrightarrow{\psi_2} & B \end{array} \quad (5)$$

be a Cartesian square of rings. That is  $A \cong \{(a_1, a_2) \in A_1 \times A_2 \mid \psi_1(a_1) = \psi_2(a_2)\}$ . Also assume that

$$A_1 \times A_2 \xrightarrow{\psi_1 - \psi_2} B$$

is surjective. Then we arrive at a short exact sequence of  $A$ -bimodules

$$0 \rightarrow A \rightarrow A_1 \oplus A_2 \rightarrow B \rightarrow 0$$

and to a short exact sequence of complexes

$$0 \rightarrow Z_*(A; A) \xrightarrow{\phi_1 \oplus \phi_2} Z_*(A; A_1 \oplus A_2) \xrightarrow{\psi_1 - \psi_2} Z_*(A; B) \rightarrow 0 \quad (6)$$

where

$$\phi_1 \oplus \phi_2(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (\phi_1(a_0), \phi_2(a_0)) \otimes a_1 \otimes \cdots \otimes a_n$$

and

$$\psi_1 - \psi_2((a_0^1, a_0^2) \otimes a_1 \otimes \cdots \otimes a_n) = \psi_1(a_0^1) \otimes a_1 \otimes \cdots \otimes a_n - \psi_2(a_0^2) \otimes a_1 \otimes \cdots \otimes a_n$$

This in turn gives rise to a long exact sequence in Hochschild homology

$$\dots \rightarrow H_*(A; A) \rightarrow H_*(A; A_1 \oplus A_2) \rightarrow H_*(A; B) \rightarrow \dots$$

It remains to relate this sequence to one of the form

$$\dots \rightarrow H_*(A; A) \rightarrow H_*(A_1; A_1) \oplus H_*(A_2; A_2) \rightarrow H_*(B; B) \rightarrow \dots$$

We are thus led to consider a change of rings formula.

Let  $\phi : A \rightarrow B$  be a homomorphism of algebras. We then get a map  $\phi^* : B\text{-Mod} \rightarrow A\text{-Mod}$  or for any of the other categories of modules, by considering any  $B$ -module  ${}_B M$  as an  $A$ -module by  $am = \phi(a)m$ . Hence the functor  $H_*(A; \phi^*(\cdot))$  acts on  $B\text{-Mod-}B$ . We want to explore when, in general, this functor coincides with  $H_*(B; \cdot)$  on  $B\text{-Mod-}B$ .

Let

$$\dots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow A$$

be a resolution of  $A$  by left  $A^e$  modules. Then

$$H_*(A; M) \cong H_*(M \otimes_{A^e} P).$$

Now extend by  $B^e$ . That is consider

$$\dots \rightarrow B^e \otimes_{A^e} P_1 \rightarrow B^e \otimes_{A^e} P_0 \rightarrow B^e \otimes_{A^e} A. \quad (7)$$

Here  $A^e$  acts on the right of  $B^e$  by  $(b_0 \otimes b_1) \cdot (a_0 \otimes a_1) = b_0 a_0 \otimes a_1 b_1$ . Now,  $B^e \otimes_{A^e} P_i$  is a projective left  $B^e$  modules. Suppose that in addition,  $B^e \otimes_{A^e} A$  which is isomorphic to  $B \otimes_A B$  is also isomorphic to  $B$  and that the sequence (7) is a resolution. Then we may use it to compute  $H_*(B; M)$ . Thus

$$H_*(B; M) \cong H_*(M \otimes_{B^e} (B^e \otimes_{A^e} P_i)) \quad (8)$$

and

$$M \otimes_{B^e} (B^e \otimes_{A^e} P_i) \cong M \otimes_{A^e} P_i$$

so

$$H_*(B; M) \cong H_*(A; \phi^*(M)).$$

Also note that

$$H_*(B^e \otimes_{A^e} P_i)$$

computes

$$H_*(A; B^e).$$

Hence we arrive at

**Theorem 1.4** *Let  $A \xrightarrow{\phi} B$  be a homomorphism of algebras. Suppose*

$$B \otimes_A B \cong B$$

and

$$H_*(A; B^e) = 0, * > 0.$$

*Then  $H_*(B; \cdot) \cong H_*(A; \phi^*(\cdot))$  on  $B\text{-Mod-}B$ .*

*The converse also holds.*

*Proof:* The only thing left to show is the converse.  $H_*(A; B^e) \cong H_*(B; B^e) \cong 0, * > 0$ . And  $B \otimes_A B \cong H_0(A; B^e) \cong H_0(B; B^e) \cong B$ .  $\square$  The conditions on the morphism  $\phi$

are well known in the vast literature on localization of non-commutative rings. They arise when one considers "localizations at torsion theories." The first condition, that  $B \otimes_A B$  is canonically isomorphic to  $B$  is exactly the condition the the morphism be an epimorphism of rings ( in the categorical sense, i.e. not necessarily surjections ). The second condition is just that  $B$  be a flat  $A$  bimodule. We thus make the following definition.

**Definition 1.5** *We call a homomorphism satisfying the hypotheses of the theorem a flat epimorphism. ( Taylor [22] considers the same condition and calls it a localization. )*

**Lemma 1.6** *Let  $A_1$  and  $A_2$  be two algebras. Then the projection  $A_1 \oplus A_2 \xrightarrow{pr_1} A_1$  is a flat epimorphism.*

*Proof:* Clearly  $A_1 \otimes_{A_1 \oplus A_2} A_1 \cong A_1$ . In the standard complex computing  $H_*(A_1 \oplus A_2; A_1^e)$  that is

$$\dots \rightarrow A_1 \otimes (A_1 \oplus A_2)^{\otimes 2} \otimes A_1 \rightarrow A_1 \otimes (A_1 \oplus A_2) \otimes A_1 \rightarrow A_1 \otimes A_1$$

one can check that

$$s(a_0 \otimes (a_1^1, a_1^2) \otimes \dots \otimes a_{n+1}) = (1 \otimes (a_0, 0) \otimes (a_1^1, a_1^2) \otimes \dots \otimes a_{n+1})$$

is a contracting homotopy. □

This gives a different way to see the following lemma from [9].

**Lemma 1.7**  $H_*(A_1 \oplus A_2; A_1 \oplus A_2) \cong H_*(A_1; A_1) \oplus H_*(A_2; A_2)$ .

Now consider the following sequence of complexes

$$Z_* A \xrightarrow{\phi_1 \oplus \phi_2} Z_* A_1 \oplus Z_* A_2 \xrightarrow{\psi_1 - \psi_2} Z_* B$$

This sequence is in general not exact in any place. But by comparing it with (6) we prove the following theorem.

**Theorem 1.8** *Suppose we have a Cartesian square of algebras (5). Suppose also that  $A_1, A_2$  and  $B$  are flat epimorphisms as  $A$ -bimodules, and that*

$$0 \rightarrow A \xrightarrow{\phi_1 \oplus \phi_2} A_1 \oplus A_2 \xrightarrow{\psi_1 - \psi_2} B \rightarrow 0 \quad (9)$$

*is exact as a sequence of  $A$ -bimodules. Then we have a Mayer-Vietoris sequence*

$$\dots \rightarrow H_i(A; A) \rightarrow H_i(A_1; A_1) \oplus H_i(A_2; A_2) \rightarrow H_i(B; B) \rightarrow \dots \quad (10)$$

*Proof:* Because  $Z_*(A; A_1 \oplus A_2) \cong Z_*(A; A_1) \oplus Z_*(A; A_2)$  we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & Z_i(A; A) & \xrightarrow{\phi_1 \oplus \phi_2} & Z_i(A; A_1) \oplus Z_i(A; A_2) & \xrightarrow{\psi_1 - \psi_2} & Z_i(A; B) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & Z_i(A; A) & \xrightarrow{\phi_1 \oplus \phi_2} & Z_i(A_1; A_1) \oplus Z_i(A_2; A_2) & \xrightarrow{\psi_1 - \psi_2} & Z_i(B; B) \end{array} \quad (11)$$

Taking homology everywhere we get the following diagram

$$\begin{array}{ccccccc} \dots & \rightarrow & H_i(A; A) & \rightarrow & H_i(A; A_1) \oplus H_i(A; A_2) & \rightarrow & H_i(A; B) \rightarrow \dots \\ & & \downarrow = & & \downarrow \cong & & \downarrow \cong \\ & & H_i(A; A) & \rightarrow & H_i(A_1; A_1) \oplus H_i(A_2; A_2) & \rightarrow & H_i(B; B) \end{array} \quad (12)$$

where the isomorphisms occur because of the flat epimorphism condition. □

We now show how this Mayer-Vietoris sequence induces one for cyclic and periodic cyclic homology. First we have a very general lemma from homological algebra.

**Lemma 1.9** *Let  $A_{**} \xrightarrow{\phi} B_{**} \xrightarrow{\psi} C$  be maps of double complexes ( either first quadrant or first and second quadrant ) such that  $\psi\phi = 0$ . Suppose that*

$$0 \rightarrow H_*^I(A) \rightarrow H_*^I(B) \rightarrow H_*^I(C) \rightarrow 0 \quad (13)$$

*is exact, where  $H_*^I(A)$  denotes the homology of  $A_{**}$  with respect to the first differential only. Then there exists a long exact sequence*

$$\dots \rightarrow H_i(A) \rightarrow H_i(B) \rightarrow H_i(C) \rightarrow H_{i-1}(A) \rightarrow \dots$$

*where  $H_*(A)$  means the homology of the associated total complex.*

*Proof:* Form the triple complex

$$D_{***} = \begin{array}{ccccccc} & & \uparrow & & & & \\ & & C_{**} & \xleftarrow{\psi} & B_{**} & \xleftarrow{-\phi} & A_{**} & \xleftarrow{0} & 0 & \xleftarrow{0} & \dots \\ & & \downarrow & & & & & & & & \end{array}$$

where  $D_{i**} = 0$  for  $i > 3$ . From  $D$  form two double complexes:

$$D_{i,j}^1 = \bigoplus_{k+l=j} D_{i,k,l}$$

and

$$D_{i,j}^2 = \bigoplus_{k+l=i} D_{k,l,j}$$

The homology of a triple complex is formed analogously to that of a double complex by forming a single total complex  $D_* = \bigoplus_{j+k+l=i} D_{j,k,l}$ , with the obvious differentials. By filtering this single complex in two ways we arrive at two spectral sequences with  $E^1$  terms given respectively by  $H_*^I(D^1)$  and  $H_*^I(D^2)$ . By the hypothesis that the sequence (13) is exact it follows that  $H_*(D^2) = 0$ , hence the second spectral sequence degenerates and  $E^1 = E^\infty = 0$  and thus  $H_*(D_{***}) = 0$ . On the other hand, for the first double complex  $D_{**}^1$ , we have  $D_{i,*}^1 = 0$  for  $i > 3$ . Hence  $E^2 = E^\infty$  which equals 0 by the previous argument. The  $E^1$  term is

$$\begin{array}{ccccccc} & & \uparrow & & & & \\ & & H_*(C) & \xleftarrow{} & H_*(B) & \xleftarrow{} & H_*(A) & \xleftarrow{0} & 0 & \xleftarrow{0} & \dots \\ & & \downarrow & & & & & & & & \end{array}$$

It follows from these two facts that there is a long exact sequence as above.  $\square$

The lemma has the following consequences for cyclic homology.

**Theorem 1.10** *Suppose we have a Cartesian square (5), that all the morphisms are flat epimorphisms and suppose finally that (20) splits as a sequence of  $A$  bimodules. Then we have the following Mayer-Vietoris sequences:*

1.

$$\dots \rightarrow H_{DR}^{i-1}(B) \rightarrow H_{DR}^i(A) \rightarrow H_{DR}^i(A_1 \oplus A_2) \rightarrow H_{DR}^i(B) \rightarrow H_{DR}^{i+1}(A) \rightarrow \dots$$

2.

$$\dots \rightarrow HC_{i+1}(B) \rightarrow HC_i(A) \rightarrow HC_i(A_1 \oplus A_2) \rightarrow HC_i(B) \rightarrow HC_{i-1}(A) \rightarrow \dots$$

3.

$$\dots \rightarrow HP_{i+1}(B) \rightarrow HP_i(A) \rightarrow HP_i(A_1 \oplus A_2) \rightarrow HP_i(B) \rightarrow HP_{i-1}(A) \rightarrow \dots$$

**Remark 1.11** *The hypothesis that (20) be split may seem artificial but it occurs very often in the  $C^\infty$  category. In the next section we remove this hypothesis by approaching the Mayer-Vietoris sequence from another angle.*

*Proof:* The splitability of (20) implies the splitability of the Mayer-Vietoris sequence for Hochschild homology of Theorem 1.8. Hence there are short exact sequences

$$0 \rightarrow H_i(A) \rightarrow H_i(A_1 \oplus A_2) \rightarrow H_i(B) \rightarrow 0$$

It is easy to check that these maps preserve the  $B$  operator and hence define maps of the complexes  $(H_*(A), B)$ . We thus arrive at the long exact sequence 1.

We have the sequence of double complexes and maps:

$$C_{**}(A) \rightarrow C_{**}(A_1) \oplus C_{**}(A_2) \rightarrow C_{**}(B)$$

One checks that it satisfies the hypotheses of Lemma 13 because of splitability condition and the sequence (10). The sequence 2. follows immediately from the lemma.

3. follows exactly as in 2. □

**Example:** As an example of a flat epimorphism consider  $V$  an affine variety with  $U_1 \subset U_2$  two affine open subsets. Then  $\mathcal{O}(U_2) \rightarrow \mathcal{O}(U_1)$  is a flat epimorphism, where  $\mathcal{O}(U)$  denotes the regular functions on  $U$ . Using this fact and the Mayer Vietoris sequences above, one can rederive the computations done in [18] for the homologies of the ring of regular functions on a smooth affine variety. But we will do the computation in the  $C^\infty$  category in section 3.

## 1.4 Relative Groups and Excision

A more classical notion of excision is that if one is given a Cartesian square (5) then certain relative groups are isomorphic. In this section we recall the notion of relative groups and show how this notion of excision fits in to our framework.

Let  $\phi : A \rightarrow B$  be a homomorphism of algebras. We can then form the algebraic mapping cone  $Z_i(A, B) = Z_i(A) \oplus Z_{i+1}(B)$  with the following differential  $b(x, y) = (b(x), b(y) - \phi(x))$  where  $(x, y) \in Z_i(A, B)$ . One can check that  $Z_*(A, B)$  is a cyclic object. We then have the following short exact sequences:

$$0 \rightarrow Z_*(B) \rightarrow Z_{*-1}(A, B) \rightarrow Z_{*-1}(A) \rightarrow 0$$

and

$$0 \rightarrow Tot_*(C_{**}(Z(B))) \rightarrow Tot_{*-1}(C_{**}(Z(A, B))) \rightarrow Tot_{*-1}(C_{**}(Z(A))) \rightarrow 0$$

The homology of  $Z_*(A, B)$ ,  $H_*(A, B)$  is called the relative Hochschild homology of  $(A, B)$  and the homology of  $Tot(C_{**}(Z(A, B)))$ ,  $HC_*(A, B)$  is the relative cyclic homology of  $(A, B)$ . From the short exact sequences above we arrive at two long exact sequences:

$$\rightarrow H_{i+1}(B) \rightarrow H_i(A, B) \rightarrow H_i(A) \rightarrow H_i(B) \rightarrow H_{i-1}(A, B) \rightarrow$$

and

$$\rightarrow HC_{i+1}(B) \rightarrow HC_i(A, B) \rightarrow HC_i(A) \rightarrow HC_i(B) \rightarrow HC_{i-1}(A, B) \rightarrow$$

Of course there a corresponding sequence for periodic cyclic homology. Finally, for our proofs below we need to introduce Hochschild homology of an algebra relative to a homomorphism of coefficients. Thus let  $M$  and  $N$  be two  $A$  bimodules and let  $\phi : M \rightarrow N$  be a bimodule homomorphism. Then define  $Z_i(A; M, N) = Z_i(A; M) \oplus Z_{i+1}(A; N)$  with a

differential defined as above. Then set  $H_i(A; M, N) = H_i(Z_i(A; M, N))$ . Because of a lack of anything to call these groups, we let the reader call them anything he wants.

Let us consider algebras  $A, A_1, A_2$  and  $B$  situated in a Cartesian square (5). Then  $I = \ker(\phi_1)$  is canonically isomorphic to  $\ker(\psi_2)$ , as ideals, and in particular as an  $A$  bimodule. Assume that the homomorphism  $\phi_1$  and  $\psi_2$  are, surjective. Finally assume that  $\phi_i$  and  $\psi_i$  are flat epimorphisms.

**Lemma 1.12** *In the following commutative diagram, the arrows labeled as isomorphisms are isomorphisms.*

$$\begin{array}{ccc} H_*(A; A, A_1) & \xrightarrow{\cong} & H_*(A, A_1) \\ \downarrow \cong & & \downarrow \\ H_*(A; A_2, B) & \xrightarrow{\cong} & H_*(A_2, B) \end{array} \quad (14)$$

*Proof:* The vertical arrow is induced by maps  $Z_*(A; A, A_1) \rightarrow Z_*(A; A_2, B)$  and because the maps  $\phi_1$  and  $\psi_2$  are surjective, there is a short exact sequence

$$0 \rightarrow Z_*(A; I, I) \rightarrow Z_*(A; A, A_1) \rightarrow Z_*(A; A_2, B) \rightarrow 0$$

These induce long exact sequences and  $Z_*(A; I, I)$  is obviously acyclic and we therefore have an isomorphism  $H_*(A; A, A_1) \cong H_*(A; A_2, B)$ .

For the horizontal arrows we use our change of rings yoga. Thus the map  $Z_*(A; A, A_1) \rightarrow Z_*(A, A_1)$  is the same as

$$Z_*(A; A) \oplus Z_{*+1}(A; A_1) \rightarrow Z_*(A; A) \oplus Z_{*+1}(A_1; A_1)$$

Now the flat epimorphism condition implies that if the differentials involved were just the direct sum, then we would have an isomorphism at the level of homology. The fact that the differential is skewed by the map  $\phi_1$  presents little complication and one can show (by a double complex argument, for example) that the map of complexes above is a quasi-isomorphism. The other horizontal arrow is obviously the same.  $\square$

**Proposition 1.13** *Suppose that all the maps in the Cartesian square are flat epimorphisms. Also assume that  $\phi_1$  and  $\psi_2$  are surjective. Then the natural maps  $H_*(A, A_1) \rightarrow H_*(A_2, B)$ ,  $HC_*(A, A_1) \rightarrow HC_*(A_2, B)$  and  $HP_*(A, A_1) \rightarrow HP_*(A_2, B)$  are isomorphisms.*

*Proof:* The case of relative Hochschild homology follows immediately from the preceding lemma. For the cyclic homology we again use our lemma 1.9 and the fact that the analogous spectral sequence from Hochschild to cyclic homology holds for the relative case.  $\square$  This formal sort of excision allows us to remove the hypothesis that the sequence (20) be exact.

**Theorem 1.14** *Suppose we have a Cartesian square of algebras (5). Assume that  $\phi_i$  and  $\psi_i$  are flat epimorphisms. Also assume that either  $\psi_1$  or  $\psi_2$  is surjective. Then the Mayer-Vietoris sequences of theorem 1.10 hold.*

*Proof:* Assume  $\psi_2$  is surjective. It follows that  $\phi_1$  is also. The Mayer-Vietoris sequence, for cyclic homology for example ( the case of periodic is exactly the same ) follow from the following diagram and a diagram chase:

$$\begin{array}{ccccccc}
 \rightarrow & HC_i(A, A_1) & \rightarrow & HC_i(A) & \rightarrow & HC_i(A_1) & \rightarrow \\
 & \downarrow \cong & & \downarrow & & \downarrow & \\
 \rightarrow & HC_i(A_2, B) & \rightarrow & HC_i(A_2) & \rightarrow & HC_i(B) & \rightarrow
 \end{array} \tag{15}$$



## 2 Topological Mayer-Vietoris

In this section we make the shift to cyclic homology of topological algebras. We are interested in studying derived functors on a category of modules over a topological algebra. The functors that we are interested in deriving are completed topological tensor products of modules over topological algebras. Here the question of which completed topological tensor product to choose is raised. There are three different tensor products we are interested in, each one with its benefits and detractions. They are all used to remedy the following situation. For  $\mathcal{F}(X)$  and  $\mathcal{F}(Y)$  some space of functions on some sort of topological spaces  $X$  and  $Y$ , there is the canonical map  $\phi : \mathcal{F}(X) \otimes \mathcal{F}(Y) \rightarrow \mathcal{F}(X \times Y)$  which is very rarely onto. The topological tensor product was invented to make  $\phi$  an isomorphism. And it depends on what sort of spaces of functions are involved. For example, if  $M$  and  $N$  are smooth manifolds then  $C^\infty(M) \hat{\otimes} C^\infty(N) \xrightarrow{\cong} C^\infty(M \times N)$  and the projective tensor product is the right one to use. While  $C_c^\infty(M) \hat{\otimes} C_c^\infty(N) \xrightarrow{\cong} C_c^\infty(M \times N)$  and the inductive tensor product is the correct one. We therefore see no reason to choose one or the other tensor product in our presentation.

When doing homological algebra over topological algebras, MacLane's relative homological algebra has been the standard method of dealing with the fact the the category of modules over a topological algebra is not abelian, . In our situation, this amounts to imposing the condition that the allowable or "admissible" short exact sequences of modules are the ones in which the underlying short exact sequence of topological vector spaces split. This means that when deriving a functor, one must find a projective resolution that is admissible.

In many of the circumstances we came upon it was very laborious to show or even doubtful that certain resolutions were admissible. We were thus led to try to relax some of the admissibility hypotheses. As we show below, if one restricts to a suitable class of topological vector spaces, then one may often do away with hypotheses concerning admissibility. Douady, in his seminar on analytic geometry was forced to consider similar difficulties. In particular, they also used the same technique of computing Tor's by non-admissible resolutions. [12]. Taylor [22] also has such results in his work about multi-variable functional calculus.

When dealing with topological tensor products, the obvious good class of topological vector spaces is the class of nuclear ones. The pleasant properties of these spaces are used in the following way, making the category of topological modules over a topological algebras a little closer to abelian. The projective tensor product behaves well with respect to surjections, while the injective tensor product behaves well with respect to injections ( see lemmas below ). It is precisely the nuclear spaces for which these two tensor products coincide.

Section 1 contains our conventions and notations concerning topological vector spaces. Section 2 covers the completed tensor products of topological vector spaces. Section 3 introduces the completed tensor product of modules over a topological algebra. It also recalls relative homological in this case. Section 4 introduces nuclear spaces into the picture and we eliminate admissibility hypotheses. Section 5 is the culmination of the

section where we derive all the Mayer-Vietoris sequences from section 1.

## 2.1 Conventions Concerning Topological Vector Spaces

All topological vector spaces will be locally convex and Hausdorff.  $E'$  will denote the dual of  $E$  equipped with the strong dual topology, that is the topology of uniform convergence on bounded sets. We let  $E'_o$  be the dual equipped with the weak topology, that is the topology of pointwise convergence. Given  $E$  and  $F$ ,  $L(E; F)$  will denote the linear space of continuous linear mappings from  $E$  to  $F$ . If  $G$  is a third space,  $\mathcal{B}(E, F; G)$  denotes the space of separately continuous bilinear maps from  $E \times F$  to  $G$  while  $B(E, F; G)$  denotes the space of jointly continuous bilinear maps. A Fréchet space is a complete metrizable locally convex topological vector space. An LF space is a topological vector space  $E$  which is the union of a countable increasing sequence of Fréchet spaces  $E_i$  such that the topology of  $E_i$  is the same as the topology induced from  $E_{i+1}$ .  $E_i$  is called a defining sequence for  $E$ .

**Definition 2.1** A topological homomorphism between topological vector spaces  $\phi : E \rightarrow F$  is a continuous homomorphism such that the image of any open set is relatively open (that is open in the induced topology of the image.) Or in other words,  $E/\ker(\phi) \xrightarrow{\phi} \text{Im}(\phi)$  is an isomorphism.

It is an easy corollary of the open mapping theorem that a continuous homomorphism between Fréchet spaces is topological if and only if its image is closed.

## 2.2 Topological Tensor Products

To make the reading a little easier, we state all the definitions and results about topological tensor products that we will be using. We have tried to give a complete list of references. Most of these results can be found in either Grothendieck [16] or Treves [24].

Let  $E$  and  $F$  be two Hausdorff locally convex topological vector spaces. As in Grothendieck [16], p.89, we say a locally convex topology  $\mathcal{T}$  on  $E \otimes F$  is *compatible* with the tensor product if

1. The bilinear map  $E \times F \rightarrow (E \otimes F, \mathcal{T})$  is separately continuous.
2. For  $e'$  and  $f'$  in  $E'$  and  $F'$  respectively,  $e' \otimes f'$  is in  $(E \otimes F, \mathcal{T})'$
3. If  $e'$  and  $f'$  range over equicontinuous sets of  $E'$  and  $F'$ , then  $e' \otimes f'$  ranges over an equicontinuous set of  $(E \otimes F, \mathcal{T})'$ .

## 2.3 The Projective Tensor Product

The *projective topology* on  $E \otimes F$  is the strongest locally convex topology which makes the canonical map  $E \times F \rightarrow E \otimes F$  continuous. We will denote this topology by  $E \otimes_{\pi} F$  and its completion by  $E \hat{\otimes} F$ . A neighborhood base of zero of  $E \otimes_{\pi} F$  consists of sets of the

form  $U \otimes V$  where  $U$  and  $V$  run over a neighborhood base of zero for  $E$  and  $F$  respectively. Let  $p$  and  $q$  be semi-norms on  $E$  and  $F$  respectively. Let  $U$  and  $V$  be their closed unit semi-balls, i.e  $U = \{e \in E | p(e) \leq 1\}$ . Let  $W$  be the balanced convex hull of  $U \otimes V$ . Then define  $(p \otimes q)(x) = \inf_{x \in \lambda W, \lambda > 0} \lambda$  for  $x \in E \otimes F$ .

**Proposition 2.2** ([24], p.435) For  $x \in E \otimes F$

$$(p \otimes q)(x) = \inf \sum_i p(e_i)q(f_i)$$

where the inf is taken over all the ways of writing  $x = \sum_i e_i \otimes f_i$  as a finite sum. Also  $(p \otimes q)(e \otimes f) = p(e)q(f)$ .

The projective tensor product has the desired universal property that one would hope for.

**Theorem 2.3** ([24], p.438) For three topological vector spaces  $E, F$  and  $G$  and given a continuous bilinear map  $B : E \times F \rightarrow G$  there exists a unique continuous linear map  $T : E \otimes_\pi F \rightarrow G$ . That is, there is an isomorphism  $B(E, F; G) \cong L(E \otimes_\pi F; G)$ . If  $G$  is complete, one has the same correspondence  $B(E, F; G) \cong L(E \hat{\otimes} F; G)$ . The projective topology is the unique one with this property.

If  $u_i : E_i \rightarrow F_i, i = 1, 2$  are two continuous linear maps and  $E_1, E_2, F_1, F_2$  are four topological vector spaces, then for any compatible topology,  $u_1 \otimes u_2 : E_1 \otimes E_2 \rightarrow F_1 \otimes F_2$  defines a continuous map. It then extends to the completion with respect to this topology.

The following property of the projective tensor product gives it its name.

**Proposition 2.4** ([24], p 441) Let  $u_i : E_i \rightarrow F_i$  be topological homomorphisms. Then if  $u_i$  maps  $E_i$  onto a dense subspace of  $F_i$  then  $u_1 \otimes u_2$  maps  $E_1 \hat{\otimes} E_2$  onto a dense subspace of  $F_1 \hat{\otimes} F_2$ . If both  $E_1$  and  $E_2$  are metrizable, then  $u_1 \otimes u_2$  maps onto  $F_1 \hat{\otimes} F_2$ .

Finally, the following property of the projective tensor product will be useful.

**Theorem 2.5** Let  $E$  and  $F$  be two Fréchet spaces. Every element  $x \in E \hat{\otimes} F$  can be written in the form

$$x = \sum_{i=1}^{\infty} \lambda_i e_i \otimes f_i$$

where  $\lambda_i$  are complex numbers satisfying  $\sum_{i=1}^{\infty} \lambda_i < 1$ , and  $e_i$  and  $f_i$  converge to zero in  $E$  and  $F$  respectively.

**Example 2.6** 1. Recall that if  $M$  is a smooth manifold, that  $C^\infty(M)$  denotes the algebra of smooth  $\mathbb{C}$ -valued functions on  $M$ . Note we do not require  $M$  to be compact. Equip  $C^\infty(M)$  with the usual semi norms

$$|f|_{K,n} = \sup\{|\partial^\alpha f(x)| | x \in K, |\alpha| \leq n\}$$

for  $K$  a compact set and using local charts on  $M$ . If  $M$  is  $\sigma$ -finite then  $C^\infty(M)$  is a Fréchet space. If  $N$  is another smooth manifold, then

$$C^\infty(M) \hat{\otimes} C^\infty(N) \cong C^\infty(M \times N)$$

2. If  $H$  is a Hilbert space, then  $H' \hat{\otimes} H \cong L^1(H)$  the space of trace class operators on  $H$ .

## 2.4 The Injective Tensor Product

$E \otimes F$  can be embedded algebraically in  $\mathcal{B}(E'_\sigma, F'_\sigma)$ .  $\mathcal{B}(E'_\sigma, F'_\sigma)$  comes equipped with the topology of bi-equicontinuous convergence, that is, the topology given by the neighborhood base of zero

$$\{B \in \mathcal{B}(E'_\sigma, F'_\sigma) \mid B(U, V) \leq 1\}$$

where  $U$  and  $V$  run over equicontinuous neighborhoods of  $E'_\sigma$  and  $F'_\sigma$  respectively. We denote the restriction of this topology from  $\mathcal{B}(E'_\sigma, F'_\sigma)$  to  $E \otimes F$  by  $E \otimes_\epsilon F$  and we call it the *injective topological tensor product*. We denote its completion by  $E \hat{\otimes} F$ .

**Lemma 2.7** *The injective topology on  $E \otimes F$  is given by the family of semi-norms:*

$$\epsilon_{U, V} = \sup_{x \in U \otimes V} |x(y)|$$

where  $U$  and  $V$  are equicontinuous subsets of  $E'_\sigma$  and  $F'_\sigma$  respectively,  $y \in E \otimes F$ .

And analogously to Proposition 2.4 we have the following

**Proposition 2.8** *Let  $u_i : E_i \rightarrow F_i, i = 1, 2$  be continuous linear maps which are injective. Then  $u_1 \otimes u_2 : E_1 \hat{\otimes} E_2 \rightarrow F_1 \hat{\otimes} F_2$  is injective.*

**Example 2.9** 1. *If  $X$  and  $Y$  are both compact Hausdorff spaces and  $C(X)$  denotes the space of continuous functions on  $X$ , then*

$$C(X) \hat{\otimes} C(Y) \cong C(X \times Y)$$

2. *If  $E$  is a complete locally convex Hausdorff topological vector space, then  $C^n(X) \hat{\otimes} E \cong C^n(X; E)$  where  $C^n$  denotes the space of  $n$  times differentiable functions on some open subset of  $\mathbb{C}^m$ .*

## 2.5 The Inductive Tensor Product

It is clear that there is a finest compatible topology on  $E \otimes F$  which we call the inductive tensor product. We denote the completion of  $E \otimes F$  under this topology by  $E \bar{\otimes} F$ . It is clear that  $(E \bar{\otimes} F)'$  is isomorphic to the space of separately continuous bilinear forms on  $E \times F$ , and that if  $G$  is complete then  $L(E \bar{\otimes} F; G) \cong \mathcal{B}(E, F; G)$ .

As a result of the Banach-Steinhaus theorem it follows that if  $E$  and  $F$  are Fréchet spaces, then a separately continuous bilinear form is jointly continuous [24] p. 352. Hence in this case follows

**Lemma 2.10** *If  $E$  and  $F$  are Fréchet spaces, then  $E \bar{\otimes} F \cong E \hat{\otimes} F$ .*

The most important property of the inductive tensor product and the one which gives it its name is given in the following theorem.

**Theorem 2.11** ([16], p. 77) *If  $E = \lim_i E_i$  and  $F = \lim_i F_i$  are both inductive limits then*

$$E \bar{\otimes} F = \lim_{i,j} E_i \bar{\otimes} F_j$$

**Example 2.12** 1. *Let  $V$  be an algebraic vector space. We may equip it with the finest locally convex topology. This is the same as the topology defined in the following way. If  $W \subset V$  is a finite dimensional subspace, it comes with a unique locally convex topology. Then  $V$  is topologized as the inductive limit over all of its finite dimensional subspaces. If  $V_1$  and  $V_2$  are two such algebraic vector spaces topologized in this manner, then*

$$V_1 \bar{\otimes} V_2 \cong V_1 \otimes V_2$$

2. *Exactly as in example 1, if  $\mathbb{C}[x_1, x_2, \dots]$  denotes a polynomial ring equipped with this strongest locally convex topology, then*

$$\mathbb{C}[x_1, \dots, x_n] \bar{\otimes} \mathbb{C}[y_1, \dots, y_m] \cong \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_m]$$

3. *If  $M$  and  $N$  are smooth manifolds and  $C^\infty_c(M)$  denotes the space of smooth functions on  $M$  with compact support, then*

$$C^\infty_c(M) \bar{\otimes} C^\infty_c(N) \cong C^\infty_c(M \times N)$$

## 2.6 Tensor Products over Algebras and Relative Homological Algebra

In the following, let  $\bar{\otimes}$  denote any compatible tensor product. Consider the following situation.  $A$  is a complete topological algebra such that  $A \times A \rightarrow A$  extends to  $A \bar{\otimes} A \rightarrow A$  and  $M \in {}_A M$ ,  $N \in M_A$ . We make the following definition.

**Definition 2.13** *A  $\bar{\otimes}$  tensor product of  $M$  and  $N$  over  $A$  is a complete topological vector space  $E$  and a continuous map  $\phi : M \times N \rightarrow E$  satisfying  $\phi(na, m) = \phi(n, am)$  for all  $a \in A$ ,  $n \in N$  and  $m \in M$ . In the case of the projective tensor product it is required to satisfy the following additional universal property: Given any  $\psi : M \times N \rightarrow F$ ,  $F$  complete, satisfying  $\psi(na, m) = \psi(n, am)$  for all  $a \in A$ ,  $n \in N$  and  $m \in M$ , there exists a unique continuous  $\mathbb{C}$ -linear homomorphism  $\tau : E \rightarrow F$  making*

$$\begin{array}{ccc} M \times N & \xrightarrow{\psi} & F \\ \downarrow \phi & \nearrow & \\ E & \xrightarrow{\tau} & \end{array}$$

*commute. We denote  $E$  by  $N \bar{\otimes}_A M$ . If we are considering tensor products over  $\mathbb{C}$  the complex number then we will use the shorthand  $N \bar{\otimes} M$ . If  $N \bar{\otimes}_A M$  exists then it is clearly unique.*

**Theorem 2.14** *Given a complete locally convex algebra  $A$  and  ${}_A M$  and  $N_A$  modules, their  $\bar{\otimes}$  tensor product exists.*

*Proof:* We start by forming the  $\otimes$  tensor product over  $\mathbb{C}$ ,  $N \otimes_{\mathbb{C}} M$ . Let  $K =$  linear span  $\{na \otimes m - n \otimes am \mid n \in N, a \in A, m \in M\}$ . Denote by  $N \otimes_A M$ ,  $N \otimes_{\mathbb{C}} M / \text{closure of } K$ .  $N \otimes_A M$  is complete as long as  $N$ , say is metrizable. If however  $N \otimes_A M$  is still not complete, we complete it. That  $N \otimes_A M$  satisfies the universal property is easy.  $\square$

**Definition 2.15** Let  $E \in \mathcal{T}$ . We call  $A \otimes E$ , (resp.  $E \otimes A$ ,  $A \otimes E \otimes A$ ) the free left (resp. right, bi)  $A$ -module over the vector space  $E$ .

The following lemma justifies this terminology.

**Lemma 2.16** Let  $E \in \mathcal{T}, M \in A - \text{Mod}$ . Then

$$\text{Hom}_{\mathbb{C}}(E, M) \cong \text{Hom}_A(A \otimes E, M)$$

Similar statements hold for the other categories of  $A$ -modules. Or in other words, the functor which associates to any topological vector space the free  $A$ -module over it, is adjoint to the functor which associates to an  $A$ -module the underlying vector space.

*Proof:* Let  $\phi \in \text{Hom}_{\mathbb{C}}(E, M)$ . Define a homomorphism  $A \times E \rightarrow M$  by

$$(a, e) \rightarrow a \cdot \phi(e).$$

Then by the universal property this extends to a map  $\phi_A : A \otimes E \rightarrow M$ . Further  $\phi_A(a \otimes e) = a \cdot \phi(e)$  and  $\{a \otimes e\}$  is a total subset of  $A \otimes E$ .  $\square$

A relative category is specified by either its projective class, or its admissible short exact sequences. We choose to define the admissible short exact sequences.

**Definition 2.17** An admissible short exact sequence of left  $A$ -modules is a sequence

$$0 \rightarrow M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \rightarrow 0$$

which splits as a short exact sequence of topological vector spaces. That is, there should exist  $\mathbb{C}$ -linear mappings  $s : M_2 \rightarrow M_1$  and  $t : M_3 \rightarrow M_2$  and satisfying  $t \circ \psi + \phi \circ s = \text{Id}_{M_2}$ .

Given the short exact sequences the projective objects are determined. Rather than choose one the possible choices, we simply define an admissible projective, or just projective, module to be one which satisfies any of the equivalent statements of the following Proposition.

**Proposition 2.18** For an  $A$ -module  ${}_A P$  the following statements are equivalent.

1. There exists a sequence

$$P \xrightarrow{i} A \otimes E \xrightarrow{j} P$$

such that  $j \circ i = \text{Id}_P$ .

2. Given an admissible short exact sequence  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  the following sequence should also be exact:

$$0 \rightarrow \text{Hom}_A(P, M_1) \rightarrow \text{Hom}_A(P, M_2) \rightarrow \text{Hom}_A(P, M_3) \rightarrow 0$$

3. There should exist a module  ${}_A Q$  such that  $P \oplus Q \cong A \check{\otimes} E$  for some  $E \in \mathcal{T}$ .

**Remark 2.19** The tensor product as we defined is not right exact. If we had been willing to allow tensor products to be incomplete and non-Hausdorff, we could have simply defined  $N \check{\otimes}_A M$  to be  $N \check{\otimes}_C M/K$ , without closing  $K$ . This would have overcome the main dissatisfaction with what we defined, by making the tensor product right exact. But as we shall see the two tensor products have the same derived functors.

**Lemma 2.20** 1. If  ${}_A M = A \check{\otimes}_C E$  where  $E \in \mathcal{T}$ , and  $N_A$  is any right  $A$ -module, then  $N \check{\otimes}_A M = N \check{\otimes}_C E$ . Hence, in this case the two possible notions of tensor product coincide.

2. If  ${}_A P$  is a projective  $A$ -module, then for any  $N_A$ ,  $N \check{\otimes}_A P \cong N \check{\otimes} P/K$  and in this case the two notions of tensor product also coincide.

**Definition 2.21** An admissible projective resolution of an  $X \in A\text{-Mod}$  is a long exact sequence

$$\dots \xrightarrow{b_1} P_1 \xrightarrow{b_0} P_0 \xrightarrow{\epsilon} X$$

of projective  $A$ -modules  $P_i$  and  $C$ -linear maps  $s_i$  so that

$$b_i \circ s_i + s_{i+1} \circ b_{i+1} = \text{Id}_{P_{i+1}}$$

**Lemma 2.22** For any  $X \in A\text{-Mod}$ , there exists an admissible projective resolution.

*Proof:* We prove this only to establish some notation and terminology. We let  $P_i = (A^{\otimes i+1}) \check{\otimes} X$  and define  $b_i : P_{i+1} \rightarrow P_i$  by

$$b_i(a_0 \otimes \dots \otimes a_i \otimes x) = \sum_{j=0}^{i-1} (-1)^j a_0 \otimes \dots \otimes a_j a_{j+1} \otimes \dots \otimes a_i \otimes x + (-1)^i a_0 \otimes \dots \otimes a_i x$$

And define  $s_i : P_i \rightarrow P_{i+1}$  by  $s_i(a_0 \otimes \dots \otimes a_i \otimes x) = 1 \otimes a_0 \otimes \dots \otimes a_i \otimes x$ . It is easy to check that this that forms an admissible projective resolution. We call this the standard resolution.  $\square$

We will denote a projective resolution of  $X$  simply by  $P \xrightarrow{\epsilon} X$

**Lemma 2.23** Let  $P \xrightarrow{\epsilon} X$  be an admissible projective resolution of  $X$  and  $Q \xrightarrow{\eta} Y$  be an admissible resolution of  $Y$ . Then given any  $A$ -module homomorphism  $\phi : X \rightarrow Y$ ,  $\phi$  may be lifted to a map of complexes. Any such lifting is unique up to homotopy.

**Corollary 2.24** Any two admissible projective resolutions of  $X$  are homotopically equivalent.

**Corollary 2.25** Let  ${}_A X, Y_A$  be two  $A$ -modules. Let  ${}_A P \xrightarrow{\epsilon} X$  be an admissible projective resolution of  $X$  and  $Q_A \xrightarrow{\eta} Y$  be an admissible projective resolution of  $Q$ . Then the following three graded groups are isomorphic and independent of the particular resolutions.





**Proposition 2.29** Let  ${}_A M_A$  be an  $A$ -bimodule. Then

1.  $H_0(A; M) \cong M \otimes A / K$
2.  $H_i(A; M) = 0, i > 0$  if  $M$  is a projective bimodule.
3. Given an admissible short exact sequence of  $A$ -bimodules

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

there exists a corresponding long exact sequence of Hochschild Homology groups.

**Theorem 2.30** Let  $A \xrightarrow{\phi} B$  be a continuous homomorphism of nuclear Fréchet (resp. LF) algebras. Suppose there exists an admissible projective resolution of  $A$  by left  $A^e$  modules  $P$  such that

$$\dots \rightarrow B^e \otimes_{A^e} P_1 \rightarrow B^e \otimes_{A^e} P_0 \rightarrow B^e \otimes_{A^e} A \rightarrow 0$$

is an admissible resolution of  $B$  by left  $B^e$  modules. Then  $H_*(B; \cdot) \cong H_*(A; \phi^*(\cdot))$  on  $B\text{-Mod-}B$ .

**Definition 2.31** A homomorphism satisfying the hypotheses of this theorem is called an admissible flat epimorphism. (Notice that in this situation, we can not simply state this condition in terms of the homology groups, but have to refer to particular resolutions. This situation will become better in the next section.)

**Theorem 2.32** Suppose we have a Cartesian square of topological algebras (5). Suppose also that all the morphisms are admissible flat epimorphisms and that

$$0 \rightarrow A \xrightarrow{\phi_1 \oplus \phi_2} A_1 \oplus A_2 \xrightarrow{\psi_1 - \psi_2} B \rightarrow 0 \quad (17)$$

is exact as a sequence of  $A$ -bimodules. Then we have a Mayer-Vietoris sequence

$$\dots \rightarrow H_i(A; A) \rightarrow H_i(A_1; A_1) \oplus H_i(A_2; A_2) \rightarrow H_i(B; B) \rightarrow \dots \quad (18)$$

**Theorem 2.33** Suppose we have a Cartesian square of topological algebras (5), that the morphisms are admissible flat epimorphisms and suppose finally that (20) splits as a sequence of  $A$  bimodules. Then we have the following Mayer-Vietoris sequences:

1.  $\dots \rightarrow H_{DR}^{i-1}(B) \rightarrow H_{DR}^i(A) \rightarrow H_{DR}^i(A_1 \oplus A_2) \rightarrow H_{DR}^i(B) \rightarrow H_{DR}^{i+1}(A) \rightarrow \dots$

2.  $\dots \rightarrow HC_{i+1}(B) \rightarrow HC_i(A) \rightarrow HC_i(A_1 \oplus A_2) \rightarrow HC_i(B) \rightarrow HC_{i-1}(A) \rightarrow \dots$

3.  $\dots \rightarrow HP_{i+1}(B) \rightarrow HP_i(A) \rightarrow HP_i(A_1 \oplus A_2) \rightarrow HP_i(B) \rightarrow HP_{i-1}(A) \rightarrow \dots$

Finally we have the following analogues of proposition 1.13.

**Proposition 2.34** Suppose we have a Cartesian square of topological algebras (5), that the morphisms are admissible flat epimorphisms. Also assume that  $\phi_1$  and  $\psi_2$  are surjective. Then the natural maps  $H_*(A, A_1) \rightarrow H_*(A_2, B)$  and  $HC_*(A, A_1) \rightarrow HC_*(A_2, B)$  are isomorphisms.

**Theorem 2.35** Assume the hypotheses of the preceding proposition. Then the sequences of Theorem 2.52 are exact without the requirement that the sequence (20) splits.

## 2.7 Nuclear Madness

Let  $E$  and  $F$  be Banach spaces. The canonical map  $E' \otimes F \rightarrow L(E; F)$  extends to a continuous linear map  $E' \hat{\otimes} F \rightarrow L(E; F)$ . If an operator  $T : E \rightarrow F$  is in the image of this map we say the operator is *nuclear*. From the structure theorem for the projective tensor product of two Fréchet spaces, it follows that every nuclear operator  $T : E \rightarrow F$  can be written in the form

$$T(x) = \sum_{i=1}^{\infty} \lambda_i e'_i(x) f_i$$

where  $\sum_{i=1}^{\infty} |\lambda_i| \leq \infty$  and  $|e'_i| \leq 1$  and  $|f_i| \leq 1$ .

**Proposition 2.36** *Let  $E$  and  $F$  be locally convex Hausdorff topological vector spaces.  $T : E \rightarrow F$  a continuous linear mapping. The following are equivalent.*

1.  $T$  factors as  $E \xrightarrow{i} E_1 \xrightarrow{\tilde{T}} F_1 \xrightarrow{j} F$  where  $E_1$  and  $F_1$  are Banach spaces, and  $\tilde{T}$  is a nuclear map.
2. There is an equicontinuous sequence  $e'_i \in E'$  and a sequence  $f_j \in F$  which is contained in a convex balanced bounded subset of  $F$  and a sequence of complex numbers  $\lambda_i$  satisfying  $\sum_{i=1}^{\infty} |\lambda_i| < \infty$  and such that

$$T(e) = \sum_{i=1}^{\infty} \lambda_i e'_i(e) f_i$$

If  $p$  is a semi norm on  $E$ , let  $E_p$  denote the completion of  $E/\ker(p)$ . Thus  $E_p$  is a Banach space. There is a canonical map  $E \rightarrow E_p$ . If  $p \leq q$  then there is a map  $E_q \rightarrow E_p$ . We say that a topological vector space  $E$  is *nuclear* if given a continuous semi-norm  $p$  on  $E$ , there exists a continuous semi-norm  $q \geq p$  such that the canonical map  $E_q \rightarrow E_p$  is nuclear. Thus a nuclear space is a certain sort of inverse limit of Banach spaces.

The most important properties of nuclear spaces for us are described in the following theorem.

**Theorem 2.37** *Let  $E$  be a locally convex Hausdorff topological vector space. The following are equivalent:*

1.  $E$  is nuclear.
2. There is an inverse system of Hilbert spaces,  $(H_i, \phi_{i,j})$  such that given  $i$  there exists a  $j \geq i$  with  $\phi_{j,i} : H_j \rightarrow H_i$  is nuclear, such that  $E \cong \varprojlim_i H_i$
3. For every locally convex Hausdorff topological vector space  $F$  the canonical map  $E \hat{\otimes} F \rightarrow E \hat{\otimes} F$  is an isomorphism.

**Remark 2.38** 1. For a Hilbert space the definition of nuclear is the same as trace class (albeit, from one Hilbert space to another).

2. The most classical examples of such inverse systems of Hilbert spaces are the Sobolev spaces. These form an inverse system satisfying the hypothesis of number 2 of the theorem (a strong form of Rellich's lemma) and they converge to show that  $C^\infty(M)$  is nuclear for  $M$  a manifold.

- Theorem 2.39**
1. A subspace of a nuclear space is nuclear.
  2. The quotient of a nuclear space by a closed subspace is nuclear.
  3. The topological product of nuclear spaces is nuclear.
  4. The countable topological sum of nuclear spaces is nuclear.
  5. The projective limit of nuclear spaces is nuclear.
  6. The countable direct product of nuclear spaces is nuclear.
  7. The projective tensor product of two nuclear spaces is nuclear.

- Example 2.40**
1. As we mentioned above, the space  $C^\infty(M)$  of smooth functions on a manifold is nuclear.
  2.  $C^\infty_c(M)$  is nuclear.
  3. A Banach space is nuclear if and only if it is finite dimensional.
  4. The space of polynomials in finitely many unknowns is nuclear, since it is the inductive limit of finite dimensional spaces.
  5. The space of rapidly decreasing functions on the real line, or the space of rapidly decreasing sequences form nuclear spaces.

Given propositions 2.4 and 2.8 we can prove the next lemma.

**Lemma 2.41** Suppose  $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$  is an exact sequence of Fréchet spaces and  $H$  is a Fréchet space. Then

$$0 \rightarrow E \hat{\otimes} H \rightarrow F \hat{\otimes} H \rightarrow G \hat{\otimes} H \rightarrow 0$$

is exact if either  $F$  or  $H$  is nuclear.

*Proof:* First suppose  $H$  is nuclear. Then the inductive and projective tensor products coincide. Therefore by viewing  $\hat{\otimes}$  as the injective tensor product  $E \hat{\otimes} H \rightarrow F \hat{\otimes} H$  is injective by the previous lemmas.  $F \hat{\otimes} H \rightarrow G \hat{\otimes} H$  is surjective by the previous lemma. Then  $E \hat{\otimes} H$  may be viewed as a sub-vector space of  $F \hat{\otimes} H$  and

$$\begin{aligned} F \hat{\otimes} H / E \hat{\otimes} H &\cong \overline{F \otimes_\pi H / E \otimes_\pi H} \\ &\cong \overline{F \otimes_\pi H} / \overline{E \otimes_\pi H} \cong \overline{F/E} \otimes_\pi \overline{H} \cong G \hat{\otimes} H \end{aligned}$$

where  $\overline{\phantom{x}}$  denotes completion. This is because for metrizable spaces  $V \subset W$  we have  $\overline{W/V} \cong \overline{W}/\overline{V}$ .

Now, if  $F$  is nuclear, then  $E$  is also nuclear being a subspace. Then the proof follows as above.  $\square$

**Lemma 2.42** Let  $\dots \xrightarrow{\partial_2} E_2 \xrightarrow{\partial_1} E_1 \xrightarrow{\partial_0} E_0 \rightarrow 0$  be an exact sequence of Fréchet spaces. Then if  $H$  is a nuclear Fréchet space then we have an exact sequence

$$\dots \rightarrow E_2 \hat{\otimes} H \rightarrow E_1 \hat{\otimes} H \rightarrow E_0 \hat{\otimes} H \rightarrow 0$$

*Proof:* First we note that since it is an exact sequence of continuous maps that  $Im(\partial_i) = ker(\partial_{i-1})$  is closed. Therefore by the remark after Definition 2.1, all the  $\partial_i$  are topological homomorphisms. Then the proof proceeds in a standard way by considering the short exact sequences  $0 \rightarrow ker(\partial_i) \rightarrow E_i \rightarrow Im(\partial_i) \rightarrow 0$ . By the previous lemma,  $0 \rightarrow ker(\partial_i) \hat{\otimes} H \rightarrow E_i \hat{\otimes} H \rightarrow Im(\partial_i) \hat{\otimes} H \rightarrow 0$  is exact. The lemma is finished by splicing these sequences together.  $\square$ .

Since, in fact many of the algebras we will deal with are not Fréchet algebras but are LF, the previous results do not apply directly. Here we extend them to cover this case. When dealing with LF spaces the appropriate tensor product to use is the inductive one.

**Proposition 2.43** *Let  $0 \rightarrow E \xrightarrow{\phi} F \xrightarrow{\psi} G \rightarrow 0$  be a short exact sequence of LF spaces where  $\phi$  and  $\psi$  are topological homomorphisms. Let  $H$  be an LF space. Suppose that either  $F$  or  $H$  is nuclear. Then*

$$0 \rightarrow E \bar{\otimes} H \rightarrow F \bar{\otimes} H \rightarrow G \bar{\otimes} H \rightarrow 0$$

*is exact.*

**Remark 2.44** *The fact that  $\phi$  and  $\psi$  are topological homomorphisms does not come free as in the case of Fréchet spaces.*

**Lemma 2.45** *Let  $\phi : E \rightarrow F$  be a continuous map between LF spaces. Then there exist defining sequences  $E_i$  for  $E$  and  $F_i$  for  $F$  such that  $\phi : E_i \rightarrow F_i$ .*

*Proof:* Let  $E_i$  and  $F_j$  be defining sequences for  $E$  and  $F$ . Fix  $E_i$  for the moment. Then  $\phi|_{E_i}$  is continuous. Let  $E_i^j = (\phi|_{E_i})^{-1}(F_j)$ . Then  $E_i = \bigcup_j E_i^j$ . Since  $E_i$  is Fréchet it follows from the Baire category theorem that one of the  $E_i^j$  has a non-empty interior, say  $E_i^N$ . It then follows that  $E_i^N$  is open in  $E_i$  and hence  $E_i^N = E_i$ . So  $\phi : E_i \rightarrow F_N$ . By reindexing the  $F_i$  we can arrange that  $\phi : E_i \rightarrow F_i$ .  $\square$

*Proof:* (of proposition) Let  $E_i$  and  $G_i$  be defining sequences for  $F$  and  $G$  such that  $\psi : F_i \rightarrow G_i$ . Since  $\psi$  is a topological homomorphism  $\psi(F_i)$  is relatively closed in  $Im(\psi) = G$ , hence is closed in  $G_i$ . Let  $G_i' = \psi(F_i)$ .  $G_i'$  is a closed subspace of  $G_i$ , and is thus Fréchet.  $\bigcup_i G_i' = \bigcup_i \psi(F_i) = G$  so  $G_i'$  is also a defining sequence for  $G$ . Let  $E_i = ker(\psi|_{F_i})$ .  $E_i$  is a defining sequence for  $E$ . We then have

$$0 \rightarrow E_i \rightarrow F_i \rightarrow G_i \rightarrow 0$$

is an exact sequence of Fréchet spaces. Let  $H_i$  be a defining sequence for  $H$ . We handle the case where  $H$  is nuclear. Then  $H_i$  is nuclear for every  $i$ . So we get that

$$0 \rightarrow E_i \hat{\otimes} H_i \rightarrow F_i \hat{\otimes} H_i \rightarrow G_i \hat{\otimes} H_i \rightarrow 0$$

is exact for every  $i$ . Since all the spaces involved are Fréchet, we may replace the projective tensor product by the inductive tensor product. Now the fact that

$$0 \rightarrow E \bar{\otimes} H \rightarrow F \bar{\otimes} H \rightarrow G \bar{\otimes} H \rightarrow 0$$

is exact follows from Theorem 2.11 and the fact that exactness is preserved by inductive limits.  $\square$



**Definition 2.50** A homomorphism satisfying the hypotheses of this theorem is called a flat epimorphism. (Notice that in this topological situation, the first condition is no longer the same thing as  $B \hat{\otimes}_A B \cong B$ .)

**Theorem 2.51** Suppose we have a Cartesian square of nuclear Fréchet (resp. LF) algebras (5). Suppose also that all the morphisms are flat epimorphisms and that

$$0 \rightarrow A \xrightarrow{\phi_1 \oplus \phi_2} A_1 \oplus A_2 \xrightarrow{\psi_1 - \psi_2} B \rightarrow 0 \quad (20)$$

is exact as a sequence of  $A$ -bimodules. Then we have a Mayer-Vietoris sequence

$$\dots \rightarrow H_i(A; A) \rightarrow H_i(A_1; A_1) \oplus H_i(A_2; A_2) \rightarrow H_i(B; B) \rightarrow \dots \quad (21)$$

**Theorem 2.52** Suppose we have a Cartesian square of nuclear Fréchet or LF algebras (5), that the morphisms are flat epimorphisms and suppose finally that (20) splits as a sequence of  $A$  bimodules. Then we have the following Mayer-Vietoris sequences:

1.

$$\dots \rightarrow H_{DR}^{i-1}(B) \rightarrow H_{DR}^i(A) \rightarrow H_{DR}^i(A_1 \oplus A_2) \rightarrow H_{DR}^i(B) \rightarrow H_{DR}^{i+1}(A) \rightarrow \dots$$

2.

$$\dots \rightarrow HC_{i+1}(B) \rightarrow HC_i(A) \rightarrow HC_i(A_1 \oplus A_2) \rightarrow HC_i(B) \rightarrow HC_{i-1}(A) \rightarrow \dots$$

3.

$$\dots \rightarrow HP_{i+1}(B) \rightarrow HP_i(A) \rightarrow HP_i(A_1 \oplus A_2) \rightarrow HP_i(B) \rightarrow HP_{i-1}(A) \rightarrow \dots$$

Finally we have the following analogues of proposition 1.13.

**Proposition 2.53** Suppose we have a Cartesian square of nuclear Fréchet or LF algebras (5), that the morphisms are flat epimorphisms. Also assume that  $\phi_1$  and  $\psi_2$  are surjective. Then the natural maps  $H_*(A, A_1) \rightarrow H_*(A_2, B)$  and  $HC_*(A, A_1) \rightarrow HC_*(A_2, B)$  are isomorphisms.

**Theorem 2.54** Assume the hypotheses of the preceding proposition. Then the sequences of Theorem 2.52 are exact without the requirement that the sequence (20) splits.

### 3 Spaces of Smooth Functions

In this section we show how to apply the results of sections one and two to specific calculations. The first section is dedicated to showing that if one wants to compute the cyclic homology of an algebra that arises as the global sections of a presheaf over a  $C^\infty$ -manifold, which is a module over the  $C^\infty$  functions, then the hypotheses of our Mayer-Vietoris sequence are fulfilled. We also take the opportunity to give a new version of Connes' computation of the Hochschild homology of the space of  $C^\infty$  functions on a smooth manifold. In section two, we introduce K-theory into the picture by recalling the construction of the Chern character map from the K-groups to the cyclic homology groups. The main theorem of this section states that Chern character is compatible as map from the Mayer-Vietoris sequence in K-theory to ours in cyclic homology. This gives fairly general criteria for the map from K-theory to periodic cyclic homology to be an isomorphism.

#### 3.1 Flat Epimorphisms

In this section, we show that for the spaces of functions we will be considering, the restriction homomorphisms are flat epimorphisms. Let  $M$  be a smooth manifold,  $U \supset V$  two open subsets of  $M$ . Recall that  $C^\infty(M)$  denotes the algebra of all smooth functions on  $M$ . It is a nuclear Fréchet algebra. Then

**Theorem 3.1** *The restriction homomorphism  $C^\infty(U) \rightarrow C^\infty(V)$  is a flat epimorphisms.*

We need the following lemma which produces a very special cut-off function.

**Lemma 3.2** [23]<sup>2</sup> *Let  $\{f_\alpha\}$  be a countable collection of functions on  $C^\infty(V), U \supset V$ . Then there exists  $\phi \in C^\infty(U)$  such that  $\phi$  and all of its derivatives vanish on  $U - V$  and for all  $\alpha$ ,  $\phi f_\alpha$  extend to  $U$  such that  $\phi f_\alpha$  and all of their derivatives vanish on  $U - V$ . Furthermore,  $\phi \neq 0$  on  $V$ .*

*Proof:* Let  $C_j$  be an exhaustive sequence, that is their union exhausts  $V$ , of compact subsets of  $V$ , such that  $C_{j-1}$  is contained in the interior of  $C_j$ . Let  $\phi_j$  be smooth functions on  $V$  which are 1 on  $C_j$ , and whose support is contained in  $C_{j+1}$  and take values in  $[0, 1]$ . Let

$$\lambda_j = \frac{1}{2^j} \inf \left\{ \frac{1}{|\phi_i|_{C_{i,i}}}, \frac{1}{|\phi_i f_\alpha|_{C_{i,i}}}, i, \alpha \leq j \right\}$$

Then

$$\sum_{i=0}^{\infty} |\lambda_i \phi_i|_{C_{n,n}} = N + \sum_{i=n}^{\infty} |\lambda_i \phi_i|_{C_{n,n}} \leq N + \frac{1}{2^n} \leq \infty$$

So define  $\phi = \sum_{i=1}^{\infty} \lambda_i \phi_i$ . Then the last computation shows that the series converges.  $\phi$  vanishes with all of its derivatives on  $U - V$  by differentiating term by term. Now

$$\sum_{i=1}^{\infty} |\lambda_i \phi_i f_\alpha|_{C_{n,n}} \leq N + \sum_{i=n}^{\infty} \frac{1}{2^{n+1}} \leq \infty$$

<sup>2</sup>I thank A. Wasserman for pointing out this reference.

So  $\sum_{i=1}^{\infty} \lambda_i \phi_i f_\alpha$  converges uniformly. Hence by extending them by 0 on  $U - V$  and differentiating term by term we see that  $\phi f_\alpha$  satisfies the conclusion of the lemma.  $\square$

*Proof:* (of Theorem) We must show that for open sets  $U \supset V$  the restriction map  $C^\infty(U) \rightarrow C^\infty(V)$  is a flat epimorphism. That is we must show

$$H_0(C^\infty(U); C^\infty(V) \hat{\otimes} C^\infty(V)) \cong C^\infty(V)$$

and

$$H_*(C^\infty(U); C^\infty(V)^\epsilon) = 0, * > 0.$$

Consider the standard complex that computes  $H_*(C^\infty(U); C^\infty(V)^\epsilon)$  along with its augmentation to  $C^\infty(V)$ . That is,

$$\begin{aligned} & \dots \rightarrow C^\infty(V) \hat{\otimes} C^\infty(U) \hat{\otimes}^2 C^\infty(V) \\ & \rightarrow C^\infty(V) \hat{\otimes} C^\infty(U) \hat{\otimes} C^\infty(V) \rightarrow C^\infty(V) \hat{\otimes} C^\infty(V) \rightarrow C^\infty(V) \rightarrow 0 \end{aligned}$$

We do not get a globally defined contracting homotopy, but we do get a "local" one. If we consider

$$a = \sum_{i=1}^{\infty} a_0^i \otimes \dots \otimes a_n^i$$

where  $a_0^i, a_n^i \in C^\infty(V)$ ,  $a_j^i \in C^\infty(U)$  for other  $j$  and if  $b(a) = 0$  then define

$$c = \sum_{i=1}^{\infty} \frac{1}{\phi} \otimes \phi a_0^i \otimes a_1^i \otimes \dots \otimes a_n^i$$

where  $\phi$  is as in the lemma for the set  $\{a_0^i\}$ . A simple calculation shows that  $b(c) = a$ . This shows both the requirements.  $\square$

For  $K = \bar{U}$  and  $L = \bar{V}$  let  $C^\infty(K)$  denote the algebra of smooth functions on  $K$  that extend to smooth functions in some neighborhood of  $K$ . Then  $C^\infty(K)$  has a couple of other descriptions. Namely,  $C^\infty(K) \cong C^\infty(M)/C^\infty_c(\overline{M-K})$ . Since  $C^\infty_c(\overline{M-K})$  is a closed subspace of  $C^\infty(M)$ ,  $C^\infty(K)$  is a nuclear Fréchet algebra. Also

$$C^\infty(K) \cong \lim_{W \supset K} C^\infty(W)$$

Here too, one sees that  $C^\infty(K)$  is a nuclear Fréchet algebra.

**Theorem 3.3** *The restriction homomorphism  $C^\infty(K) \rightarrow C^\infty(L)$  is a flat epimorphism.*

*Proof:* Again, we must show two things. The first, that  $C^\infty(L) \cong C^\infty(L) \hat{\otimes}_{C^\infty(K)} C^\infty(L)$  is even easier than in the previous case, since one can extend functions from  $L$  to  $K$ . This leaves only  $H_*(C^\infty(K); C^\infty(L) \hat{\otimes} C^\infty(L)) = 0$  for  $* > 0$  left to show. For this, note that the method of the previous proof extends to show that  $H_*(C^\infty(K); C^\infty(W) \hat{\otimes} C^\infty(W)) = 0$  for  $* > 0$  where  $W$  is an open subset of  $K$  containing  $L$ . Then we have that

$$H_*(C^\infty(K); C^\infty(L) \hat{\otimes} C^\infty(L)) = H_*(C^\infty(K); \lim_{W \supset L} C^\infty(W) \hat{\otimes} C^\infty(W))$$



= 0 for  $* > 0$ . □

We would like to point out some consequences of these results for more general situations. Let  $C^\infty$  denote the sheaf on  $M$  which assigns to an open subset  $U$  of  $M$ ,  $C^\infty(U)$ . Also, let  $C^\infty_\flat$  the presheaf which assigns to  $U$   $C^\infty(\bar{U})$ . Let  $\mathcal{F}$  be a sheaf of algebras over  $M$ . Let  $\mathcal{F}_\flat$  denote the presheaf which assigns to any open subset  $U$  of  $M$ ,  $\lim_{W \supset \bar{U}} \mathcal{F}(W)$ . Then the following theorem follows immediately from the preceding discussion.

**Theorem 3.4** *If  $\mathcal{F}$  is a sheaf of left modules over  $C^\infty$  then for each pair of open subsets  $U \supset V$ , the restriction homomorphism  $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$  is a flat epimorphism. The homomorphism  $\mathcal{F}_\flat(U) \rightarrow \mathcal{F}_\flat(V)$  is a flat epimorphism.*

We now rederive Connes' computation of the Hochschild and cyclic homologies of the algebra of smooth functions on a  $C^\infty$  manifold. Let  $U$  and  $V$  be two open subsets of  $M$ . Then using a partition of unity of the cover  $\{U, V\}$  of  $U \cup V$ , we see that the sequence

$$0 \rightarrow C^\infty(U \cup V) \rightarrow C^\infty(U) \oplus C^\infty(V) \rightarrow C^\infty(U \cap V) \rightarrow 0 \quad (22)$$

is exact as a sequence of  $C^\infty(U \cup V)$  bimodules. Also, by the theorem above, the homomorphisms from  $C^\infty(U \cup V)$  to the other algebras in the sequence are flat epimorphisms. We can therefore apply our sequence to this situation to compute the Hochschild homology of  $C^\infty(M)$ . The sequence (22) is split as a sequence of  $C^\infty(U \cup V)$  bimodules. Hence our Mayer-Vietoris sequence gives that we have a short exact sequence

$$0 \rightarrow H_*(C^\infty(U \cup V)) \rightarrow H_*(C^\infty(U)) \oplus H_*(C^\infty(V)) \rightarrow H_*(C^\infty(U \cap V)) \rightarrow 0 \quad (23)$$

Using the standard complex computing  $H_*(C^\infty(U); C^\infty(U))$  define a map

$$H_*(C^\infty(U); C^\infty(U)) \rightarrow \Omega^*(U)$$

by

$$f_0 \otimes f_1 \otimes \dots \otimes f_n \xrightarrow{\alpha} \frac{1}{n!} f_0 d(f_1) \wedge \dots \wedge d(f_n)$$

A quick calculation shows that  $\alpha \circ b = 0$  so that  $\alpha$  descends to  $H_*(C^\infty(U); C^\infty(U))$ . The lemma shows that  $\alpha$  is a quasi-isomorphism.

Armed with Theorem 2.47 we can establish the following Koszul resolution:

**Theorem 3.5** *Let  $A$  be a topological algebra and  $M$  be a left  $A$  module. Let  $I$  be an ideal generated by  $n$  elements  $(x_1, \dots, x_n)$ . Let  $I_k = (x_1, \dots, x_k)$  where the  $x_i$  commute among themselves and suppose that  $x_{k+1}$  is not a zero divisor of  $M/I_k M$ . Then the following provides a resolution of  $M/IM$ .*

$$\dots \rightarrow \bigwedge^i (y_1, \dots, y_n) \otimes M \rightarrow \dots \rightarrow \bigwedge^1 (y_1, \dots, y_n) \otimes M \rightarrow M \xrightarrow{\epsilon} M/IM$$

where the differential is

$$d(y_{j_1} \wedge \dots \wedge y_{j_i} \otimes m) = \sum_{k=1}^i (-1)^k y_{j_1} \wedge \dots \wedge \hat{y}_{j_k} \wedge \dots \wedge y_{j_i} \otimes x_{j_k} m$$

*In the case where  $M = A$  and  $A$  is nuclear Fréchet ( or LF ) the preceding is a projective resolution and can be used in computing  $\text{Tor}_*^A(A/I, \cdot)$*

**Lemma 3.6** *Let  $U$  be a star shaped subset of  $\mathbf{R}^n$ . Then  $\alpha$  is an isomorphism between  $H_*(C^\infty(U); C^\infty(U))$  and  $\Omega^*(U)$ .*

*Proof:* The proof is essentially the same as the one given by Connes' for  $C^\infty(M)$  but the details in this case are much easier because  $U$  is diffeomorphic to  $\mathbf{R}^n$ .  $C^\infty(U)^\epsilon \cong C^\infty(U \times U)$ . Let  $I = (x_1 - y_1, \dots, x_n - y_n)$ . Here  $x_i$  ( resp.  $y_i$  ) are the coordinate functions on the first ( resp. second ) copy of  $\mathbf{R}^n$ . Then  $C^\infty(\mathbf{R}^n) \cong C^\infty(\mathbf{R}^n \times \mathbf{R}^n)/I$ . Hence the Koszul resolution provides the following :

$$\begin{aligned} & \dots C^\infty(\mathbf{R}^n)^\epsilon \otimes \bigwedge^i (y_1, \dots, y_n) \rightarrow \dots \\ & \rightarrow C^\infty(\mathbf{R}^n)^\epsilon \otimes \bigwedge^1 (y_1, \dots, y_n) \rightarrow C^\infty(\mathbf{R}^n)^\epsilon \rightarrow C^\infty(\mathbf{R}^n) \rightarrow 0 \end{aligned}$$

Tensoring this resolution with  $C^\infty(\mathbf{R}^n)$  over  $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$  yields

$$\dots \xrightarrow{0} C^\infty(\mathbf{R}^n) \otimes \bigwedge^1 \mathbf{R}^n \xrightarrow{0} C^\infty(\mathbf{R}^n) \rightarrow 0$$

The homology of this is  $\Omega^*(\mathbf{R}^n)$ . The only thing left to check is that the isomorphism is induced by  $\alpha$ . This is simply a matter of comparing the standard resolution with the one given above □.

**Remark 3.7**  $\alpha$  has an inverse homomorphism  $\beta : \Omega^n(U) \rightarrow Z_n(C^\infty(U))$  defined by

$$\beta(f_0 df_1 \wedge \dots \wedge df_n) = \sum_{\sigma \in \Sigma} \epsilon(\sigma) f_0 \otimes f_{\sigma_1} \otimes \dots \otimes f_{\sigma_n}$$

We may now state our version of Connes' computation of  $H_*(C^\infty(M))$ .

**Theorem 3.8** *Let  $M$  be a finite smooth manifold. Then  $\alpha : H_*(C^\infty(M)) \rightarrow \Omega^*(M)$  is an isomorphism.*

*Proof:* Cover  $M$  by open sets, each diffeomorphic to a star-shaped region of  $\mathbf{R}^n$ . We now proceed by induction on the number  $c$  of elements in such a cover. The case  $c = 1$  is just the lemma above. So suppose the theorem true for  $c$ . Let  $U_1, \dots, U_{c+1}$  be such an open cover with  $c + 1$  elements. Let  $V = U_1 \cup \dots \cup U_c$ . Then the sequence

$$0 \rightarrow C^\infty(U_{c+1} \cup V) \rightarrow C^\infty(U_{c+1}) \oplus C^\infty(V) \rightarrow C^\infty(V \cap U_{c+1}) \rightarrow 0$$

yields as in (23) a short exact sequence which fits into the following commutative diagram

$$\begin{array}{ccc} \begin{array}{c} 0 \\ \downarrow \\ H_*(C^\infty(V \cup U_{c+1})) \\ \downarrow \\ H_*(C^\infty(V)) \oplus H_*(C^\infty(U_{c+1})) \\ \downarrow \\ H_*(C^\infty(V \cap U_{c+1})) \\ \downarrow \\ 0 \end{array} & \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\alpha \oplus \alpha} \\ \xrightarrow{\alpha} \end{array} & \begin{array}{c} 0 \\ \downarrow \\ \Omega^*(V \cup U_{c+1}) \\ \downarrow \\ \Omega^*(V) \oplus \Omega^*(U_{c+1}) \\ \downarrow \\ \Omega^*(V \cap U_{c+1}) \\ \downarrow \\ 0 \end{array} \end{array} \quad (24)$$

The maps between the second and third groups are isomorphisms because of the inductive hypothesis, since  $V \cup U_{c+1}$  can be covered by  $c$  sets diffeomorphic to a star-shaped region of  $\mathbb{R}^n$ . Hence the first groups are isomorphic. The theorem follows by induction.  $\square$

**Remark 3.9** *We do not have to require the manifold to be compact as in Connes, [10], just covered by finitely many open subsets each diffeomorphic to  $\mathbb{R}^n$ .*

*The homotopy invariance as proved in section 1 can take the following form in the topological category.*

**Proposition 3.10** *Let  $\phi_t : A \rightarrow B$  be a  $C^\infty$  path of homomorphism from topological algebras  $A$  to  $B$ . Then  $\phi_t \circ S : HC_*(A) \rightarrow HC_{*-2}(B)$  is independent of  $t$ . Also the induced map  $\phi_{t*} : H_{DR}^*(A) \rightarrow H_{DR}^*(B)$  is independent of  $t$ .*

*Proof: The  $\phi_t$  give rise to a single homomorphism*

$$\phi : A \rightarrow C^\infty([0,1]) \hat{\otimes} B$$

*and hence there is a map*

$$\phi_* : HC_*(A) \rightarrow HC_*(C^\infty([0,1]) \hat{\otimes} B) \xrightarrow[e_0]{e_1} HC_*(B)$$

*where  $e_t$  is evaluation at  $t$ . It therefore suffices to prove that*

$$HC_*(C^\infty([0,1]) \hat{\otimes} B) \xrightarrow{e_t \circ S} HC_{*-2}(B)$$

*is independent of  $t$ .  $\frac{\partial}{\partial t}$  acting on  $C^\infty([0,1])$  gives a derivation on  $C^\infty([0,1]) \hat{\otimes} B$  and by Theorem 1.2  $\mathcal{L}_{\frac{\partial}{\partial t}} \circ S$  acts like zero on cyclic homology. Therefore*

$$e_1 \circ S - e_2 \circ S = \int_0^1 \mathcal{L}_{\frac{\partial}{\partial t}} e_t \circ S dt = 0$$

$\square$

## 3.2 The Chern Character

In this section we define the Chern character from algebraic K-theory of topological algebras to cyclic and periodic cyclic homology. We then show that the Mayer-Vietoris sequence in K-theory and our Mayer-Vietoris sequence in cyclic and periodic cyclic homology are compatible.

The simplest definition of algebraic K-theory of topological algebras is that

$$K_i(A) = \pi_{i-1}(GL_\infty(A)) \cong \pi_i(BGL_\infty(A))$$

where  $GL_\infty(A)$  is endowed with its natural topology. Under suitable conditions, the Bott periodicity theorem holds. These conditions are outlined in Blackadar [4] p. 18. We refer to this book for all details concerning K-theory. We sketch only very briefly the facts we will be using. We will implicitly assume our algebras are such that periodicity does hold,

or else our results hold only for  $K_0$  and  $K_1$ . We use the description of  $K_0(A)$  in terms of idempotents in matrix algebras over  $A$  and  $K_1(A)$  in terms of invertibles in matrix algebras over  $A$ .

It is well known that to a sequence corresponding to an ideal  $I$

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$$

there corresponds a long exact sequence in K-theory which we write as a Hexagon (assuming Bott periodicity)

$$\begin{array}{ccccc} K_0(I) & \rightarrow & K_0(A) & \rightarrow & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \leftarrow & K_1(A) & \leftarrow & K_1(I) \end{array} \quad (25)$$

From this sequence we derive a Mayer-Vietoris sequence in K-theory. So consider a Cartesian square:

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & A_1 \\ \downarrow \phi_2 & & \downarrow \psi_1 \\ A_2 & \xrightarrow{\psi_2} & B \end{array} \quad (26)$$

Assume one of  $\psi_1$  or  $\psi_2$  is surjective, say  $\psi_2$ . This implies that  $\phi_1$  is also surjective. It is then easy to see that the kernels of  $\psi_2$  and  $\phi_1$  are canonically isomorphic. Let  $I$  denote the kernels of the horizontal arrows so that we end up with a diagram as follows:

$$\begin{array}{ccccccc} I & \xrightarrow{i_1} & A & \xrightarrow{\phi_1} & A_1 & & \\ \downarrow = & & \downarrow \phi_2 & & \downarrow \psi_1 & & \\ I & \xrightarrow{i_2} & A_2 & \xrightarrow{\psi_2} & B & & \end{array} \quad (27)$$

This leads to a commutative diagram where the rows are exact:

$$\begin{array}{ccccccc} \xrightarrow{\partial} & K_i(I) & \xrightarrow{i_1} & K_i(A) & \xrightarrow{\phi_1} & K_i(A_1) & \xrightarrow{\partial} \\ & \downarrow = & & \downarrow \phi_2 & & \downarrow \psi_1 & \\ \xrightarrow{\partial} & K_i(I) & \xrightarrow{i_2} & K_i(A_2) & \xrightarrow{\psi_2} & K_i(B) & \xrightarrow{\partial} \end{array} \quad (28)$$

After a little minor league diagram chasing one can splice these sequences together to get a Mayer-Vietoris sequence:

$$\xrightarrow{\partial} K_i(A) \xrightarrow{\phi_1 \oplus \phi_2} K_i(A_1 \oplus A_2) \xrightarrow{\psi_1 - \psi_2} K_i(B) \xrightarrow{\partial}$$

where the boundary map is the composite of the following portion of the diagram (28):

$$\begin{array}{ccc} & K_{i-1}(I) & \xrightarrow{i_1} & K_{i-1}(A) \\ & \uparrow & & \\ K_i(B) & \xrightarrow{\partial} & K_{i-1}(I) & \end{array} \quad (29)$$

To make this map more explicit, we have to describe the map  $\partial$  in the hexagonal sequence (25). First consider the map  $\partial : K_1(A/I) \rightarrow K_0(I)$ . Let  $[u] \in K_1(A/I)$  be a class in  $K_1(A/I)$  represented by an invertible  $u \in GL_n(A/I)$ . Then the matrix

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

is in the connected component of the identity in  $GL_{2n}(A/I)$ . Hence it lifts to an invertible  $w$  in  $GL_{2n}(A)$ . Now define

$$\partial([u]) = [w \begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix} w^{-1}] - [\begin{pmatrix} \text{Id} & 0 \\ 0 & 0 \end{pmatrix}]$$

$\partial([u])$  defines a class in  $K_0(A)$  which goes to zero in  $K_0(A/I)$  and so defines a class in  $K_0(I)$ .

To define the map  $\partial : K_0(A/I) \rightarrow K_1(I)$ , let  $[e] - [\text{Id}]$  be a class in  $K_0(A/I)$  where  $e \in M_n(A/I)$  is an idempotent and  $\text{Id}$  is the  $n \times n$  identity matrix. Then  $\partial([e] - [\text{Id}]) = [e^{2\pi i x}]$  where  $x$  is a lift of  $e$  to an element of  $M_n(A)$ , (which is not necessarily an idempotent). From this description of the boundary maps for the long exact sequence of an ideal, one can easily figure out the form of the boundary maps of the Mayer-Vietoris sequence, which we will do in practice below.

We now define the Chern character. Recall that there is a universal differential graded algebra  $\Omega_*(A)$ , see section 1, such that one has an isomorphism  $H_*(\Omega_*(A)/[\cdot, \cdot]) \cong \ker\{\tilde{B} : HC_*(A) \rightarrow H_{*+1}(A)\}$ . The Chern character is best defined via this algebra. Thus for a projection  $p \in M_n(A)$  we define  $ch_i([p]) = \frac{2^i!}{i!} p(dpdp)^i \in H_{2i}^{DR}(A)$ . This definition is based on the fact that in some precise sense,  $pdpdp$  is the curvature of the connection on the projective module,  $pA^n$ , that comes from the trivialization by  $p$ . Using the isomorphism of section 1.2 we see that this element corresponds to the element  $\frac{2^i!}{i!} p \otimes p \otimes \dots \otimes p$  ( $2i + 1$  times) in  $C_{2i}^\lambda(A)$ . Chasing this element back into the double complex  $C_{**}(A)$  we find that  $ch_i([p])$  is given by the following zig zag of elements:

$$\begin{array}{ccccccc}
 \frac{2^i!}{i!} p^{\otimes 2i+1} & \xleftarrow{\epsilon} & \dots & & & & \\
 & & & & & & \\
 & & \dots \xleftarrow{N} & 12p^{\otimes 5} & & & \\
 & & & \downarrow b & & & \\
 & & & \cdot & \xleftarrow{1-t} & 6p^{\otimes 4} & \\
 & & & & & \downarrow -b' & \\
 & & & & & \cdot & \xleftarrow{N} & -2p^{\otimes 3} & \\
 & & & & & & & \downarrow b & \\
 & & & & & & & \cdot & \xleftarrow{1-t} & p^{\otimes 2} & \\
 & & & & & & & & & \downarrow -b' & \\
 & & & & & & & & & \cdot & \xleftarrow{N} & p & 
 \end{array} \tag{30}$$

For the odd Chern characters,  $ch_i : K_1(A) \rightarrow HC_{2i+1}(A)$  we set for  $u \in GL_n(A)$ ,  $ch_i([u]) = (u^{-1} - 1)d(u - 1)d(u^{-1} - 1) \cdot d(u - 1) \in \Omega_{2i+1}(A)$ . In the same way it is easy to find an expression for  $ch_i([u])$  in the double complex.

### 3.3 Relative Cyclic Homology, K-theory and Chern Characters

In this section we show that the Chern character as defined by Connes [10] from K-theory to cyclic homology is compatible as a map from the Mayer-Vietoris sequence of K-theory to ours in cyclic homology. More particularly,

**Theorem 3.11** *Let  $A, A_1, A_2$  and  $B$  be topological algebras satisfying the hypotheses making our Mayer-Vietoris sequence hold and let them sit in a cartesian square (5) Then the following diagram is commutative:*

$$\begin{array}{ccccccc}
 \dots \rightarrow & K_1(B) & \rightarrow & K_0(A) & \rightarrow & K_0(A_1) \oplus K_0(A_2) & \\
 & \downarrow \text{ch}_l & & \downarrow \text{ch}_l & & \downarrow \text{ch}_l \oplus \text{ch}_l & \\
 \dots \rightarrow & HC_{2l+1}(B) & \rightarrow & HC_{2l}(A) & \rightarrow & HC_{2l}(A_1) \oplus HC_{2l}(A_2) & \\
 & & & & & & \\
 & \rightarrow & K_0(B) & \rightarrow & K_1(A) & \rightarrow & \dots \\
 & & \downarrow \text{ch}_l & & \downarrow \text{ch}_{2l-1} & & \dots \\
 & \rightarrow & HC_{2l}(B) & \rightarrow & HC_{2l-1}(A) & \rightarrow & \dots
 \end{array} \tag{31}$$

Whereas the boundary map in the Mayer-Vietoris sequence for Hochschild or noncommutative de Rham cohomology is easy to compute, the boundary map in the Mayer-Vietoris sequence for cyclic homology is difficult to determine since the exactness was established by spectral sequence means. We therefore look for ways to show the diagram above is commutative. Towards this end we introduce relative K-groups and relative cyclic homology groups.

Given a surjective homomorphism  $\phi : A \rightarrow B$  we define  $K_0(A, B)$  in terms of triples  $(p, q, u)$  where  $p, q \in M_n(A)$  are idempotents and  $u \in GL_n(B)$  is an equivalence of  $\phi(p)$  with  $\phi(q)$ , i.e.  $u\phi(p)u^{-1} = \phi(q)$ . One then introduces an equivalence relation on the set of such triples, thus obtaining a group  $K_0(A, B)$ . See Blackadar [4]. For  $K_1(A, B)$  we consider pairs  $(u, p)$  where  $u \in GL_n(A)$  and  $p$  is an idempotent in  $M_n(B)$  and they satisfy that  $u = e^{2\pi i x}$  where  $x \in M_n(A)$  is a pullback of  $p$ . The relative groups sit in a long exact sequence

$$\rightarrow K_1(B) \rightarrow K_0(A, B) \rightarrow K_0(A) \rightarrow K_0(B) \rightarrow K_1(A, B) \rightarrow K_1(A) \rightarrow$$

This sequence is related to the one above (25) by the strong excision theorem of K-theory which says that  $K_*(A, B) \cong K_*(I)$  where  $I = \ker(\phi)$ . The maps  $K_0(A, B) \rightarrow K_0(A)$  and  $K_1(A, B) \rightarrow K_1(A)$  are obvious. The map  $K_1(B) \rightarrow K_0(A, B)$  is given by sending  $u \in GL_n(B)$  to the triple  $(0, 0, u)$  and the map  $K_0(B) \rightarrow K_1(A, B)$  is given by sending an idempotent  $p \in M_n(B)$  to  $(e^{2\pi i x}, p)$  where  $x$  is a pullback of  $p$ .

Finally we extend the Chern characters to the relative case by defining

$$\text{ch}_l(p, q, u) = (\text{ch}_l(p) - \text{ch}_l(q), \text{ch}_l(u)) \in HC_{2l}(A, B)$$

and

$$\text{ch}_l(u, p) = (\text{ch}_l(u), \text{ch}_{l+1}(p)) \in HC_{2l+1}(A, B)$$

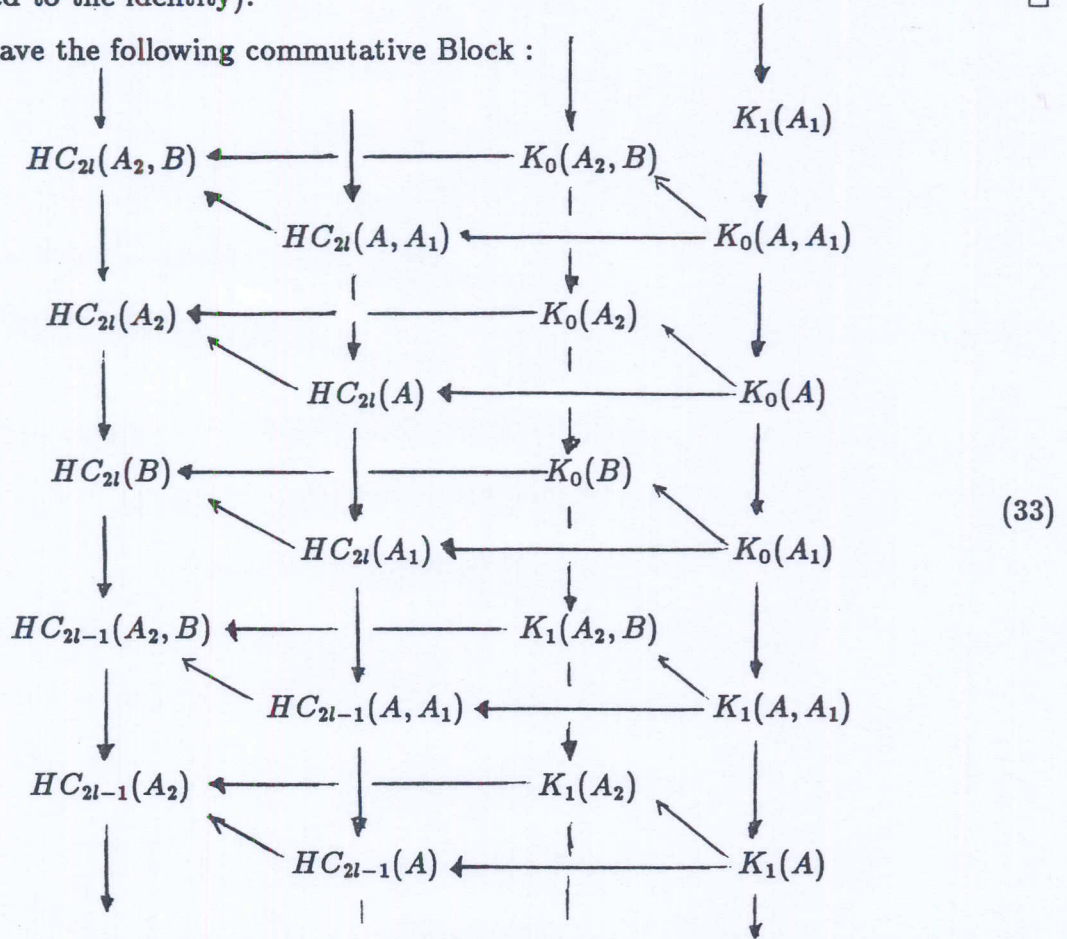
one checks that  $\text{ch}_l(p, q, u)$  and  $\text{ch}_l(u, p)$  are cycles. This uses the fact that  $e^{2\pi i p} = \text{Id}$ .

**Lemma 3.12** *The following diagram is commutative:*

$$\begin{array}{ccccccc}
 K_1(B) \xrightarrow{\partial} & K_0(A, B) & \rightarrow & K_0(A) & \rightarrow & K_0(B) \xrightarrow{\partial} & K_1(A, B) \\
 \downarrow \text{ch}_l & \downarrow \text{ch}_l & & \downarrow \text{ch}_l & & \downarrow \text{ch}_l & \downarrow \text{ch}_{l-1} \\
 HC_{2l+1}(B) \xrightarrow{\partial} & HC_{2l}(A, B) & \rightarrow & HC_{2l}(A) & \rightarrow & HC_{2l}(B) \xrightarrow{\partial} & HC_{2l-1}(A, B)
 \end{array} \tag{32}$$

*Proof:* The proof is a straight forward check. The only point to check is that  $ch_{l-1} \circ \partial(p) = \partial \circ ch_l(p)$ . This follows since  $ch_{l-1}(e^{2\pi iz})$  is zero in  $HC_{2l-1}(A)$  because  $e^{2\pi iz}$  is zero in  $K_1(A)$  (it is connected to the identity).  $\square$

We then have the following commutative Block :



In this diagram all the columns are exact. Hence, by the same diagram chase that gave the Mayer-Vietoris sequence for K-theory, we see that the diagram (31) is commutative.

## 4 The Calculations

### 4.1 The Crossed Product Algebra

Let  $G$  be a compact Lie group acting on the left of a compact smooth manifold  $M$ . We would now like to apply some of our techniques to computing the various homologies of a dense subalgebra of  $C^*(G, C(M))$ ,  $C(M)$  denotes the  $C^*$ -algebra of continuous functions on  $M$ . This  $C^*$ -algebra is of interest, in part because one has the basic isomorphism

$$K_G^*(M) \cong K_*(C^*(G, C(M)))$$

It thus provides a sort of homotopy quotient, where the standard homotopy quotient  $M_G = M \times_G EG$  is inadequate as far as K-theory is concerned. As in the previous example, the dense subalgebra of  $C^*(G, C(M))$  we are interested in, is the global sections of a sheaf of algebras, in this case over the ordinary quotient space  $M/G$ . We therefore describe the algebra  $C^*(G, C(M))$  in these terms.

First let us recall the definition of  $C^*(G, C(M))$ . Consider the vector space  $C(G, C(M))$  the space of  $C(M)$ -valued functions on  $G$ , (recall  $G$  is compact, if it were not compact, we would consider  $C_c(G, C(M))$  the space of compactly supported functions.) Let  $\alpha : G \rightarrow \text{Aut}(C(M))$  denote the action of  $G$  on  $C(M)$

$$\alpha_g(f(m)) = f(g^{-1} \cdot m)$$

Define a multiplication on  $C(G, C(M))$  by

$$u * v(g) = \int_G u(h) \alpha_h(v(h^{-1}g)) dh$$

for  $u, v \in C(G, C(M))$ . Define the  $*$  operation by  $u^*(g) = \alpha_g(u(g^{-1}))^*$ . One checks that this makes  $C(G, C(M))$  into an associative  $*$ -algebra. Now define a norm by

$$\|u\| = \int_G |u(g)| dg$$

where the norm in the integrand is the sup norm on  $M$ . The completion is then a Banach Algebra and  $C^*(G, C(M))$  or  $C^*(G, C(M), \alpha)$  if we need to distinguish the action, is the enveloping  $C^*$ -algebra. We will denote  $C^*(G, C(M))$  simply by  $C^*(G, M)$ .

Consider  $K = K(L^2(G))$  the algebra of compact operators of the Hilbert space of square integrable complex valued function on  $G$ , with respect to Haar measure.  $G$  act on  $L^2(G)$  via the left and right regular representations. Denote them by  $l$  and  $r$  respectively. It therefore acts on  $K$  by  $Adr$  or  $Adl$ , choose  $Adr$ .  $G$  also acts on  $C(M)$  via its action on  $M$ . Then  $G$  acts on  $C(M) \otimes K$  by the tensor product action. ( $\otimes$  denotes the projective  $C^*$ -tensor product, of course it really doesn't matter since  $K$  is a nuclear  $C^*$ -algebra.) Now  $C(M) \otimes K \cong C(M, K)$  the  $K$ -valued functions on  $M$ . The tensor product action then corresponds to

$$(\alpha \otimes Adr)_g f(m) = r_g \cdot f(g^{-1} \cdot m) \cdot r_g^*$$

We then have the following well known theorem.



**Theorem 4.1** *The transformation group  $C^*$ -algebra  $C^*(G, C(M))$  is isomorphic to  $\{C(M, K)\}^G$  the invariants under the tensor product action,  $\alpha \otimes \text{Ad}$ .*

From this theorem we quickly arrive at the following analogue of Swan's statement that there is an equivalence between the category of complex vector bundles on a compact space and projective modules over the ring of continuous ( or smooth ) complex-valued functions.

**Proposition 4.2** ( Green, [15] ) *There is an equivalence of categories between complex equivariant  $G$  vector bundles on a compact  $G$ -space and projective modules over the algebra  $C^*(G, C(M))$ . In particular, there is an isomorphism*

$$K_G^*(M) \cong K_*(C^*(G, C(M)))$$

**Lemma 4.3** *For  $m \in M$  and  $f \in C(M, K)^G$ ,  $f(m) \in K(L^2(G))^{G_m}$ . Furthermore  $K(L^2(G))^{G_m} \cong K(L^2(G_m))^{G_m} \otimes K(L^2(G/G_m))$*

## 4.2 Mayer-Vietoris for Group Actions

We now define a dense subalgebra of  $C^*(G, M)$ . It is built up out of simpler algebras. So consider  $(\pi, V)$ , a finite dimensional representation of  $G$ . In analogy with the expression of  $C^*(G, M)$  given by Theorem (4.1), we define

$$C^\infty(G, M, \pi) = \{C^\infty(M) \hat{\otimes}_C \text{End}(V)\}^{\alpha \otimes \text{Ad}\pi}$$

This algebra is a nuclear Fréchet algebra and is somewhat easier to deal with in part because it has an identity element. Now define

$$C^\infty(G, M) = \bigcup_{(\pi, V) \subset L^2(G)} C^\infty(G, M, \pi)$$

$C^\infty(G, M)$  is a nuclear LF algebra and therefore when considering Hochschild and cyclic homology, it is always with respect to the inductive tensor product. Also, from the fact that  $C^\infty(G, M)$  is closed under holomorphic functional calculus and that it is dense in  $C^*(G, C(M))$ , it follows that one has  $K_*(C^\infty(G, M)) \cong K_*(C^*(G, C(M)))$ . We begin with the following refinement of lemma 3.2.

**Lemma 4.4** *Let  $U \supset V$  be two open  $G$ -invariant subsets of  $M$ . Then given a sequence of functions  $\{f_i\}$  on  $V$  there exists a  $G$ -invariant function  $\phi$  on  $U$  such that  $\phi$  and all its derivatives vanish on  $U - V$  and such that  $f_i \phi$  extend to functions on  $U$  which vanish with all their derivatives on  $U - V$ .*

*Proof:* The proof proceeds as before. Let  $C_j$  be an increasing sequence of compact  $G$ -invariant subsets of  $V$  such that  $C_j$  is contained in the interior of  $C_{j+1}$  and so that  $\bigcup C_j = V$ . Then let  $0 \leq \phi_j \leq 1$  be smooth  $G$ -invariant functions on  $V$  such that  $\phi_j = 1$  on  $C_j$  and the support of  $\phi_j$  is contained in  $C_{j+1}$ . Such functions exist ( for example, by first finding

functions which are not necessarily  $G$ -invariant and then averaging ). The rest of the proof is the same as in 3.2.  $\square$

A few words need to be said about the fact that the algebra,  $C^\infty(G, M)$  does not have a unit. Homological algebra typically works for algebras with unit. The problem with non-unital algebras is that the category of modules is not abelian in general. In the case at hand however, the algebra  $C^\infty(G, M)$  is the union of algebras with unit and therefore the category of non-degenerate  $C^\infty(G, M)$ -modules is abelian and has enough projectives. (Recall that an  $A$ -module  $M$  is non-degenerate if  $AM = M$ .) Another way to handle the situation is to prove theorems about the algebras  $C^\infty(G, M, \pi)$  and deduce the corresponding theorems about  $C^\infty(G, M)$  by just taking direct limits. However one handles it, what goes through for  $C^\infty(G, M, \pi)$  goes through for  $C^\infty(G, M)$ .

**Theorem 4.5** *For any two  $G$ -invariant open sets  $U \supset V$  the following homomorphisms are flat epimorphisms:*

1.  $C^\infty(G, U, \pi) \rightarrow C^\infty(G, V, \pi)$
2.  $C^\infty(G, U) \rightarrow C^\infty(G, V)$
3.  $C^\infty(G, \bar{U}, \pi) \rightarrow C^\infty(G, \bar{V}, \pi)$
4.  $C^\infty(G, \bar{U}) \rightarrow C^\infty(G, \bar{V})$

*Proof:* Given the lemma above, the proof of theorem is easy.  $\square$

And of course as immediate consequences we have

**Theorem 4.6** *Let  $U$  and  $V$  be two open  $G$ -invariant subsets of  $M$ . Then from the Cartesian square of algebras*

$$\begin{array}{ccc} C^\infty(G, U \cup V) & \xrightarrow{\phi_1} & C^\infty(G, U) \\ \downarrow \phi_2 & & \downarrow \psi_1 \\ C^\infty(G, V) & \xrightarrow{\psi_2} & C^\infty(G, U \cap V) \end{array} \quad (34)$$

*there corresponds the Mayer-Vietoris sequences of Theorem 1.8. The sequences also hold for the algebras  $C^\infty(G, M, \pi)$ .*

*Proof:* The sequence

$$0 \rightarrow C^\infty(G, U \cup V) \rightarrow C^\infty(G, U) \oplus C^\infty(G, V) \rightarrow C^\infty(G, U \cap V) \rightarrow 0$$

is exact as can be shown using a  $G$ -invariant partition of unity. It also splits as  $C^\infty(G, U \cup V)$  bimodules, since  $G$ -invariant functions lie in the center of  $C^\infty(G, \cdot)$ . Therefore, the hypotheses of Theorem 1.8 are met.  $\square$

### 4.3 The Algebra of Invariant Functions

The algebras  $C^\infty(G, M, \pi)$  are interesting in their own right. For example, if  $\pi_{triv}$  is the trivial representation, then  $C^\infty(G, M, \pi_{triv})$  is just the space of invariant functions. And this identifies to a subset of functions on the quotient space  $G \backslash M$ . And in fact, we have the following theorem.

**Theorem 4.7** 1. *The non-commutative de Rham cohomology of  $C^\infty(G, M, \pi_{triv})$  is the ordinary de Rham cohomology of the quotient space,  $H_{DR}^i(C^\infty(G, M, \pi_{triv})) \cong H_{DR}^i(G \backslash M)$ .*

2. *The Hochschild to cyclic spectral sequence degenerates at the  $E^2$  level.*

First we recall a theorem of constant use in transformation groups.

**Theorem 4.8** [7] II.5.4 (*The Slice Theorem.*) *Let  $G$  be a compact Lie group acting on a completely regular space  $M$ . Then for each  $x \in M$  there exists a set  $S$  with  $x \in S$  such that*

1.  *$S$  is closed in  $GS$ .*
2.  *$GS$  is an open neighborhood of the orbit of  $x$ .*
3.  *$G_x S = S$ .*
4.  *$GS \cong G \times_{G_x} S$  as a  $G$ -space.*

Furthermore, if  $M$  is a smooth manifold.  $S$  can be chosen to be diffeomorphic to  $\mathbf{R}^n$  as a  $G_x$ -space and where  $G_x$  acts as a linear representation on  $\mathbf{R}^n$ .

*Proof:* (of Theorem 4.7). The proof proceeds along the same lines as in the computation of the homologies of smooth functions. Recall that a differential form  $\omega$  is basic if for all  $X \in Lie(G)$  one has  $i_X(\omega) = L_X(\omega) = 0$ . [1]. We denote the space of  $G$  basic forms on  $M$  by  $\Omega_b^*(M)$  and hope that no confusion arises because of lack of reference to  $G$ . Let  $\alpha : Z_i(C^\infty(G, U, \pi_{triv})) \rightarrow \Omega_b^*(U)$ . Where  $U$  is a  $G$ -invariant open subset of  $M$ . That  $\alpha$  actually lands in the basic forms is easy to check.  $\alpha$  descends to a map  $\alpha : H_*(C^\infty(G, U, \pi_{triv})) \rightarrow \Omega_b^*(U)$  and one checks that  $\alpha$  is a map of complexes

$$(H_*(C^\infty(G, U, \pi_{triv})), B) \rightarrow (\Omega_b^*(U), d)$$

So let  $x \in M$ . By the slice theorem, one can choose an open set  $U$  such that there is a slice  $S$  about  $x$  such that  $S$  is diffeomorphic to  $\mathbf{R}^n$  and  $U \cong GS \cong G \times_{G_x} S$ . Now let  $h_t$  be a contraction of  $S$  to a point. This is the same thing as an invariant deformation retraction of  $U$  onto  $Gx$ . This gives us a  $C^\infty$  path of homomorphisms  $\phi_t : C^\infty(G, U, \pi_{triv}) \rightarrow C^\infty(G, U, \pi_{triv})$ , such that  $\phi_0 = Id$  and  $\phi_1$  is the restriction homomorphism from  $C^\infty(G, M, \pi_{triv})$  to  $C^\infty(G, Gx, \pi_{triv})$ . And this last algebra is just the scalars since  $Gx$  is a homogeneous space. Hence, by homotopy invariance 3.10 the induced map on non-commutative de Rham cohomology is the same. That is,

$$H_{DR}^*(C^\infty(G, U, \pi_{triv})) \rightarrow H_{DR}^*(C^\infty(G, Gx, \pi_{triv})) \cong H_{DR}^*(C)$$

We can therefore see that  $\alpha$  induces an isomorphism

$$\alpha : H_{DR}^*(C^\infty(G, U, \pi_{triv})) \rightarrow H^*(\Omega_b^*(U))$$

By comparing Mayer-Vietoris sequences and doing an induction over number of elements in a good cover of  $G \backslash M$ , that is really a cover by  $G$ -invariant subsets of  $M$ , we arrive at the first result. That the cohomology of  $\Omega_b^*(M)$  is the same as that of  $G \backslash M$  is also a Mayer-Vietoris proof based on a Cech -de Rham complex, with respect to a good cover of  $G \backslash M$ .

The denegeneration of the Hochschild to cyclic spectral sequences occurs because we can map the bicomplex  $C_{**}(C^\infty(G, M, \pi_{triv}))$  to the bicomplex composed of  $\Omega_b^*(M)$  with the differential  $d$  in one direction and the differential 0 in the other. The result follows just as in Loday and Quillen [18].  $\square$

#### 4.4 Equivariant Cohomology

**Lemma 4.9** *If  $H \subset G$  is a closed subgroup then*

$$C^\infty(G, G \times_H M, \pi) \cong C^\infty(H, M, \pi|_H)$$

*Proof:* Let  $f \in C^\infty(G, G \times_H M, \pi)$ . Thus  $f : G \times_H M \rightarrow \text{End}(V)$  and  $f(g, m) = \pi_g f(e, m) \pi_{g^{-1}}$ . Hence  $f$  is determined by its value on  $M$ . Also  $f(e, h \cdot m) = f(h, m) = \pi_h f(e, m) \pi_{h^{-1}}$ . So  $f \in \{C^\infty(M) \hat{\otimes} \text{End}(V)\}^{\alpha \otimes \text{Ad} \pi|_H}$   $\square$ .

More importantly, we have an analogue of this for the algebra  $C^\infty(G, M)$ .

**Lemma 4.10** *Let  $H \subset G$  be a closed subgroup. Then we have a Morita equivalence*

$$C^\infty(G, G \times_H M) \sim C^\infty(H, M)$$

*Or more precisely  $C^\infty(G, G \times_H M) \cong C^\infty(H, M) \otimes F(L^2(G_x))$  where  $F(L^2(G_x))$  denotes the subalgebra of the compact operators which can be written as finite matrices with respect to a basis of  $L^2(G_x)$  that respects the decomposition of  $L^2(G_x)$  into irreducible representations.*

*Proof:* This follows either from lemma 4.3 or from the lemma above.  $\square$  Armed with this lemma we may finally prove

**Theorem 4.11** *The Chern character is an isomorphism between*

$$K_G^*(M) \otimes_{\mathbb{Z}} \mathbb{C} \cong K_*(C^\infty(G, M)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{ch} HP_*(C^\infty(G, M))$$

*Proof:* Again, this proof simply uses a comparison between the Mayer-Vietoris sequences in K-theory and cyclic homology and a local computation. The theorem then follows by doing an induction on the number of elements in a good cover of  $G \backslash M$ .

So we only have to do the local calculation. Because K-theory is a compactly supported cohomology theory, we find it convenient to do our calculations using the closed sets  $\bar{U}$ . Given any point  $\tilde{x} \in G \backslash M$  let  $\tilde{U}$  be a contractible open set containing it. When  $\tilde{U}$  is pulled

by to  $M$ , the slice theorem tells us that  $U \cong G \times_{G_x} S$  where  $x$  is a pullback of  $\tilde{x}$  and  $S$  is a slice that we can choose to be diffeomorphic to  $\mathbf{R}^n$ . As in the calculation of homology of the invariant functions, we find a deformation retraction of  $U$  onto  $Gx$ . By homotopy invariance, this time of  $HP_*$ , Morita invariance, and lemma 4.10 we have an isomorphism

$$\begin{aligned} HP_*(C^\infty(G, \bar{U})) &= HP_*(C^\infty(G, \overline{G \times_{G_x} S})) \rightarrow HP_*(C^\infty(G_x, \bar{S})) \\ &\cong HP_*(C^\infty(G_x, point)) \cong R(G_x) \otimes_{\mathbf{Z}} \mathbf{C} \end{aligned}$$

where  $R(G)$  denotes the representation ring of  $G$ . And tracing all the homomorphisms, one sees that the composed map

$$\begin{aligned} R(G_x) \otimes_{\mathbf{Z}} \mathbf{C} &\cong K_*(C^\infty(G_x, point)) \cong K_*(C^\infty(G, \bar{U})) \\ &\xrightarrow{ch} HP_*(C^\infty(G, \bar{U})) \rightarrow R(G_x) \otimes_{\mathbf{Z}} \mathbf{C} \end{aligned}$$

is an isomorphism. This completes the local calculation.  $\square$

We will now present some examples of computations in cyclic homology of crossed product algebras. For notation's sake, let us denote by  $R(G)$  the representation ring of  $G$  with complex coefficients.

**Example 1:** Let  $S^1$  act on the two sphere  $\mathbf{CP}^1$  by  $\lambda \cdot (z_0 : z_1) = (z_0 : \lambda z_1)$ . This action has two fixed points and no other points with non-trivial isotropy group. By decomposing  $\mathbf{CP}^1$  into the standard affine open subsets,  $U_0, U_1$ , it is easily seen by retracting  $U_i$  back to its fixed point that  $H_{DR}^*(C^\infty(S^1, U_i)) \cong R(S^1)$  in degree 0 and is zero elsewhere. Also, since  $S^1$  acts freely on  $U_0 \cap U_1$ , we see that  $H_{DR}^*(C^\infty(S^1, U_0 \cap U_1)) = \mathbf{C}$  in degree 0 and is zero else where. We therefore apply the Mayer-Vietoris sequence and find the following exact sequence

$$H_{DR}^0(C^\infty(S^1, \mathbf{CP}^1)) \rightarrow R(S^1) \oplus R(S^1) \xrightarrow{\epsilon - \epsilon} \mathbf{C} \rightarrow 0$$

where  $\epsilon$  is the map from  $R(S^1) \rightarrow \mathbf{C}$  that associates to a virtual representation its virtual dimension. Now,  $R(S^1) \cong \mathbf{C}[t, t^{-1}]$  is the Laurent polynomials in one variable. The group we are interested in

$$H_{DR}^0(C^\infty(S^1, \mathbf{CP}^1)) = \ker(R(S^1) \oplus R(S^1) \rightarrow \mathbf{C})$$

It is now easy to see that this kernel is isomorphic to the free  $R(S^1)$ -module on two generators. For we have an exact sequence

$$R(S^1) \oplus R(S^1) \xrightarrow{A} R(S^1) \oplus R(S^1) \rightarrow \mathbf{C}$$

where  $A$  is given by the matrix:

$$\begin{pmatrix} 1 & -t \\ 1 & -1 \end{pmatrix}$$

Its inverse is given by  $\frac{1}{1-t}A$ . One simply checks that if  $(x, y)$  is in the kernel of  $\epsilon - \epsilon$  then  $A$  applied to  $(x, y)$  is divisible by  $1 - t$ . We therefore see that  $H_{DR}^*(C^\infty(S^1, \mathbf{CP}^1))$  is just



The  $E^3$  differential is not zero and the  $C$  is killed by the arrow drawn. We therefore see that  $HP_0$  is a free  $R(S^1)$  module of rank three, while  $HP_1$  vanishes. The homology of the other algebras  $C^\infty(G, M, \pi)$  should be interesting to study. The periodic cyclic homology of the crossed product algebra of a compact transformation group should provide a useful tool in transformation groups. It is easy to see that know  $\mathbf{Z}$  graded equivariant cohomology theory, such as Bredon's, can have the pleasant property that a chern character (suitably defined) from equivariant K-theory to this theory is an isomorphism. Thus, the periodicity of periodic cyclic homology is seen to be a necessity for this result to be true.

## 4.5 The Algebra of Differential Operators

By now, the Hochschild and cyclic homology of the ring of differential operators have been computed independently by several authors, [4], [8],[25]. So in this section, we only summarize our results.

For convenience. let  $M$  be a smooth, connected, finite manifold of dimension  $N$ . The results of this section extend to non-finite manifolds by standard tricks. Let  $\mathcal{D}(M)$  denote the algebra of differential operators on  $M$ . If the space is understood we will simply write  $\mathcal{D}$ . The filtration by degree yields a spectral sequence converging to the Hochschild homology with  $E^1$  term given by  $H_*(\text{gr}(\mathcal{D}))$ .

Recall that if  $M$  is a differentiable manifold that a differential operator on  $M$  is an operator that can be expressed in any local coordinate system  $x_i$  as  $\sum f_\alpha \partial^\alpha$  where  $\alpha = (i_1, \dots, i_k)$  is a multi-index and  $\partial^\alpha$  is a shorthand for  $\frac{\partial}{\partial x_{i_1}} \dots \frac{\partial}{\partial x_{i_k}}$ . Equivalently, by a theorem of Peetre, a differential operator is a continuous operator  $D : C^\infty(M) \rightarrow C^\infty(M)$  such that  $\text{supp } D(f) \subset \text{supp } f$  for  $f \in C^\infty$ . This characterization of differential operator as "local" operators is exactly what we need to apply our Mayer-Vietoris sequence.  $\mathcal{D}$  is filtered by degree,  $F_i \mathcal{D} = \{D \in \mathcal{D} \mid D \text{ is of degree } \leq i\}$ . We now equip each  $F_i \mathcal{D}$  with its natural Frechet topology given by uniform convergence of all derivatives on compact sets of the coefficients of the operator. Now topologize  $\mathcal{D}$  as the inductive limit of the  $F_i \mathcal{D}$  making  $\mathcal{D}$  an LF-space. Thus, when we consider Hochschild homology, we are thinking of the homology with respect to the inductive tensor product.

Let  $U$  and  $V$  be two open subsets of  $M$ . Let  $0 \leq \phi \leq 1$  be a smooth function on  $M$  such that  $\phi = 1$  on  $U - V$  and  $0$  on  $V - U$ . Then we can use  $\phi$  to prove that the following sequence is exact:

$$0 \rightarrow \mathcal{D}(U \cup V) \rightarrow \mathcal{D}(U) \oplus \mathcal{D}(V) \rightarrow \mathcal{D}(U \cap V) \rightarrow 0$$

This is exact as a sequence of  $\mathcal{D}(U \cup V)$  bimodules. Unlike our other examples above, this sequence does not split as a sequence of bimodules. This is because  $C^\infty$  does not commute with  $\mathcal{D}$ .  $\mathcal{D}$  being the inductive limit of nuclear Frechet spaces, is nuclear LF. Hence we may apply our theorem 2.48.

**Lemma 4.12** *If  $U \supset V$  are two open subsets of  $M$ , then  $\mathcal{D}(U) \rightarrow \mathcal{D}(V)$  is a flat epimorphism.*

*Proof:* This follows since  $\mathcal{D}(U)$  is a bimodule over  $C^\infty(U)$ . □

**Corollary 4.13** For  $U$  and  $V$  there is a long exact sequence

$$\dots \rightarrow H_i(\mathcal{D}(U \cup V)) \rightarrow H_i(\mathcal{D}(U)) \oplus H_i(\mathcal{D}(V)) \rightarrow H_i(\mathcal{D}(U \cap V)) \rightarrow \dots$$

**Theorem 4.14**  $H_i(\mathcal{D}(M)) \cong H^{2N-i}(M)$

From the computation of the Hochschild homology, we see that the  $E^1$ -term of the Hochschild to cyclic spectral sequence has  $E_{p,q}^1 = H_{DR}^{2n-q+p}(M)$ . There seems to be no obvious differential at any higher term. Therefore, one's first guess is that they are all zero. This indeed is what happens.

**Theorem 4.15** ([5], [25])

For  $M$  a differentiable manifold of dimension  $N$ , the Hochschild to cyclic spectral degenerates at  $E^1$ , thus  $HC_i(\mathcal{D}(M)) \cong H^{2N-i}(M) \oplus H^{2N-i+2}(M) \oplus \dots$ .

Let  $P \xrightarrow{f} B$  be a  $C^\infty$  fiber bundle where  $P$  and  $B$  are smooth manifolds. As in Brylinski [8] consider the algebra of relative differential operators

$$\mathcal{D}(f) = \{D \in \mathcal{D}(P) \mid Df^*(\phi) = f^*(\phi)D, \phi \in C^\infty(B)\}$$

That is,  $\mathcal{D}(f)$  is the algebra of all differential operators on  $P$  that differentiate only in the fiber direction. Then  $\mathcal{D}(f)$  is again graded by degree of differential operator and  $F_0\mathcal{D}(f) = C^\infty(P)$ . Hence by Theorem 1 of Block [5]  $HP_*(\mathcal{D}(f)) \cong \bigoplus_i H^{2i+*}(P)$ . On the otherhand, by the computation of the Hochschild homology of the differential operators, the local structure of  $\mathcal{D}(f)$  and Mayer-Vietoris sequence, it is easy to arrive at the following result.

**Proposition 4.16**

$$H_{DR}^i(\mathcal{D}(f)) \cong \bigoplus_{j+k=i} H^j(B, \mathcal{K}^{2m-k}(F))$$

where  $m$  is the dimension of the fiber of  $f$ .

In fact, even more can be said. By comparing the Hochschild to cyclic spectral sequence with the Leray spectral sequence of the fibration, we arrive at many examples of algebras, for which the Hochschild to cyclic spectral sequence degenerates at arbitrarily high levels.

**Theorem 4.17** The Hochschild to cyclic spectral sequence for the algebra  $\mathcal{D}(f)$  is, after the  $E^2$  term, to a direct sum of Leray spectral sequences for the fiber bundle  $f$ .



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