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## Noncommutative Geometry

by Alain Connes

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REVIEWED BY JONATHAN BLOCK

**W**hat is geometric about noncommutative geometry?

Gelfand [Gelfand] and Grothendieck [Gr] taught us the value of treating a commutative ring as though it were the ring of functions on an underlying geometric object. But why stop there? For many years, algebraists and algebraic geometers have tried to extend the tools of their trade—localization, sheaves, differentials, schemes, and so forth—to the context of noncommutative rings. The question then becomes whether or not this is valuable.

The exciting development of the past two decades, initiated in the arena of functional analysis and differential geometry by the author of the book under review, is the appearance of a host of interesting examples of rings in which one feels there ought be geometry. These rings arose in applications to operator algebras, differential topology and geometry, and theoretical physics. As such, the “geometric” study of these rings is rewarded by the success of the applications. Another bonus of this example-based approach is that the examples suggest the tools that facilitate their study, much the same way that sheaves were invented to study topological spaces and algebraic varieties. Thus, many of the tools that come out of Connes’s theory are quite unexpected.

This beautiful, ambitious, and erudite book explains, through many examples, the phenomena, tools, and some of the applications of noncommutative geometry. *Noncommutative Geometry* is Connes’s answer to the

question, “What is geometric about noncommutative geometry?”

The mathematical framework from which Connes’s theory arose and from which it is, for the most part, still cast is the theory of operator algebras or, more specifically,  $C^*$ -algebras. A  $C^*$ -algebra is simply a subalgebra  $A \subseteq L(H)$  of the bounded linear operators on a Hilbert space  $H$  which is norm-closed and closed under the adjoint operation on operators. There is an abstract characterization of a  $C^*$ -algebra as a Banach  $*$ -algebra in which the algebra structure and Banach space structure are compatible. This compatibility is expressed by the condition that for all  $x \in A$  one has  $\|xx^*\| = \|x\|^2$ ; see [Gelfand]. However, this definition hides one of the features that makes  $C^*$ -algebras so natural: although the norm may appear to be an additional structure, it is, in fact, uniquely determined by the algebra structure and can be derived from it by the formula  $\|x\| = (\text{spectral radius } xx^*)^{1/2}$ .

Another significant feature of  $C^*$ -algebras, at least for most of the deep applications, is the spectral theorem for continuous functions acting on normal elements of  $A$ . (An element  $x$  of  $A$  is called normal if  $xx^* = x^*x$ .) Any commutative  $C^*$ -algebra is isomorphic to the ring of continuous functions  $C_0(X)$  which vanish at infinity on a locally compact set  $X$ ; see [Gelfand]. Thus, a commutative  $C^*$ -algebra contains exactly the information of a locally compact topological space. It is in this way that noncommutative  $C^*$ -algebras are assumed to play the role of noncommutative topological spaces.

A basic example of a noncommutative  $C^*$ -algebra is  $M_n(\mathbb{C})$ , the algebra of  $n \times n$  complex matrices, or a direct sum of these. (This exhausts all finite-dimensional examples, by Wedderburn’s theorem.) A slightly more interesting example is  $M_n(C_0(X))$ , the algebra of  $n \times n$  matrices of functions



on a locally compact space  $X$  which vanish at infinity.

A typical way to form  $C^*$ -algebras is to represent a  $*$ -algebra on a Hilbert space and take its norm closure. For example, let  $\Gamma$  be a discrete group. Then, the algebraic group algebra  $\mathbb{C}[\Gamma]$  is represented naturally, via the left regular representation, on  $l^2(\Gamma)$ , and its norm closure  $C_r^*(\Gamma)$  is a  $C^*$ -algebra called the reduced group  $C^*$ -algebra of  $\Gamma$ . In the case where  $\Gamma$  is abelian,  $C_r^*(\Gamma)$  is isomorphic to  $C_0(\hat{\Gamma})$ , the continuous functions which vanish at infinity on the Pontrjagin dual of  $\Gamma$ . For general  $\Gamma$ , one likes to think of  $C_r^*(\Gamma)$  as describing the noncommutative space  $\hat{\Gamma}$ , which for many nonabelian  $\Gamma$  is a very badly behaved classical topological space.

Connes's book starts out appropriately with one of the motivations of the theory of operator algebras, namely quantum physics. In the very first section, he shows how Heisenberg rediscovered matrix multiplication from empirical results [H]. Classically, observable physical quantities are represented by functions on phase space. Connes shows that if you assumed this classical picture, then the set of frequencies emitted by an atom would form a semigroup inside the real numbers  $\mathbb{R}$ ; that is, the sum of two emitted frequencies would also be one. The algebra of observables would be the convolution algebra on the group generated by this semigroup, which (since it is abelian) is then  $C_0(\hat{\Gamma})$ .

Experiments show, however, that this is not the case. They show that the frequencies can be represented by the differences of a small number of terms  $\nu_{ij} = \tau_i - \tau_j$  ( $i, j$  in some index set  $I$ , which is discrete so as to conform to the observed discreteness of possible energies of an atom). They, therefore, obey the Rydberg–Ritz combination principle,  $\nu_{ij} = \nu_{il} + \nu_{lj}$ . One then sees that the set of possible frequencies does not form a group, but rather a groupoid. (A groupoid is like a group in having an associative multiplication and inverses, but unlike a group, only certain pairs of elements can be multiplied.) Let  $\Delta = \{(i, j) | i, j \in I\}$ , and assume the product  $(i, j) \cdot (k, l)$  exists if and only if  $j = k$  and then the product is  $(i, l)$ . The algebra of observables is

then given by sums  $\sum a_{(i,j)}$  with product dictated by the combination principle,

$$(a \cdot b)_{(i,j)} = \sum_k a_{(i,k)} b_{(k,l)},$$

which one recognizes as the usual rule for matrix multiplication.

This is actually a special case of a general construction of a groupoid algebra, similar to a group algebra. Thus, if  $G$  is a groupoid,  $\mathbb{C}[G]$  consists of finite formal sums  $a = \sum_{\gamma \in \Gamma} a_\gamma a_\gamma \in \mathbb{C}$ . The multiplication is given by

$$(a \cdot b)_\gamma = \sum_{\gamma_1 \cdot \gamma_2 = \gamma} a_{\gamma_1} b_{\gamma_2}.$$

One can then form a  $C^*$ -algebra  $C_r^*(G)$  much the same way as we did for a group, by taking the norm closure in some representation. Also, let me remark that there is a version of the group and groupoid  $C^*$ -algebra when  $G$  has a topology. The matrix algebras and group algebras are, thus, special cases of groupoid algebras.

Let me try to explain some of the situations where noncommutative spaces arise and some of the examples that Connes addresses in his book. One basic operation throughout mathematics which causes considerable difficulties is the process of forming quotients. Quotients of topological spaces by subspaces or, more generally, by equivalence relations are often

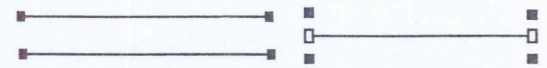
poorly behaved. They can fail to be Hausdorff or can easily have the trivial topology (only two open sets). Algebraic geometers are familiar with the process of forming moduli spaces. This process itself is a quotient process. Again, many problems are faced because the quotient of a variety is not a variety or the quotient of a scheme is not a scheme. Notions such as stack or algebraic space are generalizations of schemes invented so as to include certain quotients. One of Connes's basic ideas is that quotients are often best expressed as noncommutative spaces.

An important framework for dealing with quotients is, again, the notion of groupoids. They also arise in algebraic geometry in the theory of stacks. So, for example, an equivalence relation on a set  $X$ ,  $\mathfrak{R} \subset X \times X$ , gives rise to a groupoid (which I will also call  $\mathfrak{R}$ )

defined as follows. The elements of  $\mathfrak{R}$  are just the pairs  $(x, y) \in \mathfrak{R}$ ; the multiplication  $(x, y) \cdot (z, w)$  exists if  $y = z$ , and the product is then  $(x, w)$ , which, again, is in  $\mathfrak{R}$  by transitivity. The inverse of  $(x, y)$  is  $(y, x)$ ; and we see another feature of the notion of groupoid: there is not just one unit, but many. In the case of  $\mathfrak{R}$ , the units are the elements  $(x, x)$  and can therefore be identified with  $X$ .

In general, we can think of a groupoid as a sort of equivalence relation on the set of units. If, for example,  $X$  is a discrete set, then the set of equivalence classes of  $\mathfrak{R}$  can be naturally identified with the set of irreducible representations of  $C_r^*(\mathfrak{R})$ .

In general, we think of  $C_r^*(G)$  as the quotient of the set  $X$  of units of  $G$  by the groupoid. However, the identification of the set of equivalences with the irreducible representations breaks down. Although the quotient space may be really badly behaved, the algebra is fine. To bring things down to earth, consider the space  $[0, 1] \times \{0, 1\}$ , the disjoint union of two copies of the unit interval. Define an equivalence relation  $\sim$  by  $(t, j) \sim (s, i)$  if  $0 < s = t < 1$ .



An equivalence relation with a non-Hausdorff quotient

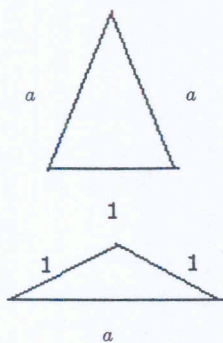
$X$  is the space on the left above, and the interiors of the two intervals get identified to a single open interval yielding the figure on the right. The groupoid algebra of the corresponding equivalence relation can be identified with  $\{f : [0, 1] \rightarrow M_2(\mathbb{C}) \mid f(0) \text{ and } f(1) \text{ are diagonal matrices}\}$ . Why is this groupoid algebra a better description of this quotient than the quotient itself? For example, the actual continuous functions on the quotient are exactly the same as on a closed interval, whereas the groupoid algebra retains a richer structure which encodes interesting features of the space.

Another example is a favorite of Connes's. Aperiodic tilings of the plane arose in the 1950s in connection with the following problem. Suppose you are given a finite number of isometry types of tiles (prototiles) and are asked if you can tile the plane with this set of



tiles. (Of course, you are given an infinite number of copies of each type of tile.) Is there an algorithm to decide if this set tiles the plane? Wang [GS] showed that there was no such decision procedure if and only if there was a set of tiles which tiled the plane, but could only tile the plane aperiodically. The search was on, and soon there was found an example of a set of prototiles which was aperiodic (that is, tiled only aperiodically). This first example had a ridiculously large number of prototiles in it. This number was gradually reduced until Penrose found his beautiful examples which had only two tiles.

An example of a set of Penrose tiles is



Here,  $a = (1 + \sqrt{5})/2$ . We will not go into why they tile the plane, but they do. In fact, they tile the plane in an infinite number of ways, an infinite number of distinct ways.

We say two tilings are equivalent if there is a rigid motion of the plane which brings one set of tiles onto the other. Connes constructs the moduli space of tilings of the plane as a noncommutative space. Although Penrose tilings are aperiodic, they possess a property called quasiperiodicity, which we will express by the fact that any finite patch of tiles in one tiling by these two prototiles occurs infinitely often in any other tiling by the same prototiles.

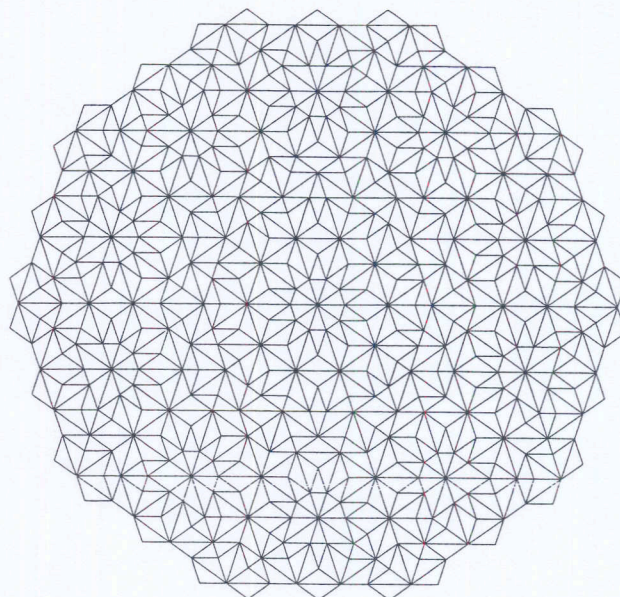
Thus, if you know all the possible ways to tile the plane with these two tiles, and you are given a tiling, it would still be impossible to determine which tiling you have in hand by looking at a finite piece. This is an expression of the fact that every tiling is arbitrarily close to every other tiling, so that the classical quotient has the trivial topology, with only two open sets.

According to Connes, the way to

make sense of this moduli space is as follows: First, one can identify the space of such tilings with  $K$ , the space of sequences  $z_n$  of 1's and 0's which don't have two 1's in a row. (I will not tell you how this is done, but it is explained in Connes's book.) Then the tiling corresponding to  $z_n$  will be equivalent to the tiling corresponding to  $z'_n$  if and only if, eventually, these two sequences coincide; that is, there is an  $n > 0$  such that  $z_j = z'_j$  for all  $j \geq n$ . Hence, the moduli space is the

quotient  $K/\mathfrak{R}$ , where  $\mathfrak{R}$  is the equivalence relation of eventual equality of sequences. One then forms the  $C^*$ -algebra of this equivalence relation (qua groupoid). This is the correct "quotient." Unlike the non-Hausdorff example above, where the algebra was not very noncommutative, this algebra is highly noncommutative. In fact, it is a simple  $C^*$ -algebra. Out of this algebra, one can read off some beautiful properties of the space of tiles. For example, for a finite patch of tiles, one can interpret the density of occurrence of this patch in any tiling as a sort of dimension associated to the patch, and because of the properties of this algebra, the density must be an element of the subgroup  $\mathbb{Z} + a\mathbb{Z}$  of  $\mathbb{R}$ .

Connes describes many more examples in his book. He also describes some ways in which noncommutative geometry enters into the index theory, the Novikov conjecture, harmonic analysis, geometry, and theoretical physics. Along the way, he develops in a tourist-friendly way the tools, like  $K$ -theory, asymptotic morphisms, and cyclic cohomology. One can read this material on many levels. The basic book is written in a way that anyone can get some of the feeling and ideas of the subject. For those who want more details, there are many appendices that cover more



Penrose tiling. From Connes, *Noncommutative Geometry*, p. 89, ©1994, Academic Press.

technicalities, and for those who want to become experts, there are many references and an extensive bibliography pointing the reader to the right place. For all these people, Connes has accomplished the wonderful feat of explaining in a simple and coherent way 20 years (or so) of his impressive work. I recommend this book most highly.

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