Excess intersection in equivariant bivariant K-theory

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Abstract - Using the formalism of bicycles we present an excess intersection formula in Kasparov's group $KK_G(X,Y)$.

Resumé - Dans le cadre de bicycles (cycles bivariants) nous donnons une formule d'intersection excessive.

1 Introduction

In [1], based on earlier work of [2], [4], the authors described Kasparov's group

$$KK_G(C_0(X), C_0(Y))$$

in geometric terms when a compact Lie group G acts on stratified spaces X and Y, and showed how to carry out the Kasparov product in the absence of transversality, using instead Bott periodicity. We proposed this as an appropriate setting for the study of equivariant intersection theory. Here this program is continued by presenting an excess intersection formula in the spirit of [7], [6] and showing how to perform the Kasparov product in the case where the intersection is not transverse but has enough structure to define a bicycle.

We then show how in the case of a local complete intersection morphism of complex algebraic varieties $f : X \to Y$ one can associate an $f! \in KK_G(X,Y)$. We then extend the excess intersection formula to this context. The formula we obtain can be viewed as the topological analogue of the excess intersection formula in algebraic geometry of Fulton and MacPherson [7].

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2 The excess intersection formula

Let G be a compact Lie group. By a stratified G-space we will mean a Thom-Mather stratified G-space [8] such that G acts differentiably on each stratum and preserves the Thom-Mather data. Consider a fiber square

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & Y_1 \\ \downarrow \psi & & \downarrow \phi \\ X & \xrightarrow{i} & Y \end{array} \tag{1}$$

of stratified G-spaces. Assume i_1 and i are Spin^c normally non-singular inclusions, [8] and [1] and that ϕ and ψ are proper. Then there exist normal bundles $\nu(i)$ and $\nu(i_1)$ of rank d and d_1 respectively which are endowed with Spin^c structures. $\nu(i_1)$ canonically maps to $\psi^*(\nu(i))$, and the fiber square (1) is called *clean* if this map is an injection. Hence we may form $\eta = \psi^*(\nu(i))/\nu(i_1)$ and this is called the excess normal bundle of the fiber square (1). This comes with a Spin^c structure that it inherits from $\nu(i)$ and $\nu(i_1)$. If the fiber square is transversal, then the canonical map is an isomorphism and $\eta = 0$. Thus η measures the deviation from transversality. Assume for convenience that η is of even rank. Consider the virtual G-vector bundle $S(\eta) = S^+(\eta) - S^-(\eta)$, where $S^\pm(\eta)$ denotes the plus and minus spinors for the bundle η. If X_1 is compact, then $S(\eta)$ is an element in $K^0_G(X_1)$. With no compactness assumption, $S(\eta)$ is an element of $KK^0_G(X_1, X_1)$. Then we have the following simple relationship between the Gysin maps of i and i_1.

Lemma 2.1 *Given a clean G-equivariant fiber square (1) in which i_1 and i are Spin^c normally non-singular inclusions and ϕ and ψ are proper, then the following diagram is commutative.*

$$\begin{array}{ccc} K^l_G(X_1) & \xrightarrow{i_{1!}} & K^{l+d}_G(Y_1) \\ \uparrow S(\eta)\psi^* & & \uparrow \phi^* \\ K^l_G(X) & \xrightarrow{i_!} & K^{l+d}_G(Y) \end{array} \tag{2}$$

That is, $\phi^* \circ i_! = i_{1!} S(\eta)\psi^*$.

Consider a general fiber square of stratified G-spaces

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & Y_1 \\ \downarrow \psi & & \downarrow \phi \\ X & \xrightarrow{g} & Y \end{array} \tag{3}$$

where g_1 and g are Spin^c normally non-singular maps, and ϕ and ψ are proper. Let $X \xrightarrow{i} Y \times \mathbf{R}^n \xrightarrow{p} Y$ be a factorization of g where i is a Spin^c normally non-singular inclusion. Then we may form the fiber square

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & Y_1 \times \mathbf{R}^n \\ \downarrow \psi & & \downarrow \phi \\ X & \xrightarrow{i} & Y \times \mathbf{R}^n \end{array} \tag{4}$$

and reduce to the previous case. Given the fiber square (3) we have the canonical map $\nu(i_1) \to \psi^*(\nu(i))$. The fiber square (3) is called clean if the associated fiber square (4) is clean. In this case $\eta = \psi^*(\nu(i)/\nu(i_1)$ is called the excess normal bundle of the fiber square (3).

Lemma 2.2 *We have the following formula* $\phi^* \circ g_! = g_{1!} S(\eta) \psi^*$.

Finally we express this in terms of the Kasparov groups. Given the clean fiber square (3) we may construct $g! \in KK_G(X, Y)$ and $g_1! \in KK_G(X_1, Y_1)$. We can also form $[\psi] \in KK_G(X, X_1)$, $[\phi] \in KK_G(Y, Y_1)$ and $S(\eta) \in KK_G(X_1, X_1)$, see [1].

Theorem 2.3 *One has the following excess intersection formula in* $KK_G(X, Y_1)$*:*

$$g! \otimes_Y [\phi] = [\psi] \otimes_{X_1} S(\eta) \otimes_{X_1} g_1!$$

Let $\Xi_0 = (Z_0, E_0, f_0, g_0)$ be a bicycle from X to Y and $\Xi_1 = (Z_1, E_1, f_1, g_1)$ a bicycle from Y to V. In [4], [5] and [1] it was shown how to take the Kasparov product of Ξ_0 and Ξ_1 when they were transverse. Furthermore, in [1], we showed how to multiply any two bicycles, even when they couldn't be put in transverse position. Here we present something of a compromise. In some cases, Ξ_0 and Ξ_1 won't be transverse but their intersection is well behaved enough to define an element of $KK_G(X, V)$. Suppose our bicycles fit into a clean fiber square

$$\begin{array}{ccc} Z & \xrightarrow{g} & Z_1 \\ \downarrow f & & \downarrow f_1 \\ Z_0 & \xrightarrow{g_0} & Y \end{array} \tag{5}$$

where Z is a stratified G-space and g is a normally non-singular Spin^c map. Then form a new bicycle $\Xi = (Z, g^* E_1 \otimes f^* E_0 \otimes S(\eta), f_0 \circ f, g_1 \circ g)$.

Theorem 2.4 *The product of the bicycles* $\Xi_0 \circ \Xi_1$ *is given by the bicycle* Ξ.

3 Locally complete intersection morphisms

In this section, let $G_{\mathbf{C}}$ be a complex reductive group. Let G be a maximal compact subgroup. Let $G_{\mathbf{C}}$ act on complex quasi-projective algebraic varieties X and Y in an algebraic manner. Then also G acts on X and Y and one can find stratifications of X and Y and control data which is compatible with the action of G. Define the group $K_{G_{\mathbf{C}}}(X)$ to be $K_G(X)$. Given two maximal compact subgroups, the corresponding K-groups are isomorphic by a unique isomorphism. A G-morphism $f : X \to Y$ is called an equivariant local complete intersection morphism if there is a factorization of f as

$$X \xrightarrow{i} M \times Y \xrightarrow{p} Y$$

where i is an equivariant regular embedding [6] , M is a non-singular G-variety, and p is the projection. We will now show how to associate to such a morphism an equivariant bicycle. The morphism $p : M \times Y \to Y$ is normally non-singular so we already know how to define $p! \in KK_G(M \times Y, Y)$ so it remains to construct $i!$ for a regular embedding $i : X \to Y$. We will in fact give two ways. The first applies only to K-theory and uses clutching ideas and works completely generally needing no form of transversality.The second may generalize to other homology theories but relies on transversality and uses a construction reminicent of MacPherson's graph construction.

Given such a regular embedding let $\nu(i)$ be its normal bundle defined as in algebraic geometry [7]. Contrary to the case of normally non-singular morphisms however, $\nu(i)$ is not necessarily homeomorphic to a regular neighborhood of X in Y. Both constructions need the following data. As in [3], there exists an algebraic G-vector bundle E and a G-invariant section $s : Y \to E$ so that X is the scheme theoretic zero set of s. This follows from the basic constructions of [3] together with some results from [9] to guarantee that they may be done equivariantly. Choose a G-invariant classical regular neighborhood such that $\bar{U}$ sits inside Y as a π-fibered subset. Let $r : \bar{U} \to X$ be the retraction of $\bar{U}$ to X. As in [3], choose a complex subbundle C of $E|_{\bar{U}}$ so that $\nu \oplus C|_X \cong E|_X$. Let $Q = E/C$ over $\bar{U}$. Let $\bar{s} : \bar{U} \to Q$ denote the natural quotient map induced by s. By shrinking U is necessary we can arrange that

$$\bar{s} : (\bar{U}, \bar{U} - X) \to (Q, Q - \{0\}) \tag{6}$$

Let $S^{\pm}$ denote the spinors on Q, $\pi : Q \to \bar{U}$ the natural projection. There exists the natural Clifford multiplication homomorphism

$$\pi^* S^+ \to \pi^* S^-$$

Now consider $\bar{s}^*\pi^*S^+ \to \bar{s}^*\pi^*S^-$ on $\bar{U}$. Since $\bar{s}$ satisfies (6), we may double $\bar{U}$ along its boundary obtaining Z. Form a vector bundle F on Z by putting $\bar{s}^*\pi^*S^+$ on one copy of $\bar{U}$ and $\bar{s}^*\pi^*S^-$ on the other copy and gluing along the boundary where Clifford multiplication is an isomorphism due to (6). The natural map $g : Z \to Y$ is normally non-singular since $\bar{U}$ was π-fibered. The retraction map $r : Z \to X$ is proper. Then $i!$ is defined by the bicycle (Z, F, r, g). We have chosen to use the spinors only because they provide a nice "assembling" of the Koszul complex of vector bundles into its even and odd parts.

The second construction relies on the same data, $E, s, Q, \bar{s}$. Replacing $\bar{s}$ by one which approximates it, we may assume that $\bar{s}$ is controlled, relative to obvious control data on Q. Consider $t_0 : \bar{U} \to Q$ the zero section. t_0 is a Spin^c normally non-singular inclusion, so according to [8], 5.3, there exists a cobordant section $t : \bar{U} \to Q$ so that $\bar{s}$ and t are transverse. Consider the set

$$X_\lambda = \{x \in \bar{U} | t(x) = \lambda \bar{s}(x)\}$$

X_λ is normally non-singularly included in Y for generic λ. For λ large, X_λ is contained in any particular regular neighborhood of X in Y. Define $i!$ to be given by the bicycle $(X_\lambda, \mathbf{1}, r, i)$ where $\mathbf{1}$ is the trivial one-dimensional vector bundle, r is the retraction given by one of our regular neighborhoods, e.g. $\bar{U}$.

For an lci morphism $f : X \to Y$, factor it as above, $f = p \circ i$. Then define $f! = i! \otimes_{M \times Y} p!$.

Proposition 3.1 *Let $f : X \to Y$ be an equivariant locally complete intersection morphism of quasi-projective G-varieties. Then*

1. *$f! \in KK_G(X, Y)$ is independent of all choices in its construction. In particular, the two constructions define the same class in $KK_G(X, Y)$.*
2. *If $f_1 : X \to Y$ and $f_2 : Y \to Z$ are two equivariant lci morphisms, then*

$$(f_2 \circ f_1)! = f_1! \otimes_Y f_2!$$

We now have the following analogue of the excess intersection formulas above.

Theorem 3.2 *Consider a fiber square*

$$\begin{array}{ccc} X_1 & \xrightarrow{g_1} & Y_1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} \\ X & \xrightarrow{g} & Y \end{array} \tag{7}$$

which is Cartesian in the sense of algebraic geometry, where all the spaces are quasi-projective algebraic G-varieties and the maps g and g_1 are local complete intersection morphisms. Then one has the formula

$$g! \otimes [\phi] = [\psi] \otimes S(\eta) \otimes g_1!$$

in $KK_G(X, Y_1)$.

Note that in the algebraic case, with the normal bundles defined as in algebraic geometry, the canonical map of normal bundles is always injective [7].

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